LUND UNIVERSITY

The three colors of nature

- This talk will be about the strong force and the three "colors" of nature
- Motivation: QCD is the "strongest" force
- If there would be infinitely many colors calculations would be so much simpler
- Dealing with three colors
- Three colors in parton showers
- Group theory based bases
- Conclusion and outlook

Lund
April 10, 2013
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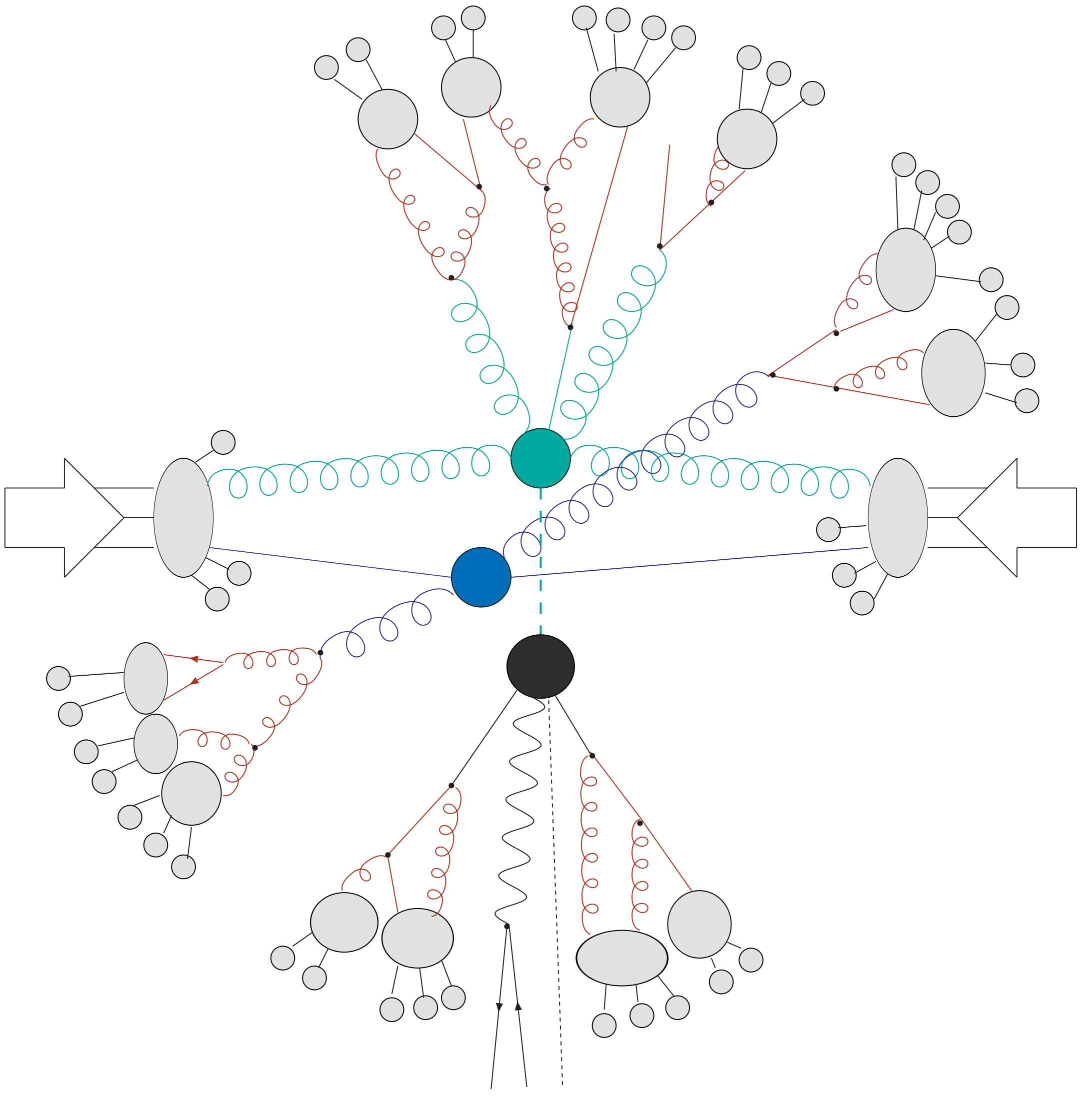
QCD is the "strongest" force

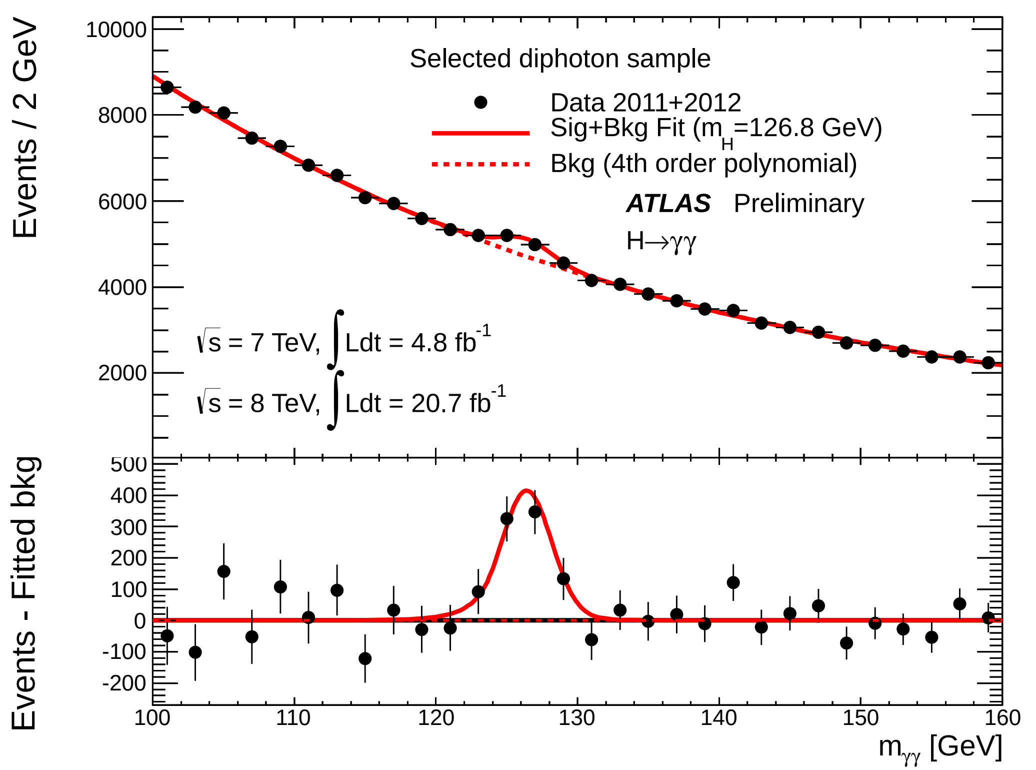
- For normal collider energies the strong force is about a factor 10 stronger than the electroweak force
- The gravitational force is in comparison completely negligible as long as we are far from the Planck scale 10^{19} GeV
- To first approximation the Large Hadron Collider is a QCD machine

Most activity at the LHC is strong



LHC is built to discover new physics

ATLAS-CONF-2013-012



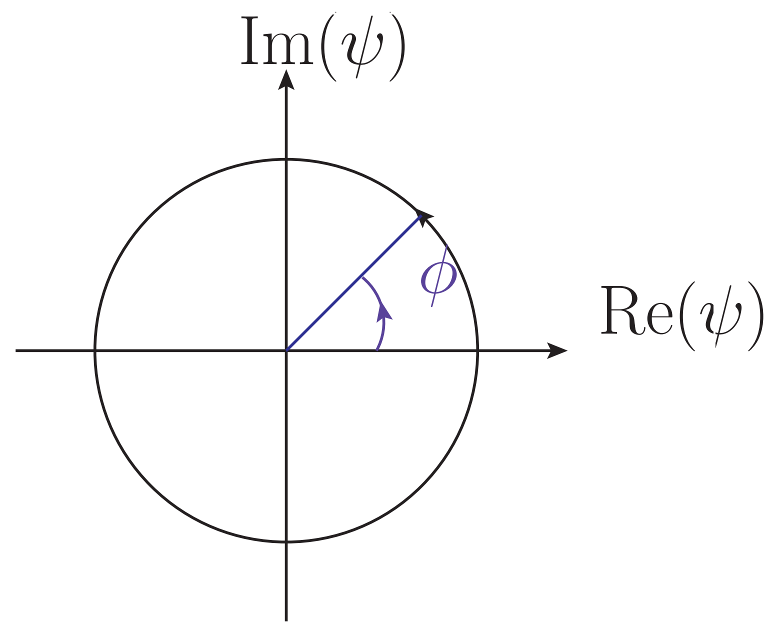
- To recognize new physics it is absolutely essential to understand the manifestations of QCD

So, how does QCD work?

Warm-up: Quantum electrodynamics

Quantum electrodynamics can be seen as coming from invariance under change of phase: physics behave the same if all particles wavefunctions are multiplied with a phase

$$\psi \rightarrow e^{i\phi}\psi$$



Interactions can be derived by letting the phase ϕ vary from point to point. The photon, giving the interaction, is related to the phase.

