Infrared logarithms in Effective Field Theories

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Saint-Petersburg, 19 Mar.
IR Logarithms in EFTs

- Introduction. Effective field theories (EFTs). Structure of effective Lagrangians and low-energy expansion
- LLogs from unitarity and analyticity
- Connection with the renormalization-group
- Geometric formulation of RG-equations

Low-energy asymptotic for the pion parton distributions

- Singularities in the chiral expansion of non-local operators
- Low-x and large-b asymptotic for pion parton distribution

Conclusion
Effective Field Theories (EFTs) is the way to construct "the most general possible S-matrix consistent with perturbative unitarity, analyticity and assumed symmetry principles" [Weinberg, 79]. EFT is the quantum field theory (QFT) with "the most general possible Lagrangian with assumed symmetry principles".

\[
\mathcal{L}^{EFT} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots
\]

- \(\mathcal{L}_2\): terms with 2 derivatives in ChPT, 2 constants
- \(\mathcal{L}_4\): terms with 4 derivatives in ChPT, \(\sim 10\) constants
- \(\mathcal{L}_6\): terms with 6 derivatives in ChPT, \(\sim 90\) constants

EFTs are non-renormalizable field theories, and the structure of their perturbative expansion differs radically from renormalizable ones.

ChPT – Chiral Perturbation Theory
Introduction. Structure of low-energy EFT expansion

Perturbative EFT series has the form:

\[
A(s, t) = c_1 \frac{E^2}{\Lambda^2} + \frac{E^4}{\Lambda^4} \left( c_2 \ln \left( \frac{\mu^2}{E^2} \right) + c_3 \right) + \frac{E^6}{\Lambda^6} (\ldots) + \ldots
\]

- Tree order only \( \mathcal{L}_2 \) parameters
- 1-loop with \( \mathcal{L}_2 \) parameters + tree order with \( \mathcal{L}_4 \) parameters

\( E^2 \) is a generic momentum parameter, i.e. \( s, t, m^2, \ldots \).

\[
A(s, t) = \sum_{n=1}^{\infty} \omega_n \left( \frac{E^2}{\Lambda^2} \right)^n \ln^{n-1} \left( \frac{\mu^2}{E^2} \right) + \sum_{n=2}^{\infty} \varpi_n \left( \frac{E^2}{\Lambda^2} \right)^n \ln^{n-2} \left( \frac{\mu^2}{E^2} \right) + \ldots
\]

- Leading Logarithms (LLogs) contains only \( \mathcal{L}_2 \) parameters
- Next-to-Leading Logarithms (NLLLogs) contains \( \mathcal{L}_2 + \mathcal{L}_4 \) parameters

In contrast to renormalizable theories, where LLogs fix the leading asymptotic term, LLog contribution has no specials.
Unitarity and LLog coefficients

Let us consider EFT of scalar fields $\phi^i$ with the symmetry group $G$.

$$S_2 = \int d^4x \left( \frac{1}{2} g_{ij} (\phi) \partial_\mu \phi^i \partial_\mu \phi^j - \frac{1}{6} R_{ik,jl} \phi^k \phi^l \partial_\mu \phi^i \partial_\mu \phi^j + \ldots \right),$$

$g_{ij}$ is the group metric, $R_{ij,kl}$ is the Riemann tensor.

$$\langle \phi^d \phi^c | S | \phi^b \phi^a \rangle = I + i2\pi(4\pi)^4 \delta \left( \sum_{i=1}^4 p_i \right) \sum_I \sum_l (2l + 1) P_l \left( 1 + \frac{2t}{s} \right) t^I_l(s),$$

$P^a_{bcd}$ is the projector onto representation $I$, $s$ and $t$ are Mandelstam variables.

The partial amplitude $t^I_l(s)$ has only one dimensional parameter, and very simple analytic properties.

