Tok 1:10 | 10

1

$\bar{a} = (\mu x^3 + xy^2 + xz^2,\ x^2 y - yz^2,\ x^2 z - y^2 z)$

a) μ så $\nabla\cdot\bar{a} = 0$?

$\nabla\cdot\bar{a} = 3\mu x^2 + y^2 + z^2 + x^2 - z^2 + x^2 - y^2 = (3\mu + 2)x^2$

$\overset{?}{=} 0 \Rightarrow \boxed{\mu = -2/3}$

b) $\nabla\times\bar{a} = \bar{0}$? $\bar{a} = \nabla\phi$?

(x): $\phi = -\frac{1}{6}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \psi_1(y,z)$

(y): $\phi = \frac{1}{2}x^2y^2 - \frac{1}{2}y^2z^2 + \psi_2(x,z)$

(z): $\phi = \frac{1}{2}x^2z^2 - \frac{1}{2}y^2z^2 + \psi_3(x,y)$

$\Leftarrow \boxed{\phi = -\frac{1}{6}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 - \frac{1}{2}y^2z^2}$ duger

$\Rightarrow \nabla\times\bar{a} = 0$ [alt. visa direkt $\nabla\times\bar{a} = \bar{0}$]

c) (Se b) $\phi = \ldots$

d) $\int_{C} \bar{a}\cdot d\bar{r} = \int_C \nabla\phi\cdot d\bar{r} = \int d\phi \overset{\text{oberoende av väg}}{=} [\phi]_{(0,0,0)}^{(1,1,1)}$

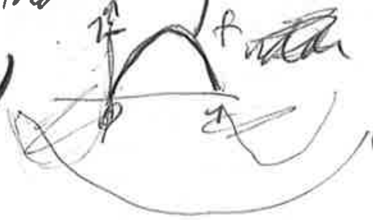
$= \phi(1,1,1) - \phi(0,0,0) = (-\frac{1}{6} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}) - (0)$

$= \boxed{1/3}$

$f(x) = 4x(1-x)$ på $[0,1]$, i F.S. [illegible]

udda el. jämn? Konv. snabbast överlag.

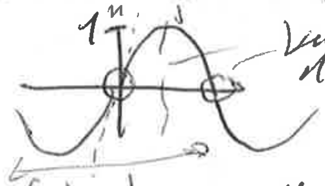
② a)



$\begin{cases} SS: f_u \text{ jämn? udda } \to \text{ bara udda } n \\ CS: f_j \text{ jämn? udda } \to \text{ bara jämna } n \end{cases}$

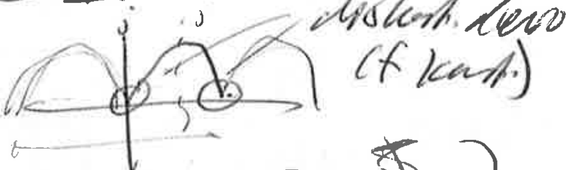
~~[illegible]~~

b)



(⇒ bara udda n)

$f_u = \sum_{\substack{n=1 \\ (udda)}}^{\infty} b_n \sin(n\pi x)$



(⇒ bara jämna n)

$f_j = \frac{a_0}{2} + \sum_{\substack{n=2 \\ (j=)}}^{\infty} a_n \cos(n\pi x)$

$\begin{cases} \text{sinserie} \Rightarrow \text{kontin. } f', \text{ disk. } f'' \Rightarrow b_n \sim 1/n^3 \\ \text{cos-serie} \Rightarrow \text{kontin. } f, \text{ disk. } f' \Rightarrow a_n \sim 1/n^2 \end{cases}$

⇒ **sin-serie bäst!** $b_n \sim 1/n^3$

c)

$b_n = 2\int_0^1 f(x) \sin n\pi x \, dx = 2\int_0^1 4x(1-x) \sin n\pi x \, dx$

$= 8\int_0^1 x(1-x)\sin(n\pi x)\,dx = \left[-8\frac{x(1-x)}{n\pi}\cos(n\pi x)\right]_0^1$

$+ \frac{8}{n\pi}\int_0^1 (1-2x)\cos(n\pi x) = \frac{8}{n^2\pi^2}\left[(1-2x)\sin(n\pi x)\right]_0^1$

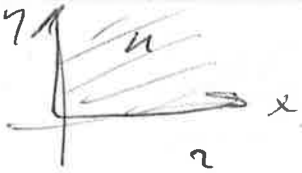
$- \frac{8}{n^2\pi^2}\int_0^1 -2\cdot\sin(n\pi x)\,dx = \frac{-16}{n^3\pi^3}\left[\cos(n\pi x)\right]_0^1$

$= \frac{16}{n^3\pi^3}\left[1-\cos(n\pi)\right] = \frac{16}{n^3\pi^3}\left(1-(-1)^n\right) = \frac{32}{n^3\pi^3}$, n udda (0 j=)

$\Rightarrow \boxed{f_{udda}(x) = \frac{32}{\pi^3}\sum_{\substack{n=1 \\ (udda)}}^{\infty} \frac{1}{n^3}\sin(n\pi x)}$

Koll: rimligt! $b_1 \approx 1$

$\left.\begin{matrix} \text{bara udda} \\ b_n \sim 1/n^3 \\ b_1 \approx 1 \end{matrix}\right\}$ OK!