Quantum chromodynamics

The strong force can similarly be seen as coming from rotations in a three dimensional complex space. Each particle has a "red" a "green" and a "blue" component, and if all particle fields are rotated as

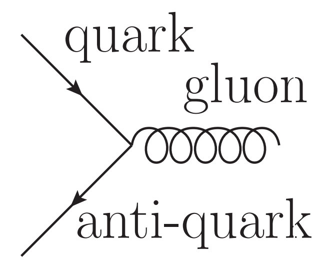
$$\begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix} \to \exp\left[i \begin{pmatrix} \lambda_3 + \lambda_8 & \lambda_1 - i\lambda_2 & \lambda_4 - i\lambda_5 \\ \lambda_1 + i\lambda_2 & \lambda_8 - \lambda_3 & \lambda_6 - i\lambda_7 \\ \lambda_4 + i\lambda_5 & \lambda_6 + i\lambda_7 & -2\lambda_8 \end{pmatrix} \right] \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$$

physics doesn't change. Again, interaction can be derived by choosing different rotations in different points in spacetime. Here we have 8 d.o.f., $\lambda_1 \ldots \lambda_8$ corresponds to 8 gluons. The various rotations don't commute so QCD is non-Abelian $\to$ gluons interact with gluons.

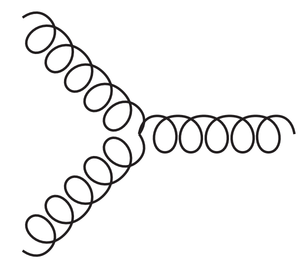
The strong interaction

The interaction can be described by three vertices

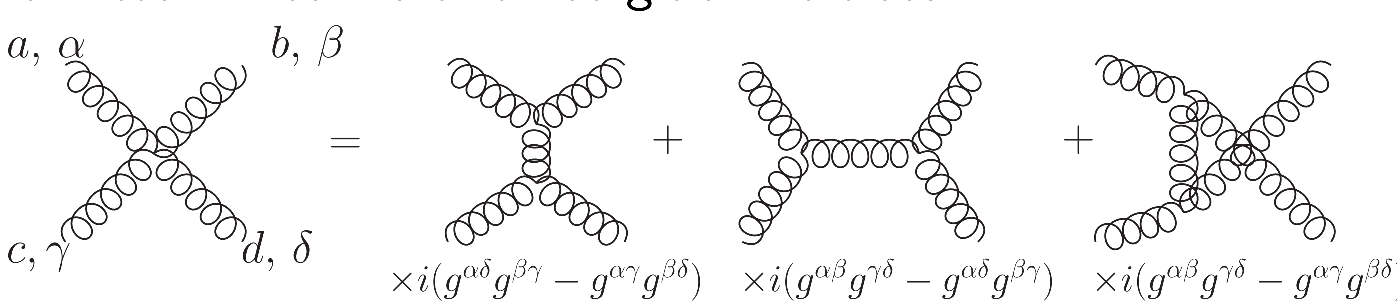
- A quark and an anti-quark can form a gluon, or a gluon can split into a $q\bar{q}$-pair, or a quark or an anti-quark can radiate a gluon



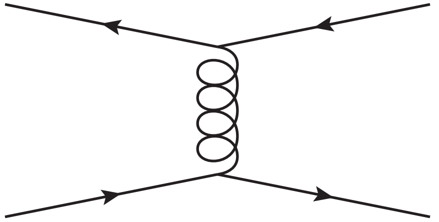
- A gluon can split into two gluons or two gluons can merge into one



- There is also a four-gluon vertex, but this can be rewritten in terms of three gluon vertices

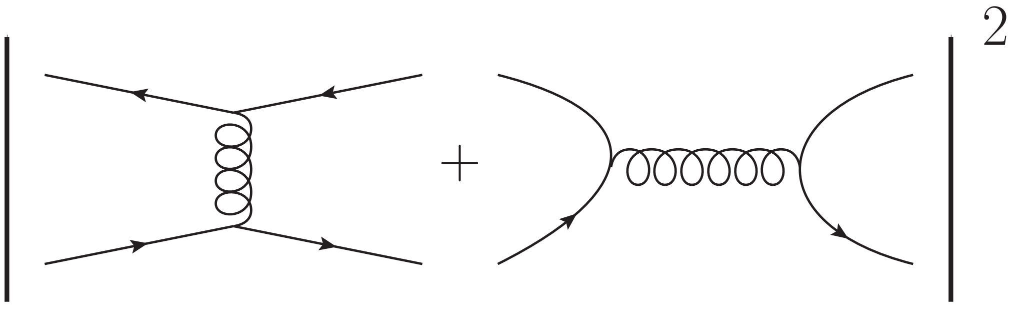


- In general the strong interaction is given by joining these vertices to Feynman diagrams



- By adding all Feynman diagrams that correspond to one process and squaring we can calculate the probability for that process to take place

probability $\propto$



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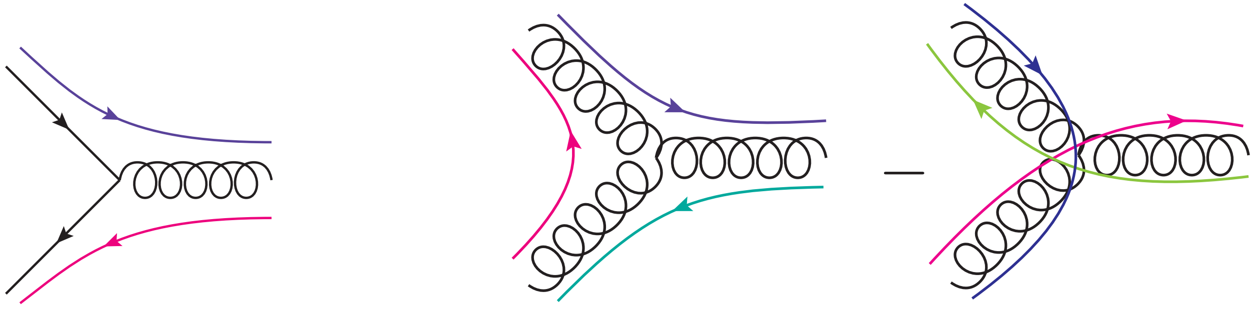
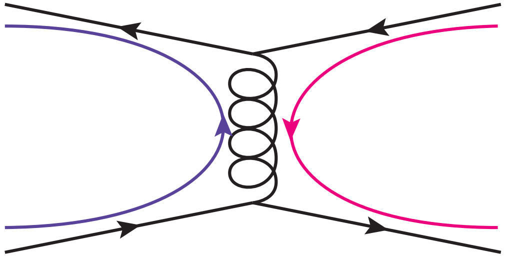
- The latter is done in parton showers such as PYTHIA
- To get the best of both types of calculations one also tries to match the two types of calculations to each other (Stefan Prestel's thesis)

- Of course we must specify what the Feynman diagrams mean
- So, how do we compute with Feynman diagrams?
- They have some momentum and spin part, same as in QED $\rightarrow$ different talk (QFT course)
- They have a color part

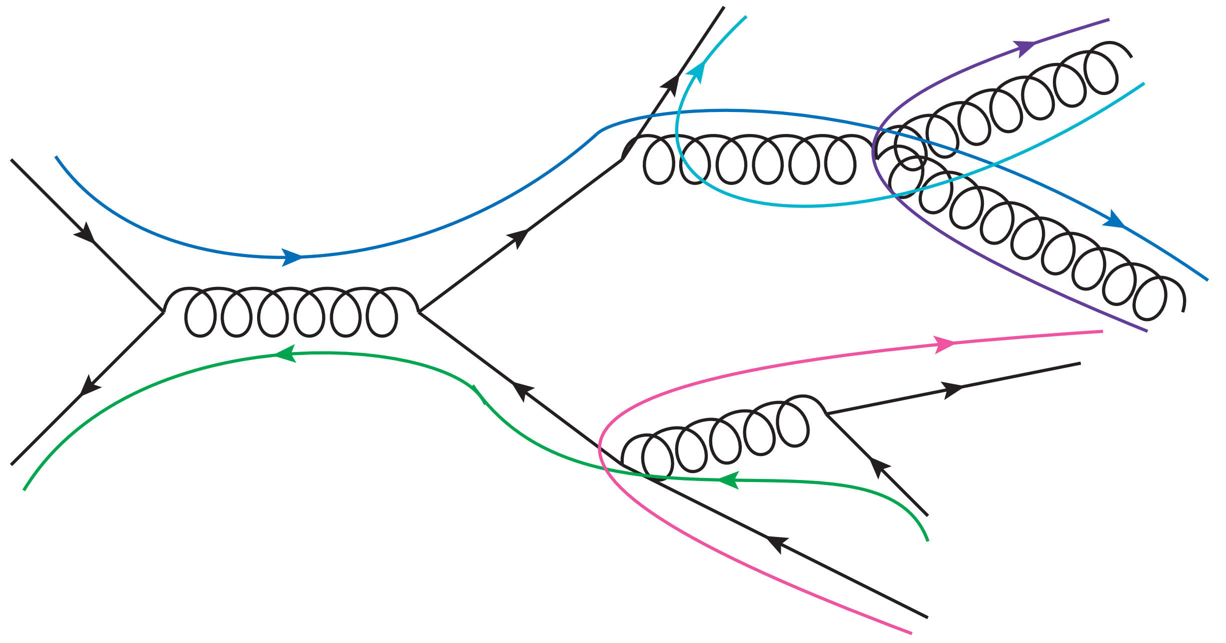
- For an infinite number of colors it is simple
- For each quark we have a color, for each anti-quark we have an anti-color, and for each gluon one color and one anti-color
- The vertices are