![Graphical representation of partial amplitudes](image)
\[ t^I_l = \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{s}{(4\pi F')^2} \right)^n \omega^I_{nl} \ln^{n-1} \left( \frac{\Lambda^2}{|s|} \right) + \mathcal{O} \left( s^n \ln^{n-2} \right) \]

- The 2-particle cuts results only from LLogs
- The coefficient \( \omega^I_{nl} \) (the "general" LLog coefficient) can be found by summing discontinuities over left- and right- cuts.
- The right-cut discontinuity is given by unitarity relation

\[
\text{Disc } t^I_l (s) = |t^I_l (s)|^2 + \mathcal{O} \text{ (Inelastic part } \sim \text{ NLLog)}, \quad s > 0.
\]

- The left-cut discontinuity can be found by analytical continuation of unitarity relation (alike imaginary part of Roy equation)

\[
\text{Disc } t^I_l (s) = \sum_{l', I'} O^I_{s u} \frac{2(2l' + 1)}{s} \int_0^{-s} ds' P_l \left( \frac{s + 2s'}{-s} \right) P_{l'} \left( \frac{2s + s'}{-s'} \right) \text{Disc } t^{I'}_{l'} (s'), \quad s < 0.
\]
The LLog coefficients are given by the recursive equation

\[
\omega_{nl}^I = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{l=0}^{n} \frac{1}{2} \left( \delta_{ll'} \delta_{ll'} + C_{st} \Omega_{n}^{ll'} + C_{su}(-1)^{l+l'} \Omega_{n}^{ll'} \right) \frac{\omega_{i,l'}^I \omega_{n-i,l'}^I}{2l'+1}
\]

\(\Omega_{n}^{ll'}\) is the crossing matrix in the partial wave space at LLog approximation, and \(C_{st}\) are crossing matrices in group space.

It is universal form of equation on LLog coefficients

\[
\tilde{\omega}_{n} = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (I + C_{st} + C_{su}) \cdot (\tilde{\omega}_{i} \tilde{\omega}_{n-i})
\]

One needs to calculate only the crossing matrices and boundary conditions \(\tilde{\omega}_{1}\).

The boundary conditions are given by tree-order of the amplitude.

[J.Koschinski,M.Polyakov, AV, 1004.2197]
Example calculation: $O(N)$-model

The Lagrangian of the Weinberg model:

$$\mathcal{L}_2 = \left( \delta_{ij} + \frac{\phi_i \phi_j}{F^2 - \phi^2} \right) \partial_\mu \phi^i \partial_\mu \phi^j. $$

There are 3 isospin spaces ($I = 0, 1, 2$) with projectors.

$$P_0^{abcd} = \frac{1}{N} \delta^{ab} \delta^{cd}, \quad P_1^{abcd} = \frac{1}{2} \left( \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right), \quad P_2^{abcd} = \frac{1}{2} \left( \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) - \frac{1}{N} \delta^{ab} \delta^{cd}$$

Boundary conditions \( \{ \omega_0^{10}, \omega_1^{11}, \omega_2^{10} \} = \{ N - 1, 1, -1 \} \). Then the simple calculation gives

$$\frac{\omega_0^{10}}{N - 1} = \left\{ 1, \frac{N}{2} - \frac{1}{9}, \frac{N^2}{4} - \frac{61N}{144} + \frac{59}{144}, \frac{N^3}{8} - \frac{631N^2}{2700} + \frac{46279N}{194400} - \frac{13309}{194400} \right\}$$

- The results agrees with known 1 and 2-loop calculations
- The large-N approximation and its correction are correct.
- The equation is very fast evaluated on computer, e.g. calculation of $\omega_{100}$ ($N = 3$) takes $\sim 10$ min.
The equation in renormalizable theory

In renormalizable theories the LLog approximation can be obtained by solving 1-loop RG equation.