20 minőség

(3) $u(x,y)$, $x,y \geq 0$ (!)

PDE: $y u'_x + x u'_y + 2u = (x+y)^2$

a) visz. általános megoldás: $u = xy + (x-y)^2 f(x^2-y^2)$

módszer: $u = u_p + v$ (homog.)

$u_p = xy$?

$y u'_x = y^2$
$+ x u'_y = + x^2$
$2u = 2xy$
$\Sigma = (x+y)^2$ OK

v: $y v'_x + x v'_y + 2v = 0$

$v = v_p \cdot w$ (homog. 2v)

$v_p = (x-y)^2$?

$y v'_x = 2y(x-y)$
$x v'_y = -2x(x-y)$ } $-2(x-y)^2$
$2v = 2(x-y)^2$
$\Sigma = 0$ OK!

w: $y w'_x + x w'_y = 0$

$w_p = p = x^2 - y^2$?

$y w'_x = 2xy$
$x w'_y = -2xy$
$\Sigma = 0$ OK

$\Rightarrow w_{\text{ált}} = f(x^2-y^2)$

$\Rightarrow v_{\text{ált}} = v_p \cdot f(x^2-y^2) = (x-y)^2 f(x^2-y^2)$

$\Rightarrow u_{\text{ált}} = u_p + v_{\text{ált}} = xy + (x-y)^2 f(x^2-y^2)$!

b) i) $u(x,0) \overset{x \geq 0}{=} x^2 f(x^2) = 1$, $x \geq 0$ $\Rightarrow$ $f(\xi) = \frac{1}{\xi}$, $\xi \geq 0$

ii) $u(0,y) \overset{y \geq 0}{=} y^2 f(-y^2) = 1$, $y \geq 0$ $\Rightarrow$ $f(\xi) = -\frac{1}{\xi}$, $\xi \leq 0$

$\Rightarrow f(\xi) = \frac{1}{|\xi|}$ pl. $\mathbb{R} \setminus \{0\}$

$\Rightarrow u(x,y) = xy + (x-y)^2 \frac{1}{|x^2-y^2|}$ (ha $x \neq y$)

$= xy + \frac{|x-y|^2}{|x-y|(x+y)} = xy + \frac{|x-y|}{(x+y)}$

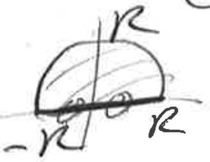
~~kontin.~~

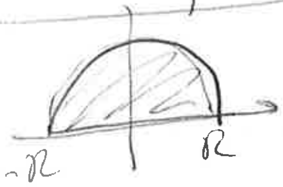
c) $\Rightarrow u$ kontin. kiterj. $x = y$?

(4)

$u(x,y)$ PDE $\nabla^2 u = 0$ i halvdisk $x^2+y^2<R^2$, $y>0$

RV $u=0$ på randen $y=0$, $-R<x<R$



a)  planpolära (r,φ) $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

b) $\nabla^2 u = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2$

$u = R\Phi \Rightarrow R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0$

$\Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = 0$

$\Rightarrow \underbrace{r^2\frac{R''}{R} + r\frac{R'}{R}}_{m^2} + \underbrace{\frac{\Phi''}{\Phi}}_{-m^2} = 0$

$\Phi'' + m^2\Phi = 0 \Rightarrow \Phi \propto e^{\pm im\varphi}$ RV $\Phi(0)=\Phi(\pi)=0 \Rightarrow \Phi = \sin(m\varphi)$, $m \in \mathbb{Z}_+$

$r^2R'' + rR' - m^2R = 0$, $R \sim r^\alpha \Rightarrow \alpha^2 = m^2$, $\alpha = \pm m$ $\Rightarrow \alpha = m$ (kont. i $r=0$)

$\Rightarrow \boxed{u_{sep}(r,\varphi) = B_m r^m \sin(m\varphi)}$, $m = 1,2,3,\dots$

c) $u_{allm}(r,\varphi) = \sum_{m=1}^{\infty} B_m r^m \sin m\varphi$

d) $u = y$ på övre randen $(r=R)$

$\therefore u(R,\varphi) = \sum_m B_m R^m \sin m\varphi \overset{!}{=} y = R\sin\varphi$

$\{\sin m\varphi\}$ fullst. bas av def. på $[0,\pi]$

$\Rightarrow B_m = 0$ utom $B_1 = 1$

$\Rightarrow \boxed{u(r,\varphi) = r\sin\varphi = y}$ i hela Ω

(Koll: $u=0$ på x-axeln, $\nabla^2 u = 0$, $u=y$ på cirkeln: OK)