- For the Feynman diagrams we just join the lines

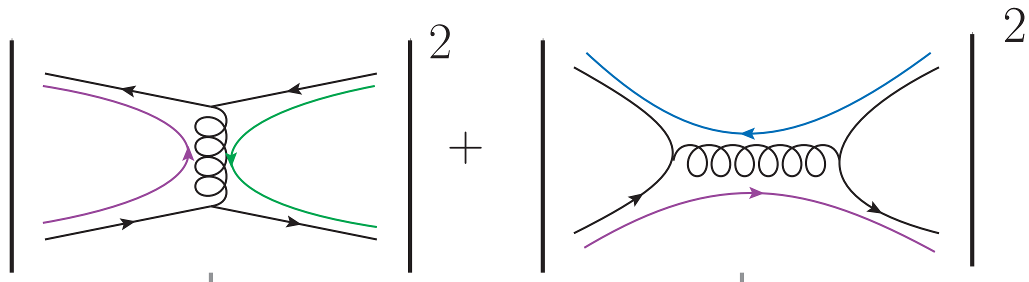


- Even if we have something more complicated the drawing of color-lines is easy



For each quark-gluon vertex there is a unique way of drawing the color lines and for each triple gluon vertex one randomly picks one configuration

- In QCD color is confined, so we never observe individual colors, we always average over initial colors and sum over outgoing
- Since all lines have different color and cannot interfere the summing/averaging the color part is easy



$\frac{1}{N_c}\frac{1}{N_c}N_cN_c$ $\times$momentum part

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Each quark-line gives just a factor N_c

- Furthermore diagrams tend not to interfere as interference tends to be forbidden by the color structure

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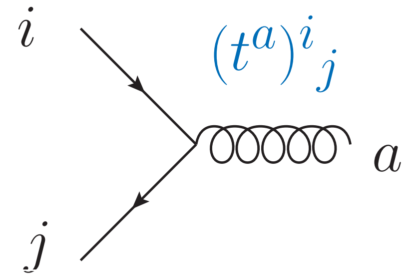
- It turns out that, for various reasons, the color suppressed terms tend to be suppressed by two powers of N_c and $1/9$ is a fairly small number

- On the other hand, we may have *many* color suppressed terms and sometimes we do have $1/N_c \to$ it would be nice to do better

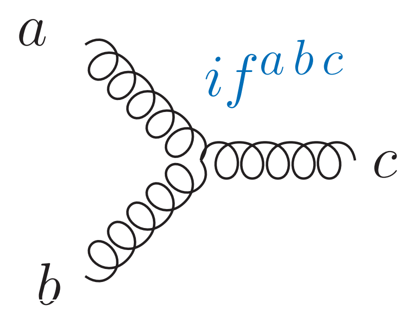
In nature $N_c = 3$

In nature, for $N_c = 3$, how do the quarks and gluons really couple to each other?

- Each of the three quark colors couples to each of the three anti-quark colors and one of the 8 gluons using a matrix. As there are 8 gluons there are 8 different matrices. They are the SU(3) versions of the SU(2) Pauli-matrices used for describing spin, the generators.

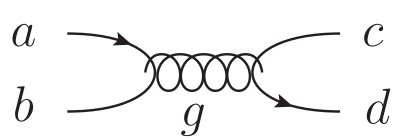


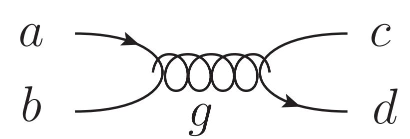
- Each set of three gluons couple to each other via some real constants f^{abc} out of which some vanish (structure constants)



- The four-gluon vertex is rewritten in three gluon vertices

- For an amplitude described by a Feynman diagram, we have to sum over all the internal lines (as always in quantum mechanics; the corresponding particles are not observed)
- The color part of an amplitude A

$A = \sum_g$ 

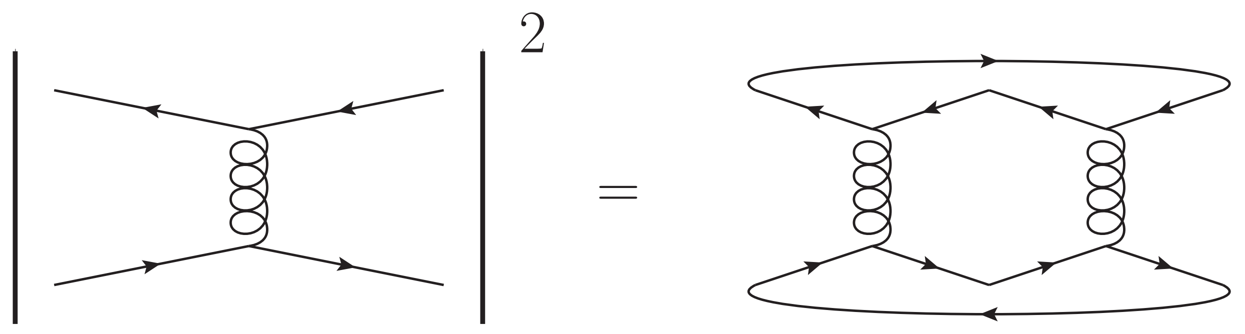
$=$ 

can be written as

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d$$

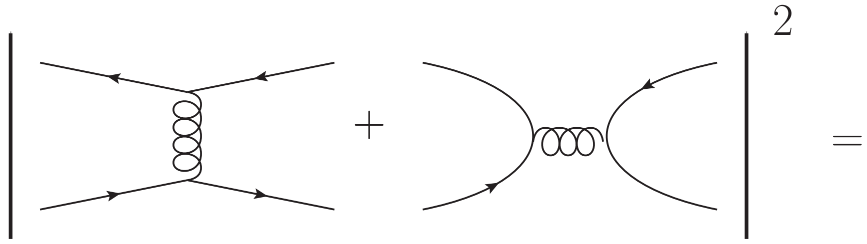
where we *sum* over the possibilities of *each* internal line

- Actually, as mentioned, in QCD we always sum over the possible colors of the external lines as well (different from spin which we may actually measure)
- If we implicitly sum over the possible color options of all lines (quark-lines and gluon-lines) which are internal, calculations of color structure in QCD boils down to trying to calculate graphs like



color summed/averaged

- Actually, it's more complicated, since now the different Feynman diagrams interfere



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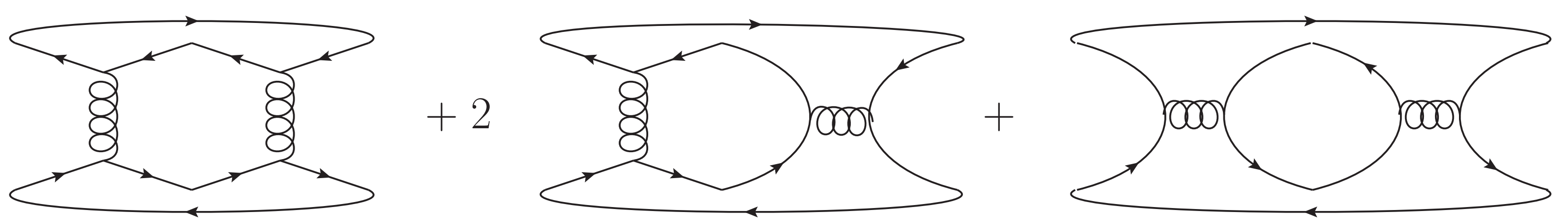
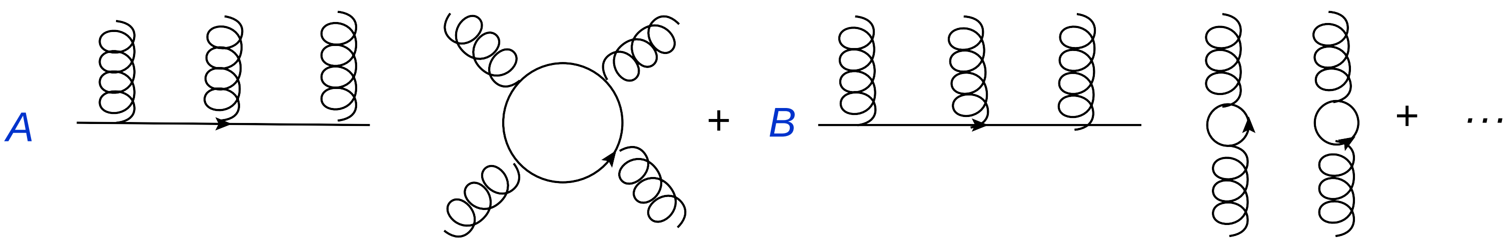
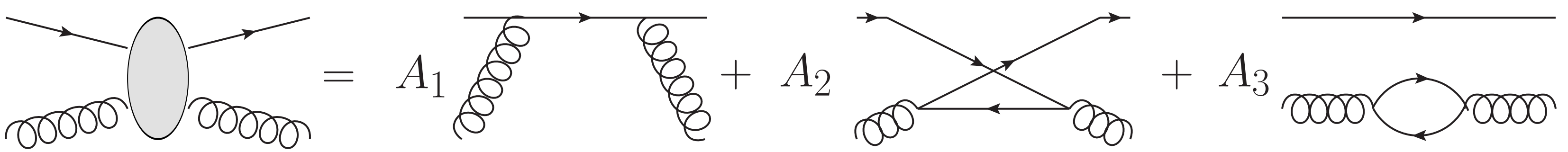