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi^i \partial_{\mu} \phi^i - \frac{\lambda_0}{4!} (\phi^2)^2 \]

\[ \mu^2 \frac{\partial}{\partial \mu^2} \lambda(\mu^2) = \beta(\lambda) = \frac{N + 2}{8} \lambda(\mu^2)^2 + \mathcal{O}(\lambda^2) \quad \implies \quad A(s, t) = \lambda(\mu^2) = \frac{\lambda_0}{1 - \frac{N+2}{8} \lambda_0 \ln(\mu^2/s)} \]

The same solution comes from the recursive equations (the \(\beta\)-function is the sum of crossing matrices):

\[ \omega_n = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \frac{(N + 2)}{4} \omega_i \omega_{n-i} \quad \implies \quad A(s, t) = \sum_{n=1}^{\infty} \lambda_0^n \ln^{n-1} \left( \frac{\mu^2}{s} \right) \omega_n \]

The recursive equations are the particular form of renormalization group equations. For the EFT (i.e. non-renormalizable QFT) it takes the form:

\[ \omega_{nl} = \frac{1}{n-1} \sum_{i=1}^{n-1} \beta_n^{l \ell'} \omega_{l \ell'} \omega_{n-i, \ell'} \quad \implies \quad \mu^2 \frac{\partial}{\partial \mu^2} A(s, t) = \int_{-s}^{0} dt_1 dt_2 A(s, t_1) A(s, t_2) K(s, t; t_1, t_2) \]
The RG equations in EFTs

In EFTs one has infinite number of coupling constants and $\beta$-functions. (see detail discussion [Buchler,Colangelo,0309049])

\[
\begin{align*}
\mu^2 \frac{\partial}{\partial \mu^2} g_1 &= 0, & \text{the lowest order constant is not running} \\
\mu^2 \frac{\partial}{\partial \mu^2} g_{2C} &= \beta(1, 1, C) g_1 g_1, & \text{the second order constant is running through } g_1 \\
\mu^2 \frac{\partial}{\partial \mu^2} g_{nC} &= \sum_{i=1}^{n-1} \beta(i, A; n-i, B/C) g_{iA} g_{n-i,B} + \mathcal{O}(g^3), & \text{the equation for 1-loop run of the } g_{nA}.
\end{align*}
\]

The $\beta$ function is the pole-part of diagrams:

\[\mu^2 \frac{d}{d\mu^2} A(s, t, \mu^2, g) = \left( \mu^2 \frac{\partial}{\partial \mu^2} + \sum_{n=1}^{\infty} \sum_{A} \beta(g_{nA}) \frac{\partial}{\partial g_{nA}} \right) A(s, t, \mu^2, g)\]
\[
\mu^2 \frac{d}{d\mu^2} A(s, t, \mu^2, g) = \left( \mu^2 \frac{\partial}{\partial \mu^2} + \sum_{n=1}^{\infty} \sum_{A} \beta(g_{nA}) \frac{\partial}{\partial g_{nA}} \right) A(s, t, \mu^2, g)
\]

\[
A(s, t, \mu_0^2, g_0) = \sum_{n,A} g_{0,nA} V_{nA}(s, t) \quad \text{-- boundary condition}
\]

The system can be, indeed, reformulated in terms of integral or recursive equation:

\[
A(s, t, \mu^2, g) = \sum_{n,A} \omega_{nA} V_{nA}(s, t) g_1^n \ln^{n-1} \left( \frac{\mu^2}{\mu_0^2} \right)
\]

\[
\omega_{nC} = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{AB} \beta(i, A; n-i, B/C) \omega_{i,A} \omega_{n-i,B}
\]

[Kivel,Polyakov,AV,0809.3236][Kivel,Polyakov,AV,0904.3008]

Such analysis can be applied only in massless EFT.

