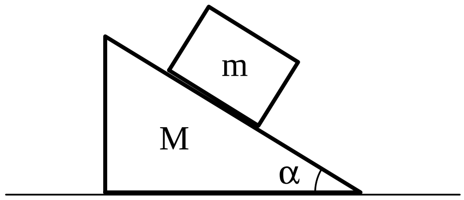
Exercises for Fyta12:1 – Classical Mechanics

1. Set up and solve Lagrange's equations for a one-dimensional harmonic oscillator – a particle with mass m attached to a spring with spring constant k.

2. Set up and solve Lagrange's equations for throwing a small object in a homogeneous gravity field with no air resistance.

3. Two blocks with equal mass m are connected with an inelastic string. One of the blocks is placed on a table, the other hangs over the edge. Determine the acceleration of the system, if **(a)** the mass of the string in neglected, **(b)** the string has a non-zero mass m' and length ℓ.

4. A block with mass m lies on a triangular block (with acute angle α and mass M), which in turn lies on a horisontal table (see figure). There is no friction. Set up and solve Lagrange's equations under the assumption that the system is initially at rest.



5. A homogeneous ball with mass m and radius R rolls down a triangular wedge of mass M, freely movable on a frictionless table. The acute angle of the wedge is θ. The ball rolls without sliding. Determine the acceleration of the wedge.

6. Write down Lagrange's equations for the motion of an elastic pendulum: A particle of mass m that hangs in a spring with spring constant k and natural length ℓ, and is free to move in a vertical plane through the other (fixed) endpoint of the spring. Approximate for small oscillations around the equilibrium position, and solve the equations.

***7.** Determine, using (constrained) variational calculus, the largest area that can be enclosed by a plane curve of fixed length. Use cartesian coordinates.
Tip: Tha area enclosed by a curve γ in the xy-plane can be written as $\frac{1}{2}\oint_\gamma(\mathbf{r}\times\mathrm{d}\mathbf{r})_z = \frac{1}{2}\oint_\gamma x\,\mathrm{d}y - y\,\mathrm{d}x$ (Why?)

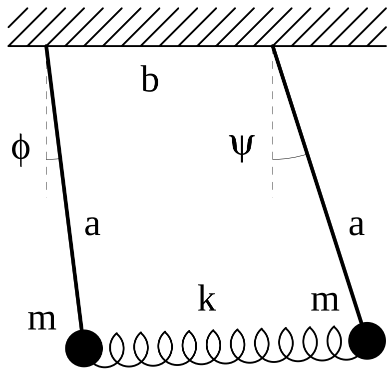
8. A spring is horisontally attached between a wall and a homogeneous right-angled triangular prism of mass M. The prism is free to slide along a horisontal table on its longer cathete. A particle of mass m is attached to one end of a spring, the other end of which is attached to the upper edge of the prism. The particle slides without friction on the hypothenuse of the prism, that forms the angle α with the horisontal plane. Both springs have spring constant k. Set up Lagrange's equations.

9. The suspension point of a mathematical pendulum is lifted vertically (with vanishing initial speed) with a constant acceleration a. Determine the frequency for small oscillations of the pendulum. Contemplate the result.

10. A mass m is suspended in a spring of spring constant k. The spring's mass M cannot be neglected. The elongation of the spring, however, is small as compared to the natural length. Set up Lagrange's equations and determine the period of the motion.
Tip: Assume that the spring is homogeneously deformed, so that the velocity of each part is proportional to its distance to the fixed endpoint.

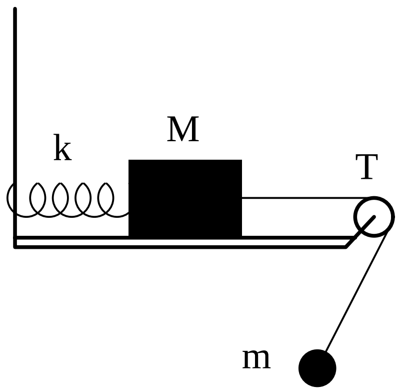
11. A particle, A, is suspended in a vertical spring with the other end attached to a fixed point P. Another particle, B, hangs in a vertical spring attached to A. A and B move along the vertical line through P, and oscillate around their equilibrium positions under the influence of gravity. Both particles have mass m and both springs have spring constant mk^2. Determine the normal mode frequencies.

12. Two pendulums, each consisting of a mass m attached to a weigthless rod of length a, are positioned on the same height, but with a horisontal displacement b. The two masses are connected by a spring of spring constant $k = mg/a$ and natural length b. The pendulums can swing without friction in their common vertical plane. Set up Lagrange's equations under the assumption that the spring can be considered horisontal during the entire motion. Determine the normal mode frequencies for small oscillations around the equilibrium position.



***13.** A chunk of soap slides inside an inverted cone with top angle 2α. The axis of the cone is vertical. Consider the soap as a pointlike particle with mass m and investigate its motion in different special cases. Analyse small deviations from a stationary trajectory.