- It is easy to argue that the color space is a vector space and one way of dealing with the amplitudes in color space is to find a basis
- In general an amplitude can be written as linear combination of different color structures, like



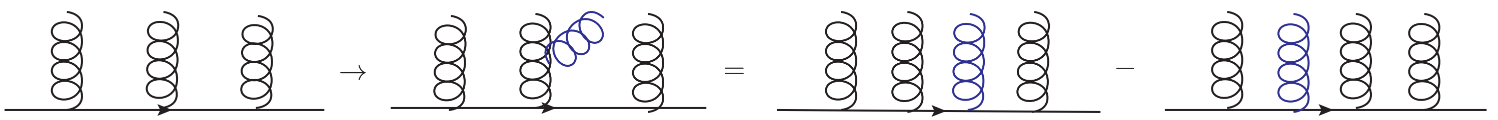
- For example for $qg \to qg$



(an incoming quark is the same as an outgoing anti-quark)

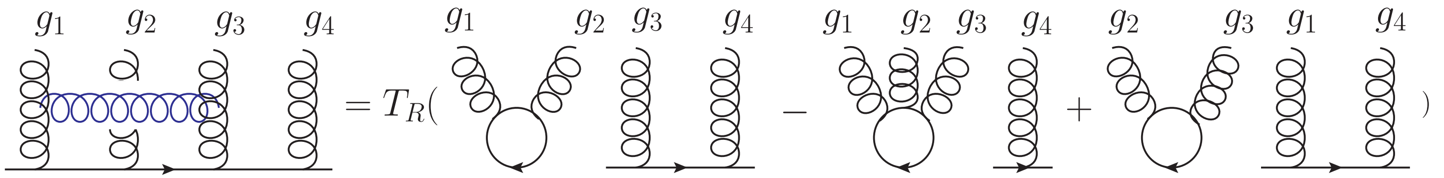
It has some nice properties

- The effect of gluon emission is easily described:



Convention: + when inserting after, minus when inserting before.

- So is the effect of gluon exchange:



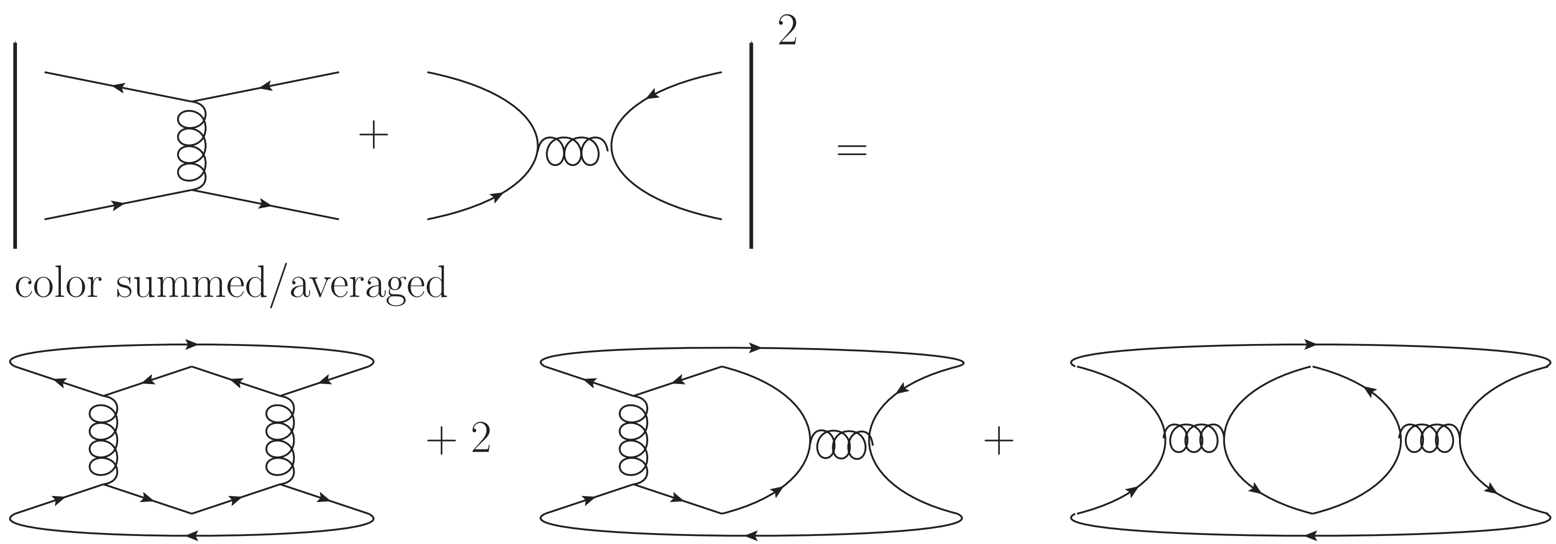
Convention: + when inserting after, - when inserting before

Three colors in parton showers

In collaboration with Simon Plätzer (DESY)

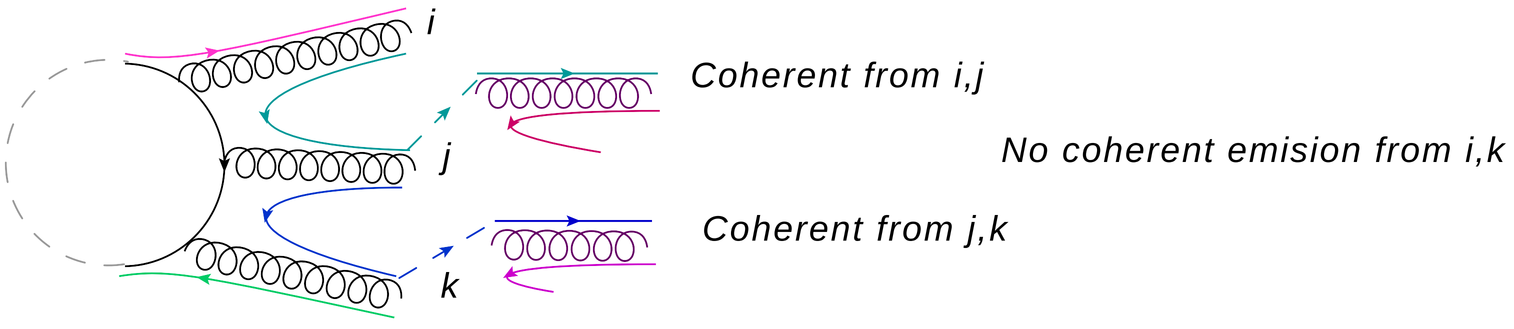
- Today all major event generators work with an infinite number of colors in the parton shower part
- I think it's time to change that. Together with Simon Plätzer I have published the first $N_c = 3$ parton shower results.
  (JHEP 1207 (2012) 042, arXiv:1108.6180)
- To accomplish this we had to deal with several challenges

- We had to calculate all the color terms, graphs like



- We know how to deal with this, but it takes quite some time when we have many colored partons

- A standard parton shower works in the approximation $N_c \to \infty$
- In this limit only color connected gluons radiate coherently



- We had to reformulate the parton shower into a framework where a new gluon can be emitted from any pair of gluons (evolution with amplitude information)
- We had to deal with negative contributions to the radiation probability

- The used type of “basis” is non-orthogonal and overcomplete
  (for more than N_c gluons plus $q\overline{q}$-pairs)
- ... and the number of “basis” vectors grows as a factorial in $N_g + N_{q\overline{q}}$
  $\rightarrow$ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond ~ 8 gluons plus $q\overline{q}$-pairs

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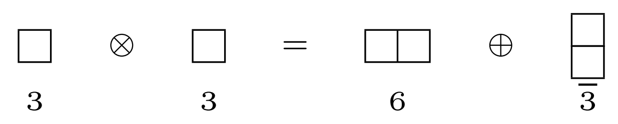
- Hard to go beyond ~ 8 gluons plus $q\overline{q}$-pairs
- Would be nice with minimal orthogonal basis

