FYTA12/FYTB03

Exercise sheet 1

Friday, January 22th, 2016

Warm-up

Exercise 1: Free fall

A particle with mass m starts at rest in $z = 0$ and falls down in the Earth's (constant) gravitational field.
(a) Write down the Lagrangian.
(b) Write down the (Euler-)Lagrange equation for z.
(c) Solve the equation.

Exercises

Exercise 2: One-dimensional harmonic oscillator

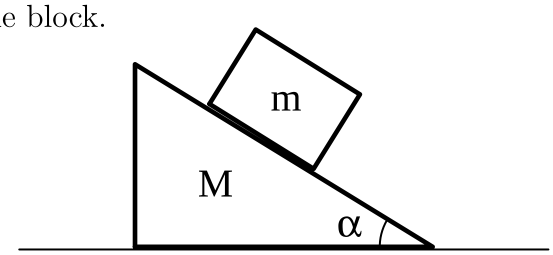
Write down and solve Lagrange's equations for a one-dimensional harmonic oscillator – a particle with mass m attached to a spring with spring constant k.

Exercise 3: Two connected blocks

Two blocks with equal mass m are connected with an inelastic string. One of the blocks is placed on a table, the other hangs over the edge. Determine the acceleration of the masses, if
(a) the mass of the string is neglected
(b) the string has a non-zero mass m' and length ℓ.
There is no friction.

Exercise 4: Block on sliding wedge

A block with mass m lies on a wedge (with mass M and angle α w.r.t. the table), which in turn lies on a horizontal table (see figure). There is no friction. Write down and solve Lagrange's equations under the assumption that the system is initially at rest. Use a horizontal coordinate x for the position of the wedge and a coordinate y along the surface of the wedge to describe the position of the block.



Exercise 5: Ball on sliding wedge

A homogeneous ball with mass m and radius R rolls down a wedge (with mass M and angle α w.r.t. a table). There is no friction between the wedge and the table, and the ball rolls without sliding. Determine the acceleration of the wedge. The exercise should be solved using Lagrange's equation.

Exercise 6: Euler-Lagrange equations for several integration variables

We have shown that a necessary (but not sufficient) condition for a (continuously differentiable) function

$$\int f(y_1(x), ..., y_n(x), y_1'(x), ..., y_n'(x), x)dx \tag{1}$$

to have an extremum is

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0 \;\forall\; i. \tag{2}$$

This gives us n Euler-Lagrange equations, one for each y_i. Prove that if each y_i in turn depends on m different variables of integration $x_1, ..., x_m$ a condition for extremum of the multiple integral

$$\int ... \int f\Big(y_1...y_n; y_1^{(1)}...y_1^{(m)}; ......; y_n^{(1)}...y_n^{(m)}; x_1...x_m\Big)dx_1...dx_m \tag{3}$$

(where $y_i^{(j)}$ denotes the derivative of y_i w.r.t. the variable x_j) is that

$$\frac{\partial f}{\partial y_i} - \sum_{j=1}^{m} \frac{\partial^{tot}}{\partial x_j}\frac{\partial f}{\partial\left(\frac{\partial y_i}{\partial x_j}\right)} = 0 \quad \forall\; i. \tag{4}$$

Comment: This equation is often written with usual partial derivatives after the sum, however, the derivatives act on all x_j dependencies, explicit as well as implicit via all y and partial derivatives.

Answers:

(1) (a) $L = E_k - E_p = m\dot{z}^2/2 - (-mgz)$ (if z is counted positive downwards)
(b) Well, you have to write one line by yourself.
(c) The one line from (b) should give you an easy equation, which I assume you know the answer to.

(2) $x = A\cos(\omega t + \phi)$, $\qquad \omega = (k/m)^{1/2}$

(3) (a) $\ddot{x} = g/2$ $\qquad$ (b) $\ddot{x} = \frac{g(m+m'\frac{x}{\ell})}{2m+m'}$

(4) If x increases to the right we have:
$\ddot{y} = g\frac{(M+m)\sin\alpha}{M+m\sin^2\alpha}$ giving $y = g\frac{(M+m)\sin\alpha}{M+m\sin^2\alpha}\frac{t^2}{2}$, and $x = -y\frac{m}{M+m}\cos\alpha = -g\frac{m\sin\alpha\cos\alpha}{M+m\sin^2\alpha}\frac{t^2}{2}$.

(5) $\pm\frac{mg\sin\alpha\cos\alpha}{\frac{7}{5}(M+m)-m\cos^2\alpha}$, where the sign depends on the orientation.