14. A block with mass M lies on a slippery horizontal table. The block is connected to a solid wall via a spring with spring constant k. From the block a thin, weightless and inelastic string is drawn over a weigthless [**lttrrlig trissa**] T. In its other end the string holds a particle with mass m, that can swing in a vertical plane. Set up Lagrange's equations for oscillations around the equilibrium configuration, and solve them for the case of small scillations. The distance between the [**trissa**] and the particle at equilibrium is ℓ and the distance between the block and the edge of the table is large enough that the block stays on the table. The string can be assumed not to touch the table.

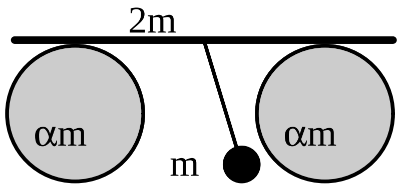


15. A particle of mass m moves without friction along the spiral

$$x = b\cos(\theta)$$
$$y = b\sin(\theta)$$
$$z = a\theta$$

where a and b are constants and the z-axis is vertical. In addition to gravity, the particle is attracted towards the origin with a force proportional to the distance to the origin. Determine the particle's motion, if it starts at rest from the position $x = b$, $y = z = 0$.

16. A horizontal homogeneous rod lies tranversely across two identical circular (homogeneous) cylinders, that can rotate without friction around their axes. The rod rolls on the cylinders without sliding. Each cylinder has mass αm and radius R, while the rod has mass $2m$. A mathematical pendulum with mass m and length ℓ



is attached to the center of the rod, which initially is centered between the cylinders. Set up Lagrange's equations. Approximate to small oscillations. For initial conditions, use that the velocity of the rod is zero, and that the pendulum is released at an angle φ_0 relative to the vertical. The rod is assumed long enough that it always rests on the cylinders.

17. The carbon dioxide molecule can be considered a linear molecule with a central carbon atom, bound to two oxygen atoms with a pair of identical springs in opposing directions. Study the longitudinal motion of the molecule. One of the normal mode frequencies vanishes. What does that represent physically? Calculate a numerical value for the ratio between the two other (non-zero) normal mode frequencies of the molecule.

18. Three identical springs are connected in a row and then attached between a pair of walls. Two identical masses are attached at the connection points between the central spring and the two others. Determine the possible periods of oscillation of the system.

***19.** Generalize the previous problem to the case of N identical springs with $N - 1$ identical masses in between.

Answers:

1. $x = A\cos(\omega t + \varphi), \qquad \omega = (k/m)^{1/2}$

2. $x = v_{0x}t, \qquad y = v_{0y}t - \frac{gt^2}{2}$

3. (a) $\ddot{x} = g/2 \qquad$ (b) $\ddot{x} = \frac{g\left(m+m'\frac{x}{\ell}\right)}{2m+m'}$

4. With x the position of the large block, and y the position of the small block relative to the large block along the slope:
$\ddot{y} = g\frac{(M+m)\sin\alpha}{M+m\sin^2\alpha}, \qquad$ and $\qquad x = -y\frac{m}{M+m}\cos\alpha.$

5. $\frac{mg\sin\theta\cos\theta}{\frac{7}{5}(M+m)-m\cos^2\theta}$, to the left.

6. $\ddot{x} + \omega_x^2 x = 0$, $\ddot{y} + \omega_y^2 y = 0$, where x is the horizontal and y the vertical deviation from the equilibrium position, with the normal frequencies given by $\omega_x^2 = k/m$, $\omega_y^2 = \left((k/m)^{-1} + (g/l)^{-1}\right)^{-1}$, so $\omega_y < \omega_x$. Best done in polar coordinates.

7. A circle.

8. $(M+m)\ddot{x} + m\ddot{y}\cos\alpha + kx = 0; \qquad m\ddot{y} + m\ddot{x}\cos\alpha - mg\sin\alpha + ky = 0$

9. $\omega = \left(\frac{g+a}{\ell}\right)^{1/2}$

10. $\omega = \left(\frac{k}{m+M/3}\right)^{1/2}$

11. $\omega = k\sqrt{\frac{3\pm\sqrt{5}}{2}} = k\frac{\sqrt{5}\pm1}{2}$

12. $\omega_1 = (g/a)^{1/2}, \qquad \omega_2 = (3g/a)^{1/2}$

13. Stationary trajectories: $\dot{\theta} = \frac{h}{r^2\sin^2\alpha}$, h constant. For small deviations from a stationary trajectory one obtains the frequency of oscillation, $\omega = \dot{\theta}\sqrt{3\sin^2\alpha}$.

14. M: $\omega = \left(\frac{k}{M+m}\right)^{1/2}; \qquad m$: $\omega = (g/\ell)^{1/2}$

15. $\theta = \frac{mg}{ka}(\cos\omega t - 1); \qquad \omega = \left(\frac{ka^2}{m(a^2+b^2)}\right)^{1/2}$

16. $\varphi = \varphi_0\cos\omega t; \qquad x = \frac{\ell\varphi_0}{3+\alpha}\left(1-\cos\omega t\right); \qquad \omega = \left(\frac{g(3+\alpha)}{\ell(2+\alpha)}\right)^{1/2}$

17. 1.915

18. $(k/m)^{1/2}, \qquad (3k/m)^{1/2}$