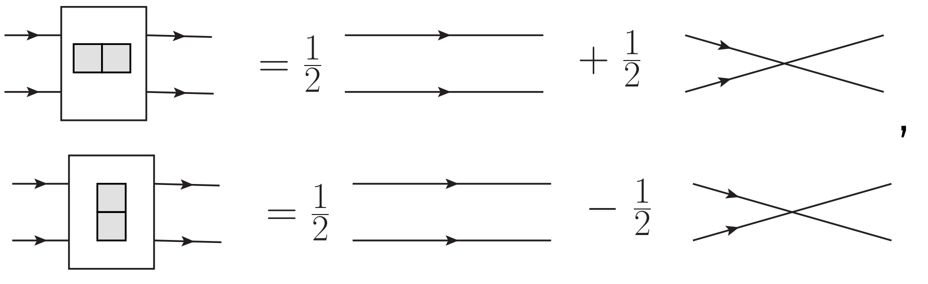
Orthogonal multiplet bases

In collaboration with Stefan Keppeler (Tübingen)

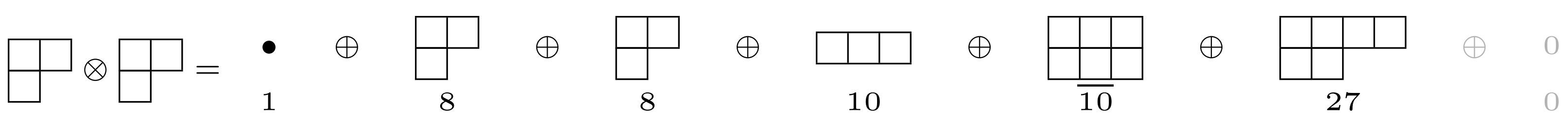
- One way of dealing with the color space is to use group theory
- The color space may be decomposed into irreducible representations, enumerated using Young tableaux multiplication
- For quarks we can construct orthogonal basis vectors using Young tableaux ...at least from the Hermitian quark projectors
- For example, for $qq \to qq$ we have



and the corresponding basis vectors



- In QCD we have quarks, anti-quarks and gluons
  $\rightarrow$ No obvious way to construct corresponding states
- Basis vectors can be enumerated using Young tableaux multiplication



- As color is conserved an incoming multiplet of a certain kind can only go to an outgoing multiplet of the same kind,
  $1 \rightarrow 1, 8 \rightarrow 8...$
  Charge conjugation implies that some vectors only occur together...

- The problem is the construction of the corresponding basis vectors; the Young tableaux operate with "quark-units"
- This may sound like a problem that should have been solved a long time ago, actually recently solved by me and Stefan Keppeler (JHEP09(2012)124, arXiv:1207.0609)
- This way of dealing with color space could potentially speed up calculations in QCD very significantly, but it remains to find quick algorithms for sorting QCD color structure in this basis

Number of basis vectors

In general, for many partons the size of the vector space is much smaller for $N_c = 3$, compared to for $N_c \to \infty$

Case	Vectors $N_c = 3$	Vectors, general case
4 gluons	8	9
6 gluons	145	265
8 gluons	3 598	14 833
10 gluons	107 160	1 334 961

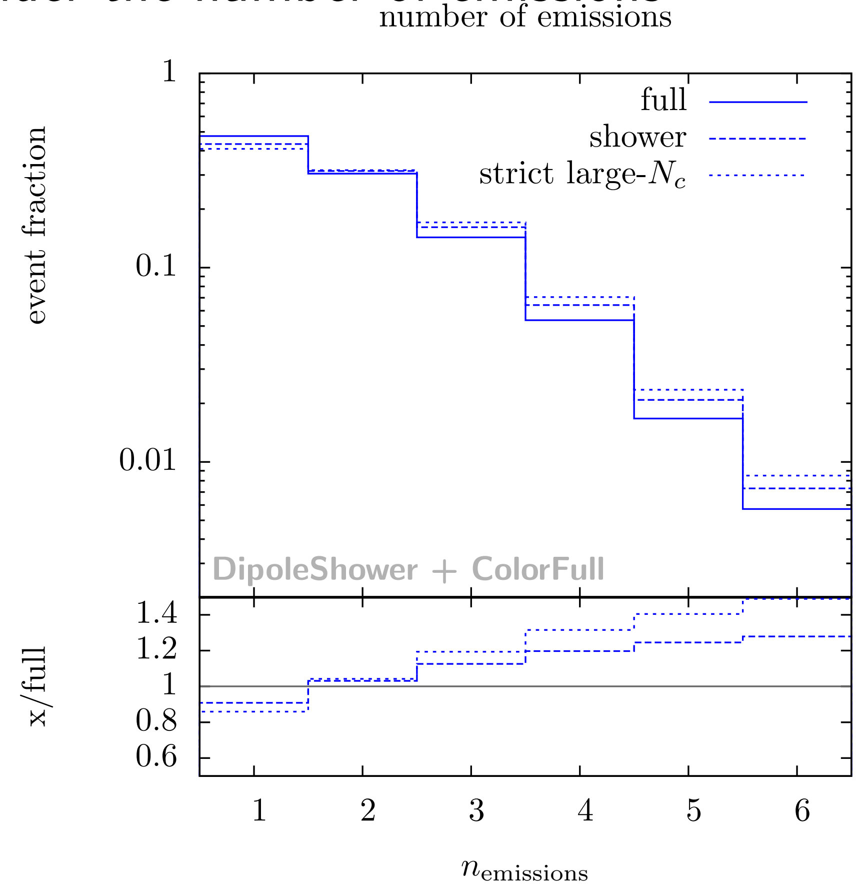
Number of basis vectors for $N_g \to N_g$ gluons *without* imposing vectors to appear in charge conjugation invariant combinations

Conclusion and outlook

- To first approximation the LHC is a QCD machine $\rightarrow$ to recognize new particles at the LHC we must know the manifestations of QCD
- In nature the strong force has three color charges
- One way of dealing with three colors is to construct orthogonal group-theory based bases (first recipe by me and Stefan)
- For calculations the 3 colors of nature is often approximated by infinitely many
- In particular this is the case for parton showers
- Me and Simon have written a first parton shower with 3 colors
- There is a lot more to do, both with color structure calculations in general and with parton showers

Backup: Number of emissions

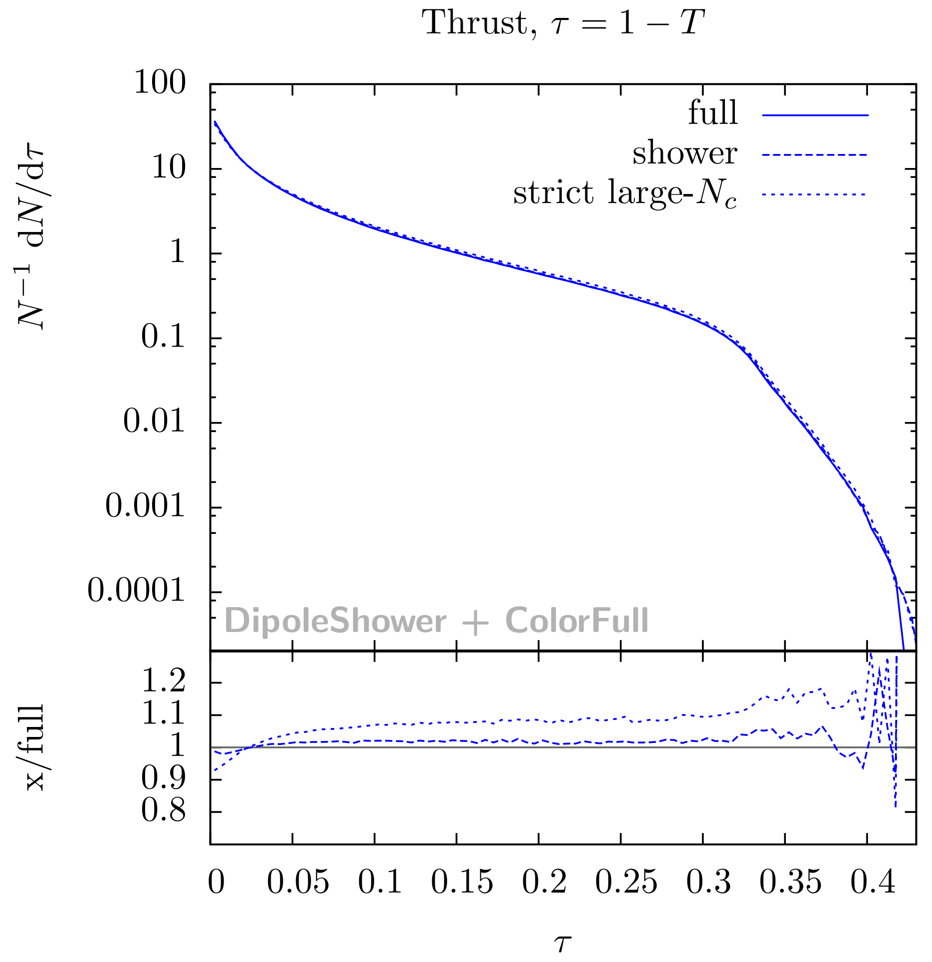
First, simply consider the number of emissions



... this is not an observable, but it is a genuine uncertainty on the number of emissions in the perturbative part of a parton shower