The presents of mass breaks the hierarchy of equations and the system can not be solved (see details [Bijnens,Carloni,1008.3499]). The presence of mass also breaks the unitarity consideration, since \( \ln (m^2) \) are "invisible" in complex s-plane.
1-loop $\beta$-functions for $\sigma$-model

In $D = 2$ the run of $\sigma$-model is defined by the famous Friedan (Ricci-flow) equation:

$$\mu^2 \frac{\partial}{\partial \mu^2} g_{ij}(\phi) = \frac{1}{8\pi} R_{ij} + \ldots \quad (D = 2)$$

For the higher dimension of base manifold the lowest order metric is not run. But the higher order tensors runs:

$$S = \int d^4 x \, \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial_\mu \phi^j + h_{ij} \partial^2 \phi^i \partial^2 \phi^j + T_{ijkl}^{(1)}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j \partial_\rho \phi^k \partial_\sigma \phi^l + \ldots$$

$$= \int d^4 x \, \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial_\mu \phi^j + \sum_{n,l} T_{ijkl}^{(nl)}(\phi)(\phi^i \nabla_{\mu_1} \ldots \nabla_{\mu_l} \phi^j) \partial^{2(n-l)}(\phi^k \nabla_{\mu_1} \ldots \nabla_{\mu_l} \phi^l) + T_{ijklmn}^{(nl)}$$

Their 1-loop run be found by recursive equation in the form:

$$\mu^2 \frac{\partial}{\partial \mu^2} T_{abcd}^{(nl)} \approx (\omega_{nl})_{abcd} + \ldots$$

$$(\omega_{nl})_{abcd} = \frac{1}{2(n-1)} \left[ \sum_{i=1}^{n-1} (\omega_{il})_{ab\alpha\beta} (\omega_{n-i,l})_{\beta\alpha}^{cd} \right]_{2l+1}$$

$$(\omega_{il})_{ab\alpha\beta} (\omega_{n-i,l})_{\beta\alpha}^{cd} + \sum_{i=1}^{n-1} \sum_{l'=0}^{n} (\omega_{il'})_{ad\alpha\beta} (\omega_{n-i,l'})_{\beta\alpha}^{ce} \Omega_{l'l}^{c} + (\omega_{il'})_{ac\alpha\beta} (\omega_{n-i,l'})_{\beta\alpha}^{bd} (-1)^{l+l'} \Omega_{n}^{l'l} \right]_{2l'+1}.$$
Boundary conditions:

\[
(\omega_{10})_{abcd} = -\frac{1}{2}(R_{acbd} + R_{adbc}), \quad (\omega_{11})_{abcd} = \frac{1}{2}R_{abcd}.
\]

The $n = 2$ evolution is well-known (e.g. [Percacci,Zanusso,0910.0851])

\[
\mu^2 \frac{\partial T_{(21)}^{abcd}}{\partial \mu^2} = \frac{1}{(4\pi)^2} \left( \frac{1}{2} \left( R_{a\beta_1 c\beta_2} R_{b\beta_1 d}^{\beta_2} + R_{a\beta_1 c\beta_2} R_{b\beta_2}^{\beta_1 d} \right) \\
+ \frac{1}{12} \left( R_{ad\beta_1 \beta_2} R_{bc}^{\beta_1 \beta_2} - R_{ab\beta_1 \beta_2} R_{cd}^{\beta_1 \beta_2} \right) \right)
\]

$n = 3$ evolution is proportional to $R^3$, etc. [Polyakov,AV,1012.4205].
Status LLog in EFTs

LLogs in renormalizable QFT

\[ A(s) = \alpha + \alpha^2 (a_1 L + b_1) + \alpha^3 (a_2 L^2 + b_2 L + c_2) + \ldots, \quad \alpha \propto \frac{\alpha_0}{L} \]

\[ A(s) \propto A(\alpha_0) + \mathcal{O}\left(\frac{1}{L}\right), \quad L = \ln (|s|) \]

In renormalizable QFTs LLog approximation gives the leading asymptotic with logarithmic accuracy.

LLogs in EFT

\[ A(s) = \frac{s}{F^2} + \frac{s^2}{F^4} (a_1 L + b_1) + \frac{s^3}{F^6} (a_2 L^2 + b_2 L + c_2) + \ldots, \quad s \ll F^2, \quad sL \sim s \]

In EFTs logarithms gives very small corrections, often less then finite part. There is no reason to keep special attention to them.

