(1) $\bar{a} = (y^2 - x^2,\ 2xy + 2yz,\ y^2 - z^2)$

a) konst. div.fr.?

$\nabla \times \bar{a} = (2y - 2y,\ 0 - 0,\ 2y - 2y) = \bar{0}$ ✓

$\nabla \cdot \bar{a} = (-2x) + (2x + 2z) + (-2z) = 0$ ✓

b)

$\phi'_x = y^2 - x^2 \Rightarrow \phi = \underline{xy^2 - \frac{1}{3}x^3} + \tilde{\phi}(y,z)$

$\phi'_y = 2xy + 2yz \Rightarrow \tilde{\phi}'_y = 2yz \Rightarrow \tilde{\phi}(y,z) = \underline{y^2 z} + \hat{\phi}(z)$

$\phi'_z = y^2 - z^2 \Rightarrow \hat{\phi}'_z = -z^2 \Rightarrow \hat{\phi}(z) = \underline{-\frac{1}{3}z^3} + d$

$\Rightarrow \boxed{\phi = xy^2 - \frac{1}{3}x^3 + y^2 z - \frac{1}{3}z^3}$ (volím $d = 0$)

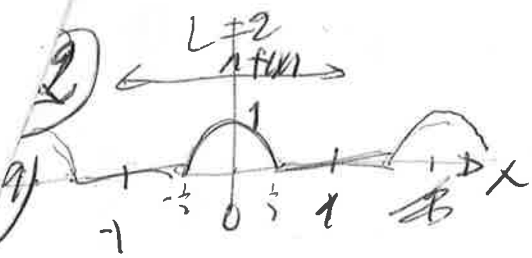
c) $\int_{(0,0,0)}^{(1,1,1)} \bar{a} \cdot d\bar{r} = \int \nabla\phi \cdot d\bar{r} = \int d\phi = [\phi] =$

$= \phi_{(1,1,1)} - \phi_{(0,0,0)} = \left(2 - \frac{2}{3}\right) - 0 = \frac{4}{3}$

10 —

2)

a) $L=2$



$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

b) f jämn $\rightarrow$ bara cos, inga sin ($\forall\, b_n = 0$)

c) $$a_n = \frac{2}{L}\int f(x)\cos(n\pi x)\,dx = \int_{-1/2}^{1/2} \cos(\pi x)\cos(n\pi x)\,dx =$$

$$= \frac{1}{2}\int_{-1/2}^{1/2}\left[\cos(n+1)\pi x + \cos(n-1)\pi x\right]dx \overset{n\neq 1}{=} \frac{1}{2}\left[\frac{\sin(n+1)\pi x}{(n+1)\pi} + \frac{\sin(n-1)\pi x}{(n-1)\pi}\right]_{-1/2}^{1/2}$$

$$= \left(\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)\pi} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)\pi}\right) = \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{(n+1)\pi} + \frac{(\;)-(\;)}{(n-1)\pi}$$

$$= \frac{\cos\frac{n\pi}{2}}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{2\cos\frac{n\pi}{2}}{\pi(n^2-1)} = \begin{cases} 0, & n \text{ udda}, n \neq 1 \\ -\frac{2(-)^m}{\pi(4m^2-1)}, & n = 2m \end{cases}$$

$n=1$? $a_1 = \int_{-1/2}^{1/2} \cos^2 \pi x = \frac{1}{2}$

$$\Rightarrow f(x) = \frac{1}{2}\cos\pi x + \sum_{m=1}^{\infty} -\frac{2(-)^m}{\pi(4m^2-1)}\cos(2m\pi x) + \frac{1}{\pi}$$

$$\boxed{f(x) = \frac{1}{\pi} + \frac{1}{2}\cos\pi x - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-)^m}{4m^2-1}\cos(2m\pi x)}$$

$a_0 = \frac{2}{\pi}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3\pi}$

d) i) rimlig? konv. som $\sim \frac{1}{n^2}$: ok, (f' diskont.) $n \neq 1$

ii) flesta udda $a_n = 0$? $\max(c, 0) = \frac{1}{2}c + \frac{1}{2}|c|$

(−) (+)

$\Rightarrow$ inga udda n

$\rightarrow$ bara $n = 2m$

10

3 PDE: $x^2 u'_x - y^2 u'_y = x^2$, $u = u_p + u_h$

a) $\underline{u_h}$: $x^2 u'_x - y^2 u'_y = 0$, $u = f(p)$, p = konst. lösg:

$x^2 p'_x - y^2 p'_y = 0$

$\underline{p = \text{konst}}$: $dx\, p'_x + dy\, p'_y = 0$ $\quad \frac{dx}{dy} = -\frac{x^2}{y^2}$

$\frac{dx}{x^2} + \frac{dy}{y^2} = 0 \quad \frac{1}{x} + \frac{1}{y} = \text{konst.}$

$\underline{\text{välj}}$ $p = \frac{1}{x} + \frac{1}{y} \Rightarrow \underline{u_h = f\left(\frac{1}{x} + \frac{1}{y}\right)}$

b) $\underline{u_p}$: ~~testa $u_p(x)$~~ $\Rightarrow u'_x = 1$ $\xrightarrow{\text{välj}}$ $\underline{u_p = x}$

$\Rightarrow$ Allm. lösg $\underline{u(x,y) = x + f\left(\frac{1}{x} + \frac{1}{y}\right)}$

c) BC: $u(1,y) = y$, $y \in \mathbb{R}$ $\quad u(1,y) = 1 + f\left(1 + \frac{1}{y}\right) = y$, $\forall y$