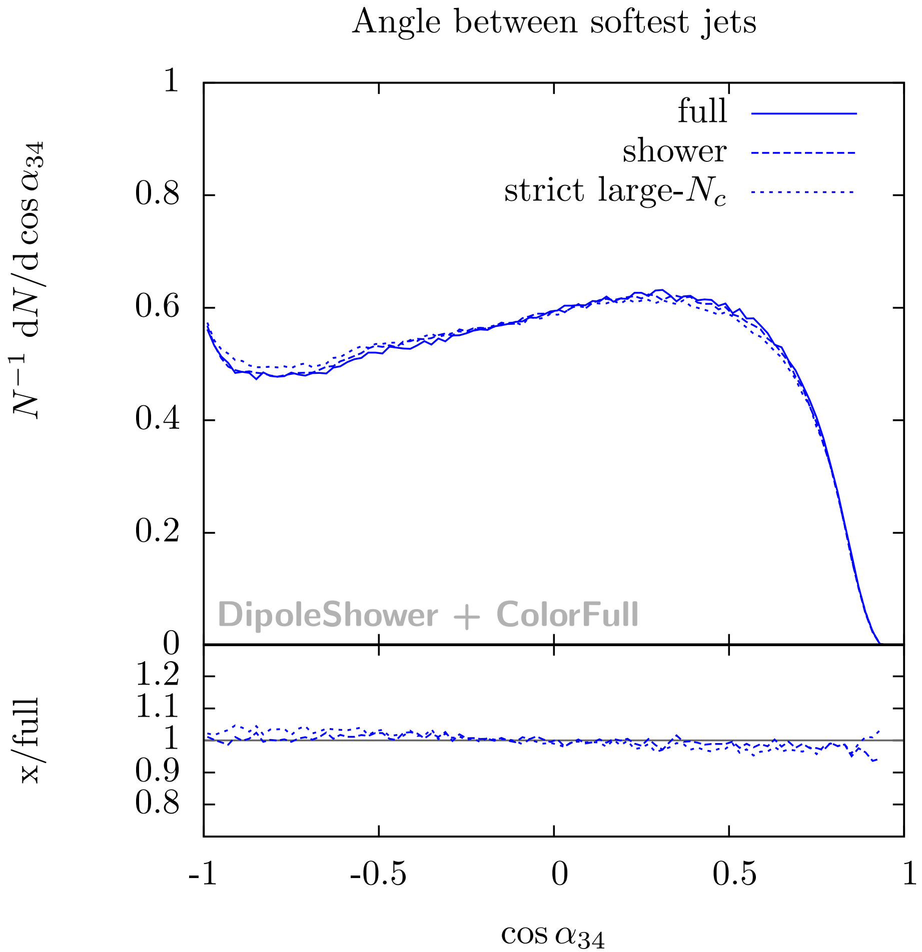
Backup: Thrust

For standard observables small effects, here thrust $T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n}|}{\sum_i |\mathbf{p_i}|}$



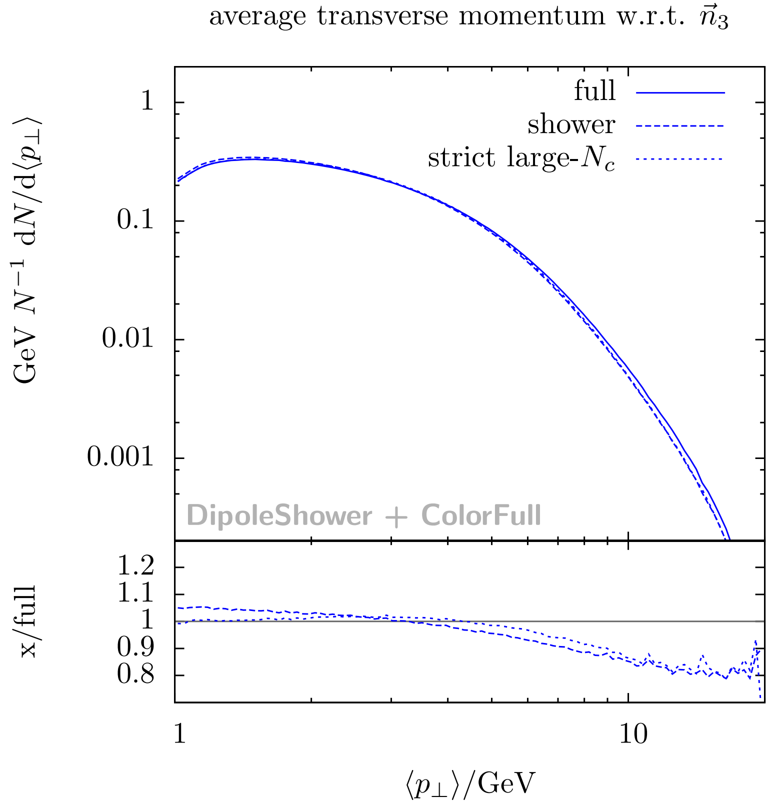
Backup: Angular distribution

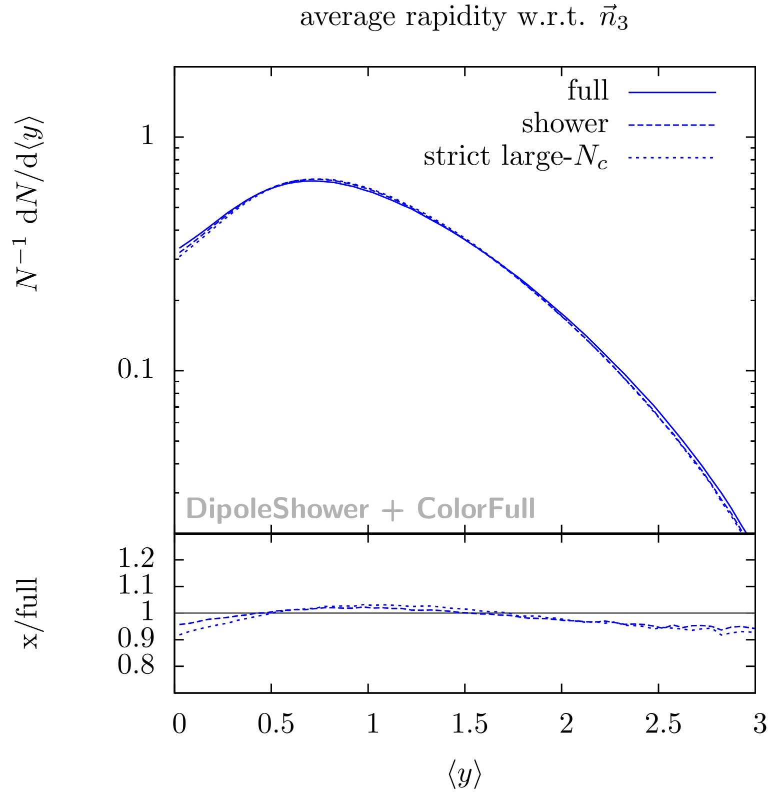
Cosine of angle between third and fourth jet



Backup: Some tailored observables

For tailored observables we find larger differences





Average transverse momentum and rapidity of softer particles with respect to the thrust axis defined by the three hardest partons

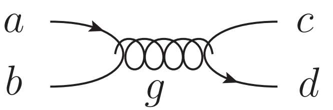
Backup: The color space

- We never observe individual colors $\rightarrow$ we are only interested in color summed quantities
- For given external partons, the color space is a finite dimensional vector space equipped with a scalar product

$$< A, B >= \sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots}$$

Example: If

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d = \sum_g$$



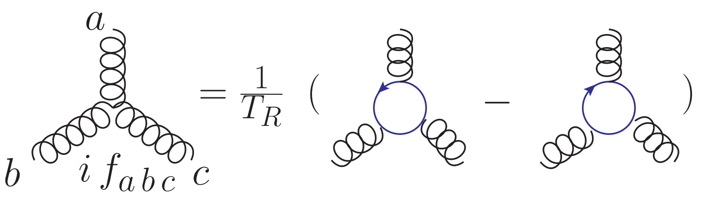
,

then $< A|A >= \sum_{a,b,c,d,g,h} (t^g)^b{}_a (t^g)^d{}_c (t^h)^a{}_b (t^h)^c{}_d$

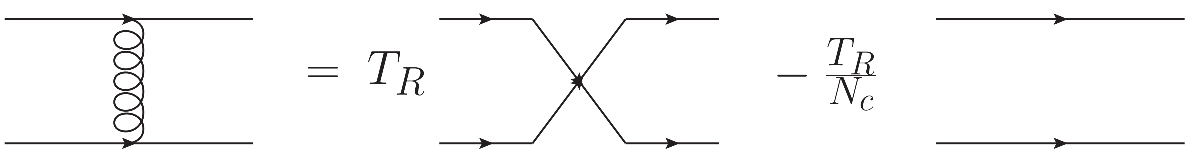
- We may use any basis (spanning set)

Backup: The standard treatment

- Every 4g vertex can be replaced by 3g vertices
- Every 3g vertex can be replaced using:



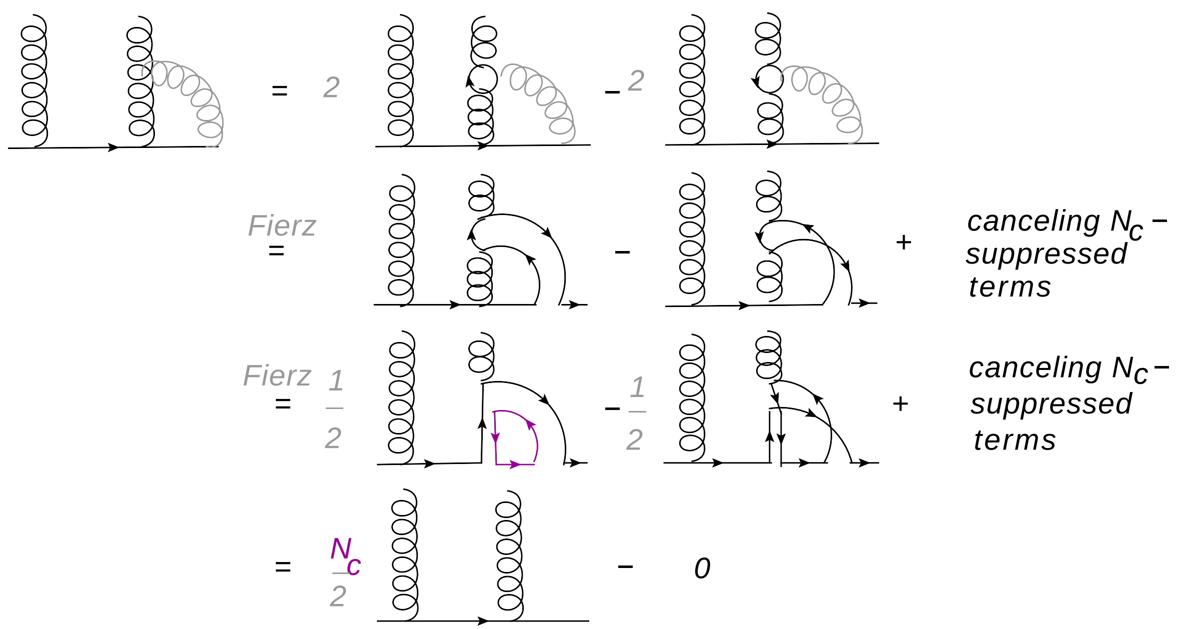
- After this every internal gluon can be removed using:



- This can be applied to any QCD amplitude, tree level or beyond

Backup: Gluon exchange

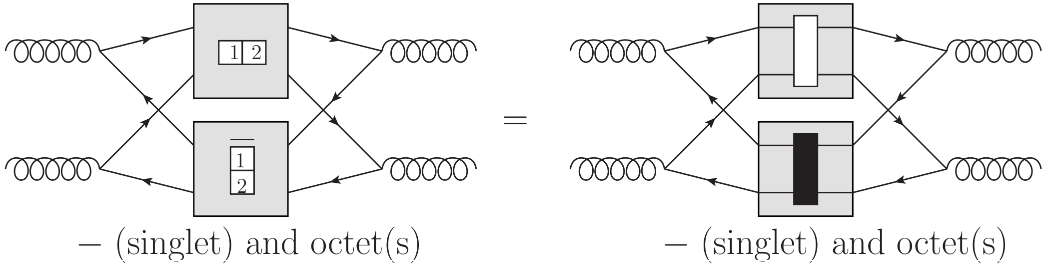
A gluon exchange in this basis "directly" i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



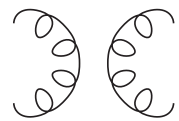
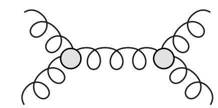
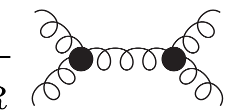
- N_c-enhancement possible only for near by partons
  $\rightarrow$ only "color neighbors" radiate in the $N_c \rightarrow \infty$ limit

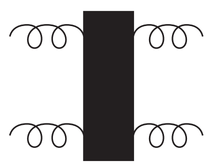
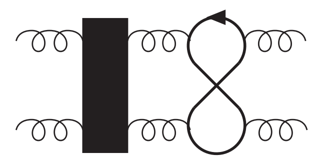
Backup: Projector construction

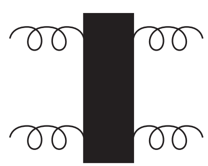
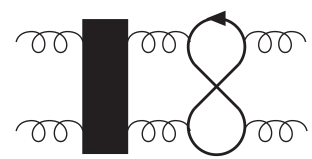
- For two gluons, there are two octet projectors, one singlet projector, and 4 new projectors, 10, $\overline{10}, 27$, and for general N_c, "0"
- It turns out that the new projectors can be seen as corresponding to different symmetries w.r.t. quark and anti-quark units, for example the decuplet can be seen as corresponding to

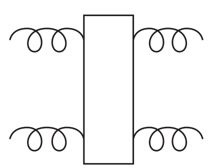
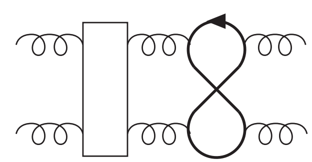


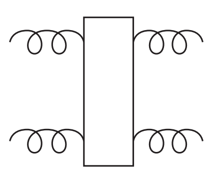
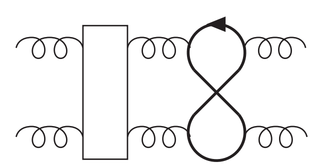
Similarly the anti-decuplet corresponds to $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \otimes \overline{\begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array}}$, the 27-plet corresponds to $\begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array} \otimes \overline{\begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array}}$ and the 0-plet to $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \otimes \overline{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}}$

$\mathbf{P}^1 = \frac{1}{N_c^2-1}$  , $\mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2-4)}$  , $\mathbf{P}^{8a} = \frac{1}{2N_cT_R}$  ,

$\mathbf{P}^{10} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

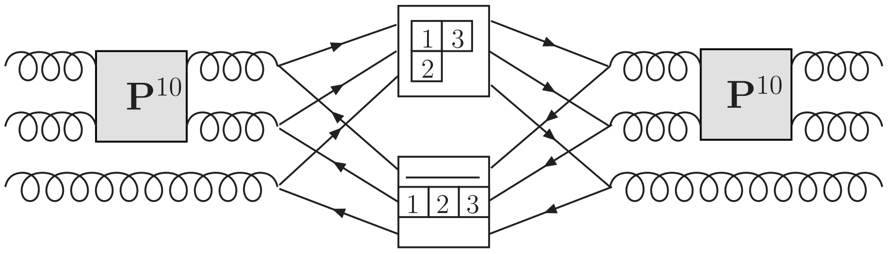
$\mathbf{P}^{\overline{10}} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{1}{2}\mathbf{P}^{8a}$

$\mathbf{P}^{27} = \frac{1}{2}$  $+ \frac{1}{2T_R^2}$  $- \frac{N_c-2}{2N_c}\mathbf{P}^{8s} - \frac{N_c-1}{2N_c}\mathbf{P}^1$

$\mathbf{P}^{0} = \frac{1}{2}$  $- \frac{1}{2T_R^2}$  $- \frac{N_c+2}{2N_c}\mathbf{P}^{8s} - \frac{N_c+1}{2N_c}\mathbf{P}^1$

Key observation:

- Starting in a given multiplet, corresponding to some $q\overline{q}$ symmetries, such as 10, from $\boxed{1|2} \otimes \overline{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}}$, it turns out that for each way of attaching a quark box to $\boxed{1|2}$ and an anti-quark box to $\overline{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}}$, to there is at most one new multiplet! For example, the projector $\mathbf{P}^{10,35}$ can be seen as coming from

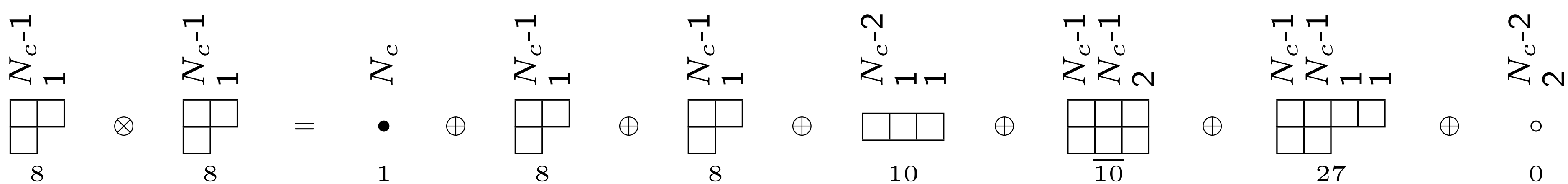


after having projected out "old" multiplets

- In fact, for large enough N_c, there is precisely one new multiplet for each set of $q\overline{q}$ symmetries

It turns out that the proof of this is really interesting:

- We find that the irreducible representations in $g^{\otimes n_g}$ for varying N_c stand in a one to one, or one to zero correspondence to each other! (For each SU(3) multiplet there is an SU(5) version, but not vice versa.)
- Every multiplet in $g^{\otimes n_g}$ can be labeled in an N_c-independent way using the lengths of the *columns*. For example



I have not seen this column notation elsewhere... have you?

