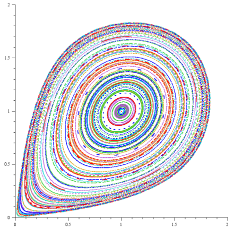

Quasiperiodic Patterns in Biology and Elsewhere

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Talk plan

- Background
- Continuous RD systems
  - Multiple species, D dim.
  - Single species, one dim.
  - Patterns and Newton
  - Stability and Schrödinger - nogo
- Discrete RD systems
  - Cellular networks
  - One species, one dim. lattice
  - Discrete Newton!
- Alternative dynamics: Active transport
  - Auxin example - simple model
  - Same patterns, better stability
- Quasi-periodic patterns from discrete spatial Newton eq.
  - Area-preserving maps
  - Auxin patterns, quasi-periodicity and conservative maps
- Conclusions

Continuous RD – dynamics

- Multispecies, D dim: general RD eqs. for $\{u_i(\mathbf{r}, t)\}$:

$$\dot{u}_i = D_i \nabla^2 u_i + f_i(u_1, \ldots, u_n)$$

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- Has a Lyapunov function $S =$ Lagrange action, decreases!

$$S = \int \left(\tfrac{1}{2} u'^2 - U(u) \right) dx$$
$$\Rightarrow \dot{S} = - \int \dot{u}^2 dx \leq 0$$

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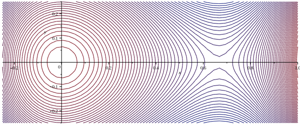
- Pattern $u(x)$ defined by level curve of H in phase space u, u':
  $\Rightarrow$ typically periodic patterns:



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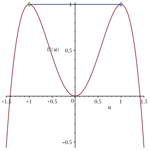
- Only chance if u monotonously interpolates between two deg. max of $U(u)$!



So **no stable** oscillatory patterns!

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- Most general case: consider RD version for multiple species on arbitrary graph G:

$$\dot{\mathbf{u}}_i = \mathbf{D} \sum_{j \in \mathcal{N}(i)} (\mathbf{u}_j - \mathbf{u}_i) - \mathbf{f}(\mathbf{u})$$

where $\mathcal{N}(i)$ defines the set of neighbor nodes of node i.

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in simplified notation, where u_-, u, u_+ are short for u_{i-1}, u_i, u_{i+1}.

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  in simplified notation, where u_-, u, u_+ are short for u_{i-1}, u_i, u_{i+1}.
- Properties similar to continuous case:
  – Lyapunov: $S = \sum(-uu_+ + F(u))$, with $F' = f$
  – Stat. sol's, patterns stable if local minima of S.

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- So **no-go?**

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  - more complicated dynamics
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  - active transport replacing the local reaction term,
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  - active transport replacing the local reaction term,
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- Written out in its full glory, the simplest auxin model dynamics reads:

$$\dot{u} = D\left(u_{+} - 2u + u_{-}\right) + T\left(\frac{uu_{+}}{u_{++} + u} - u + \frac{uu_{-}}{u + u_{--}}\right)$$

where D and T are parameters governing the rate of diffusive and active transport, respectively.

Auxin model – continued

- More compactly, the auxin dynamics can be written as a continuity eqn.:

$$\dot{u} = -\Delta_+ I = -\Delta_+ u u_- \Delta_- \left(\frac{D}{u} - \frac{T}{u_+ + u_-} \right),$$

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- Note linearity of eqs. $\Rightarrow$ can rescale solutions.
- Pattern eqs. follow from $I = 0$:

$$\frac{D}{u} - \frac{T}{u_+ + u_-} = -C$$

  where C is a positive integration constant.

Example – auxin model

- **Miracle 1**: the auxin pattern eq. can be turned around to yield a discrete Newton type pattern eq. with a simple, rational $f(u)$, here given by:

$$f(u) = \frac{2u}{\mu + Ku}$$

where $\mu \in [0, 1] = 2D/T$ gives the relative balance between passive and active transport, while $K > 0$ is a mere rescaling of C.

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- By rescaling u, K can be rescaled to an arbitrary number, suitably chosen as $1 - \mu$, yielding the auxin pattern eqs. in the standard version:

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- This eq. will turn out to have some **unusual properties**!

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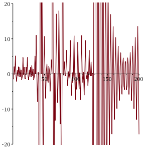
- *Discrete* Newtonian patterns normally are **not** conservative!



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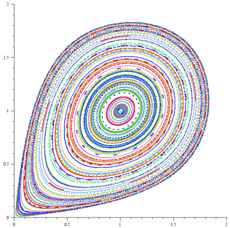

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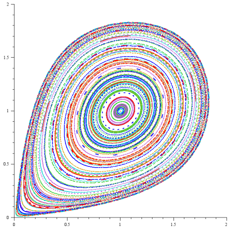

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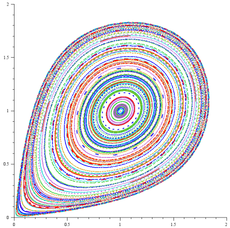

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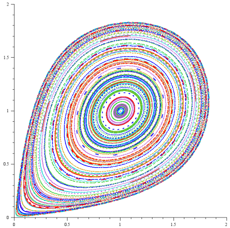

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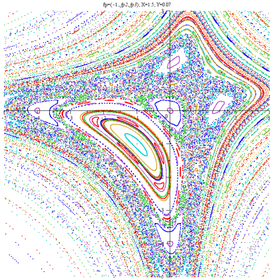
- Compare to the **chaotic** trajectories of a generic, non-integrable Newton eqs. in u, v plane:



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- H symmetric, finally $\Rightarrow$ H must be **biquadratic**:

$$H(u, v) = Au^2 v^2 - Buv(u+v) + C(u^2+v^2) + Duv - E(u+v) + F$$

Auxin map conservative?

- Working backwards from general conserved $H(u, v)$:
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- ...but multiplying upstairs/downstairs by $(1-\mu)u - (2+\mu)$ helps! (Miracle nr 2!)

$$f(u) = \frac{2(1-\mu)u^2 - 2(2+\mu)u}{(1-\mu)^2u^2 - 2(1-\mu)u - \mu(2+\mu)}$$

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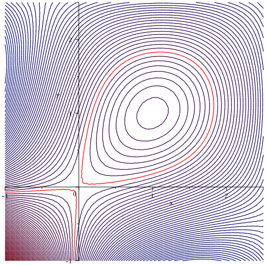

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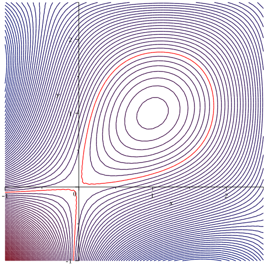

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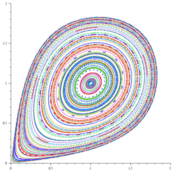

- ...smooth, non-chaotic patterns!

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- Active transport helps stability of Newtonian patterns.
- Unexpected spatial conservation law in auxin system.
- Q: What with higher dim.? Similar special properties for certain choices of reaction term? Applications in astronomy?
- Extra take-home lesson: Be careful when simulating one-dim. Newton eqs: conservation law spoilt, except for a special integrable class of f!

That's all, folks!

THANK YOU!