$f\left(1 + \frac{1}{y}\right) = y - 1 \Leftrightarrow f(1+y) = \frac{1}{y} - 1 \Leftrightarrow f(y) = \frac{1}{y-1} - 1$

$\underline{f(y) = \frac{1}{y-1} - 1} \Rightarrow f\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{1}{\frac{1}{x} + \frac{1}{y} - 1} - 1$

$= \frac{xy}{x+y-xy} - 1 \Rightarrow$

$$\boxed{u(x,y) = x - 1 + \frac{xy}{x+y-xy}} \left(= x - 2 + \frac{x+y}{x+y-xy} \right)$$

(testa BC: $u(1,y) = \frac{y}{1+y-y} = y$)

4) $\nabla^2 u \Rightarrow$ in $R: x^2+y^2 \le R^2$ $(\,15\ \sim)$

a) sep: pl. pol. (r,φ) $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$\nabla^2 u = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2\right) u = 0$

$u = R(r)\,\Phi(\varphi) \Rightarrow$

$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = 0$ $\quad m \in \mathbb{Z}$ per. in φ

$\underbrace{r^2\frac{R''}{R} + r\frac{R'}{R}}_{m^2} + \underbrace{\frac{\Phi''}{\Phi}}_{-m^2} = 0 \Rightarrow \begin{cases} \Phi'' + m^2\Phi = 0 \\ r^2R'' + rR' - m^2 R = 0 \end{cases}$

$\underline{\Phi}$: $\Phi \propto e^{\pm i m\varphi}$ ($m=0$: $\Phi \propto \varphi$: [illegible])

$\underline{R}$: prova $r^\alpha \Rightarrow \alpha(\alpha-1) + \alpha - m^2 = 0 \Rightarrow \underline{\alpha = \pm m}$

$R \propto \begin{cases} r^{\pm m}, & m \neq 0 \\ 1, \ln r, & m = 0 \end{cases}$ (r^{-m}, $\ln r$ [illegible] in $r=0$)

$\Rightarrow \underline{R} \propto r^m,\ m = 0, 1, \ldots$

$\Rightarrow u_m = r^m(A\cos m\varphi + B\sin m\varphi),\quad m \in \mathbb{N}$

b) $\Rightarrow$ $\boxed{\underline{u} = \sum_{m=0}^{\infty} r^m (A_m \cos m\varphi + B_m \sin m\varphi)}$

c) $\underline{u(r=R)} = 2x^2 = 2R^2\cos^2\varphi \stackrel{!}{=} R^2 + R^2\cos 2\varphi$

$u(R,\varphi) = \sum_0^\infty R^m(A_m \cos m\varphi + B_m \sin m\varphi) \Rightarrow$

$= A_0 + R^2 A_2 \cos 2\varphi + \ldots = \underline{R^2} + \underline{R^2}\cos 2\varphi$

$\Rightarrow A_0 = R^2,\ A_2 = 1$, altre $= 0$, $\Rightarrow$

$u(r,\varphi) = R^2 + r^2\cos 2\varphi = R^2 + r^2(2\cos^2\varphi - 1)$

$= R^2 + 2x^2 - x^2 - y^2 = \boxed{R^2 + x^2 - y^2}$

(5) $N = \#$ Würfe bis 6'n

a) $p_n = \Pr(N=n)$?

$(p_0=0)$ $\quad p_1 = p\left(=\frac{1}{6}\right)$, $\quad p_2 = (1-p)p$, $\quad p_3 = (1-p)^2 p$, etc

$$\underline{p_n = (1-p)^{n-1}\cdot p} \quad (n>0)$$

b) i) mw? $\mathbb{E}(N) = \sum_1^\infty n p_n = p\cdot\sum_{1(0)}^\infty n(1-p)^{n-1}$

$$= p\,\partial_x \frac{1}{1-x}\Big|_{x=1-p} = p\cdot\frac{1}{(1-x)^2}\Big|_{x=1-p} = \frac{1}{p}$$

$$\boxed{\mathbb{E}(N)=6}$$

ii) $\mathbb{E}(N(N-1)) = \sum_1^\infty n(n-1)p_n = p\sum_1^\infty n(n-1)(1-p)^{n-1} = p(1-p)\sum_1 n(n-1)\ldots$

$$= p(1-p)\,\partial_x^2\frac{1}{1-x}\Big|_{x=1-p} = p(1-p)\frac{2}{(1-x)^3}\Big|_{x=1-p} = p(1-p)\cdot\frac{2}{p^3} = \frac{2(1-p)}{p^2}$$

$$= \frac{2\cdot 5/6}{(1/6)^2} = 2\cdot 6\cdot 5 = 60 = \mathbb{E}(N(N-1))$$

$\left[= \frac{2-p}{p^2} = \frac{1+q}{(1-q)^2}\right]$

$$= \mathbb{E}(N^2) = 60+6 = 66$$

$$\Rightarrow \sigma_N^2 \,(=\mathrm{Var}(N)) = 66-6^2 = \underline{30}$$

$$\Rightarrow \boxed{\sigma_N = \sqrt{30}}$$

c) (μ, σ) für $M = \#$ Würfe bis 2 Sechsen

NB: $M = N_1 + N_2$, N_1, N_2 unabh. (gleich vert.)

(N_1: 1te 6, N_2: bis 2te 6)

$\Rightarrow \begin{cases}\mathbb{E}(M) = 2\mathbb{E}(N) \\ V(M) = 2V(N)\end{cases} \Rightarrow$ $\mathbb{E}(M) = 2\cdot 6 = \boxed{12} \Rightarrow \mu_M = 12$

$V(M) = 2\cdot 30 = 60 \Rightarrow \sigma_M = \sqrt{60}$