- Calculations are done using my Mathematica package, ColorMath, arXiv:1211.2099, Eur. Phys. J. C 73:2310 (2013)
- Intended to be an easy to use Mathematica package for color summed calculations in QCD, $SU(N_c)$

In[1]:= `Get["/data/Documents/Annatjobb/Color/Mathematica/ColorMathv5.m"]`

In[2]:= $\text{Amplitude} = T\, t^{\{g\}q1}{}_{q3}\, t^{\{g\}q4}{}_{q2} + S\, t^{\{g\}q1}{}_{q2}\, t^{\{g\}q4}{}_{q3};$

In[3]:= `Amplitude Conjugate[Amplitude /. g→ g2] // CSimplify // Simplify`

Out[3]= $$\frac{\left(-1+N_C^2\right)\,\left(\text{Conjugate}[S]\,\left(-T+S\,N_C\right)+\text{Conjugate}[T]\,\left(-S+T\,N_C\right)\right)\,T_F^2}{N_C}$$

- In this way we have constructed the projection operators onto irreducible subspaces for $3g \to 3g$
- There are 51 projectors, reducing to 29 for SU(3)
- From these we have constructed an orthogonal (normalized) basis for the $6g$ space, by letting any instance of a given multiplet go to any other instance of the same multiplet. For general N_c there are 265 basis vectors. Crossing out tensors that do not appear for $N_c = 3$, we get a minimal basis with 145 basis vectors.

There's also a reduction from charge conjugation

- The size of the vector spaces asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c
- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[n_q, N_g] = N_{\text{vec}}[n_q, N_g - 1](N_g - 1 + n_q) + N_{\text{vec}}[n_q, N_g - 2](N_g - 1)$$

where

$$\begin{aligned} N_{\text{vec}}[n_q, 0] &= n_q! \\ N_{\text{vec}}[n_q, 1] &= n_q n_q! \end{aligned}$$

As the multiplet basis also is orthogonal it has the potential to very significantly speed up exact calculations in QCD!

Backup: Some example projectors

$$\mathbf{P}^{8a,8a}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R^2}\frac{1}{4N_c^2} i f_{g_1\,g_2\,i_1} i f_{i_1\,g_3\,i_2} i f_{g_4\,g_5\,i_3} i f_{i_3\,g_6\,i_2}$$

$$\mathbf{P}^{8s,27}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{1}{T_R}\frac{N_c}{2(N_c^2-4)} d_{g_1\,g_2\,i_1}\mathbf{P}^{27}_{i_1\,g_3\,i_2\,g_6} d_{i_2\,g_4\,g_5}$$

$$\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} = \frac{4(N_c+1)}{N_c^2(N_c+3)}\mathbf{P}^{27}_{g_1\,g_2\,i_1\,g_3}\mathbf{P}^{27}_{i_1\,g_6\,g_4\,g_5}$$

$$\begin{aligned}\mathbf{P}^{27,64=c111c111}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} &= \frac{1}{T_R^3}\mathbf{T}^{27,64}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} - \frac{N_c^2}{162(N_c+1)(N_c+2)}\mathbf{P}^{27,8}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6} \\ &- \frac{N_c^2-N_c-2}{81N_c\,(N_c+2)}\mathbf{P}^{27,27s}_{g_1\,g_2\,g_3\,g_4\,g_5\,g_6}\end{aligned}$$

Backup: Three gluon multiplets

$SU(3)$ dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
	$((45)^{8s}6)^{1}$	$2\times((45)^{8s}6)^{8s \text{ or } a}$	$((45)^{8s}6)^{10}$	$((45)^{8s}6)^{\overline{10}}$	$((45)^{8s}6)^{27}$	$((45)^{8s}6)^{0}$
	$((45)^{8a}6)^{1}$	$2\times((45)^{8a}6)^{8s \text{ or } a}$	$((45)^{8a}6)^{10}$	$((45)^{8a}6)^{\overline{10}}$	$((45)^{8a}6)^{27}$	$((45)^{8a}6)^{0}$
		$((45)^{10}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{10}6)^{27}$	$((45)^{10}6)^{0}$
		$((45)^{\overline{10}}6)^{8}$	$((45)^{10}6)^{10}$	$((45)^{\overline{10}}6)^{\overline{10}}$	$((45)^{\overline{10}}6)^{27}$	$((45)^{\overline{10}}6)^{0}$
		$((45)^{27}6)^{8}$	$((45)^{27}6)^{10}$	$((45)^{27}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$
		$((45)^{0}6)^{8}$	$((45)^{0}6)^{10}$	$((45)^{0}6)^{\overline{10}}$	$((45)^{27}6)^{27}$	$((45)^{0}6)^{0}$

$SU(3)$ dim	64	35	$\overline{35}$	0
Multiplet	c111c111	c111c21	c21c111	c21c21
	$((45)^{27}6)^{64}$	$((45)^{10}6)^{35}$	$((45)^{\overline{10}}6)^{\overline{35}}$	$((45)^{10}6)^{c21c21}$
		$((45)^{27}6)^{35}$	$((45)^{27}6)^{\overline{35}}$	$((45)^{\overline{10}}6)^{c21c21}$
				$((45)^{27}6)^{c21c21}$
				$((45)^{0}6)^{c21c21}$

$SU(3)$ dim	0	0	0	0	0
Multiplet	c111c3	c3c111	c21c3	c3c21	c3c3
	$((45)^{10}6)^{c111c3}$	$((45)^{\overline{10}}6)^{c3c111}$	$((45)^{10}6)^{c21c3}$	$((45)^{\overline{10}}6)^{c3c21}$	$((45)^{0}6)^{c3c3}$
			$((45)^{0}6)^{c21c3}$	$((45)^{0}6)^{c3c21}$	

Multiplets for $g_4 \otimes g_5 \otimes g_6$

Backup: First occurrence

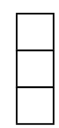
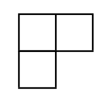
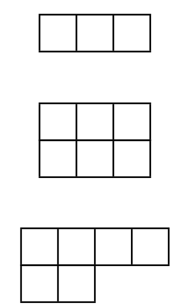
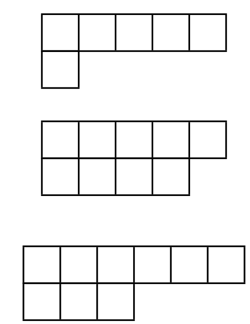
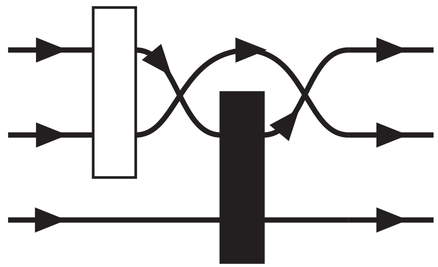
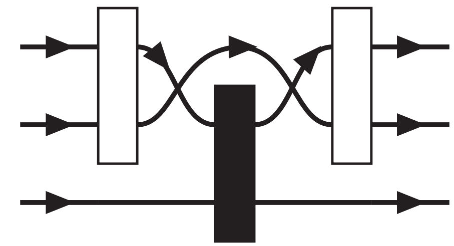
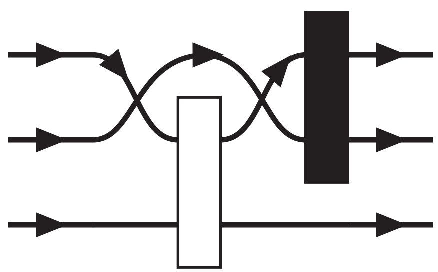
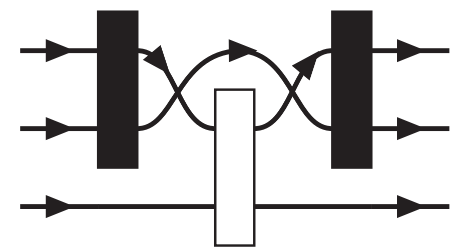
n_f	0	1	2	3
$SU(3)$ Young diagrams	$\bullet =$ 			

Table 1: Examples of $SU(3)$ Young diagrams sorted according to their first occurrence n_f.

Backup: Importance of Hermitian projectors

$\mathbf{P}_Y^{6,8} = \frac{4}{3}$  , $\mathbf{P}^{6,8} = \frac{4}{3}$ 

$\mathbf{P}_Y^{\overline{3},8} = \frac{4}{3}$  , $\mathbf{P}^{\overline{3},8} = \frac{4}{3}$ 

The standard Young projection operators $\mathbf{P}_Y^{6,8}$ and $\mathbf{P}_Y^{\overline{3},8}$ compared to their Hermitian versions $\mathbf{P}^{6,8}$ and $\mathbf{P}^{\overline{3},8}$.

Clearly $\mathbf{P}^{6,8\dagger}\mathbf{P}^{\overline{3},8} = \mathbf{P}^{6,8}\mathbf{P}^{\overline{3},8} = 0$. However, as can be seen from the symmetries, $\mathbf{P}_Y^{6,8\dagger}\mathbf{P}_Y^{\overline{3},8} \neq 0$.