But situation changes in the presence of additional dimensional parameter.
LLog in presence of additional dimensional parameter

\[ A(s, \lambda) = \frac{s}{F^2} + \frac{s^2 \lambda}{F^2} (a_1 L + b_1) + \frac{s^2}{F^2} (\tilde{a}_1 L + \tilde{b}_1) + \frac{s^3 \lambda^2}{F^2} (a_2 L^2 + b_2 L + c_2) + .. \]

\[ s \ll F^2, \quad s \lambda \sim 1 \]

\[ A(s, \lambda) = \frac{s}{F^2} \tilde{A}(s\lambda L) + \mathcal{O} \left( \frac{1}{L} \right) + \mathcal{O} \left( \frac{s^2}{F^4} \right) \]

Such situation is normal for the non-local operators.
Pion parton distribution in ChPT

Lowest order Chiral Perturbation Theory (ChPT)

\[ \mathcal{L}_2 = \frac{F_\pi^2}{2} \text{tr} \left( \partial_\mu U \partial_\mu U^\dagger + m^2 U + U^\dagger \right), \quad U = \exp(it^a \pi^a). \]

The parton distributions cannot be calculated within framework of QCD, but their low-energy behavior can be find within EFTs.

The pion generalized parton distribution (GPD) in ChPT

The quark operator can be matched to ChPT [Kivel,Polyakov,0203264]

\[ H(x, \Delta_\perp, \xi) = \int \frac{d\lambda}{2\pi} e^{-ixP+\lambda \langle \pi(p')|O(\lambda)|\pi(p)\rangle} \frac{P}{2}(p+p'), \quad \Delta^2 = (p'-p)^2, \quad \xi = \frac{(p'-p)+}{(p'+p)+} \]

\[ O(\lambda) = \frac{iF^2}{4} \mathcal{F}(\beta, \alpha) \star \left[ U \left( \frac{\alpha + \beta}{2} \lambda n \right) (n \xrightarrow{\partial}) U^\dagger \left( \frac{\alpha - \beta}{2} \lambda n \right) \right] \]

* is convolution in \((\alpha, \beta)\), and \(\mathcal{F}\) is the double distribution (DD) ansatz in chiral limit.

\[ \int_{-1+|\beta|}^{1-|\beta|} d\alpha \mathcal{F}(\beta, \alpha) = q^o (\beta) \]
Singular terms in PDF

The presence of additional dimensional parameter $\lambda$ leads to singular contributions ($\Delta = 0, \xi = 0$) [Kivel,Polyakov,0203164][Diehl,Manashov,Schafer,0608113]:

\[ q(x) = \bar{q}(x) + a_\lambda \ln \left( \frac{1}{a_\lambda} \right) \left( \bar{q}(x) - \delta(x) \right) + ... \]

\[ a_\lambda = \frac{m_\pi^2}{(4\pi F_\pi)^2}. \]

There are stronger singularities at higher orders [Kivel,Polyakov,0707.2208]:

\[ q(x) = q^{\text{reg}}(x) + \sum_n C_n \langle x^n \rangle \left( a_\lambda \ln \left( \frac{1}{a_\lambda} \right) \right)^n \delta^{(n-1)}(x) + \mathcal{O}(\ln^{-1}) + \mathcal{O}\left( \delta^{(n-2)}(x) \right) \]

At small $x$ then $a_\lambda \ln (a_\lambda) \delta(x) \sim a_\lambda^0$ or at big light-cone distances $\lambda \sim (a_\lambda \ln (a_\lambda))^{-1}$ these terms give the main contribution.

Only explicit summation of singular terms removes unphysical singularities from PDF.
Singular terms in GPD

The presence of additional dimensional parameter $\lambda$ leads to singular contributions ($\Delta = 0, \xi = 0$) [Kivel,Polyakov,0203164][Diehl,Manashov,Schafer,0608113]:

$$H(x, \xi, \Delta) = H(x, \xi) - \int_{-1}^{1} d\eta \frac{R \ln R}{2\xi} \partial_\eta \left( \eta \mathcal{F} \ast \delta \left( \beta \eta + \alpha - \frac{x}{\xi} \right) \right) + \ldots , \quad R = \frac{m^2 - \Delta^2 (1 - \eta^2)}{(4\pi F_\pi)^2}.$$  

There are stronger singularities at higher orders [Kivel,Polyakov,0707.2208]:

$$H(x, \xi, \Delta) = H^{\text{reg}} + \sum_n \tilde{D}_n \int_{-1}^{1} d\eta \left( \frac{R \ln R}{\xi} \right)^n \partial_\eta^n \left( \eta \mathcal{F} \ast \delta \left( \beta \eta + \alpha - \frac{x}{\xi} \right) \right) + \mathcal{O}(\xi^{-n-1})$$

Such expression to unphysical singularities in kinematical parameter $\xi$.

Only explicit summation of singular terms removes unphysical singularities from GPD.
\[ \delta q^I(x) = \sum_{n} \frac{C^I_n}{n!} \langle x^n \rangle \left(a_x \ln \left( \frac{1}{a_x} \right) \right)^n \delta^{(n-1)}(x) \]

Only the massless part of \(\pi\pi\) scattering graph contributes to coefficient \(C^I_n\). We can apply our methods of LLog calculation

\[ C^I_n = \frac{(2n)!}{2(n!)^2} \omega^I_{nn} \]

### Approximate solution of equation

The summation leads to smooth function which gives significant contributions at \(x \lesssim 10^{-3}\)

\[ q(x) = \bar{q}(x) + \frac{1}{A} [\bar{q} * f] \left( \frac{|x|}{\epsilon} \right) + \mathcal{O} \left( \frac{1}{\ln(\epsilon)} \right), \]

\[ \epsilon = \frac{A m^2_{\pi}}{(4\pi F_{\pi})^2} \ln \left( \frac{(4\pi F_{\pi})^2}{m^2_{\pi}} \right) \simeq 0.066. \]

\[ f(z) = \theta(z < 1) (c_0 + c_1 - c_1 z). \]

Transverse size of pion

$q(x, b^2)$ has the meaning of pion matter density at distance $b$ from the center of mass.

$$\delta q(x, b^2) = \int_{-1}^{1} d\eta \int \frac{dz}{z} q(z) K\left(z, \frac{x + m^2 b^2}{1 - \eta^2}\right) \left(1 + \mathcal{O}\left(\frac{1}{\ln(b^2)}\right)\right)$$

$$\epsilon_b = \frac{\ln \left(b^2 (4\pi F)^2\right)}{(4\pi F)^2 b^2}$$

$$q(x) \sim \frac{1}{x^\alpha},$$

$$x \to 0, \quad \langle b^2 \rangle \sim \frac{1}{x^\alpha},$$

Very low-$x$ behave as usual

intermediate low-$x$ contribute stronger

Standard asymptotic PDF asymptotic
Two independent methods to obtain the logarithmic behavior of massless EFTs were presented.

- Unitarity method, which uses only unitarity relation and analytic properties of amplitude
- RG method based on principe on renormalization group invariance, involves loop-calculations for high-derivative vertices.
- The methods can be applied to any amplitude, see form-factor case at [Kivel,Polyakov,AV,0904.3008]

The recursive equations can be reformulated in geometrical, "Friedan"-like form.
- One-loop run of higher derivative operators tensors in $\sigma$-model

The method is used to compute the leading chiral behavior of pion partonic distributions
- The solution of the singular terms problem – resummation
- Various asymptotic behaviors of pion partonic distributions ($\Delta^2 \to 0$, $b^2 \to \infty$) are investigated.