Leading chiral logarithm for the nucleon mass

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Chiral perturbation theory (ChPT) is low-energy effective field theory.

\[ \mathcal{A} = \mathcal{A}^{(0)} + \frac{q^2}{(4\pi F_\pi)^2} \mathcal{A}^{(1)} + \left( \frac{q^2}{(4\pi F_\pi)^2} \right)^2 \mathcal{A}^{(2)} + \left( \frac{q^2}{(4\pi F_\pi)^2} \right)^3 \mathcal{A}^{(3)} + \ldots \]

\[ q = \text{momenta, masses, etc} \lesssim m_\pi \]

- Infinite effective Lagrangian with dimensional couplings (non-renormalizable theory).
- Ordering in dimension allows to systematically calculate low-energy expansion.
- With increasing chiral order the number of low-energy constants (LECs) grows.

\[ \mathcal{L}^{N+\pi}_{\text{ChPT}} = \mathcal{L}^{(0)}(F_\pi, m_\pi, g_A) + \mathcal{L}^{(1)}(c_1, \ldots, c_4) + \mathcal{L}^{(2)}(l_1, \ldots, l_{10}, d_1, \ldots, d_{23}) + \ldots \]
Logarithmic structure of chiral expansion

\[ \mathcal{A} = \mathcal{A}^{(0)} + \frac{q^2}{(4\pi F_\pi)^2} \mathcal{A}^{(1,1)} L + \left( \frac{q^2}{(4\pi F_\pi)^2} \right)^2 \mathcal{A}^{(2,2)} L^2 + \left( \frac{q^2}{(4\pi F_\pi)^2} \right)^3 \mathcal{A}^{(3,3)} L^3 + \cdots \]

\[ L = \ln \left( \frac{q^2}{\mu^2} \right) \]

- Leading logarithms (LLogs) involves only the leading order Lagrangian.
- Instead of evaluation of \( n \)-loop diagrams one can apply renormalization group (RG) [Büchner, Colangelo, 03]
Chiral Logs from renormalization group

The renormalization scale invariance implies

\[ A(\mu^2) = \exp \left( \ln \left( \frac{\mu^2}{\mu_0^2} \right) \sum_{n=0}^{\infty} \beta^{(n)} \frac{\partial}{\partial c^{(n)}} \right) A(\mu_0^2) \]

The LECs obey the equation

\[ \mu^2 \frac{d}{d\mu^2} c^{(n)} = \beta^{(n)} \left[ c^{(n-1)}, \ldots, c^{(0)} \right] \leftarrow \text{infinite set of equations} \]

With help of these equations one can extract the logarithmical contributions with minimum efforts.

- For LLog contributions one needs only the one-loop $\beta$-functions for LECs of chiral order up to order of calculation.
- For NLLLog contribution one needs one- and two-loop $\beta$-function
- etc.
LLogs in meson ChPT

The renormalization group technique has been successfully applied in meson sector.

\[ \mathcal{L}^{(0)}_\pi = \frac{F^2}{4} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger + m^2 \left( U + U^\dagger \right) \right) \]

- The physical pion mass has been calculated up to six-loop order [Bijnens, et al, 2012]

\[ m_{\text{phys}}^2 = m^2 \left( 1 - \frac{1}{2} L + \frac{17}{8} L^2 - \frac{103}{24} L^3 + \frac{24367}{1152} L^4 - \frac{8821}{144} L^5 + \frac{1922964667}{6220800} L^6 + \cdots \right), \]

here: \[ L = \frac{m^2}{(4\pi F)^2} \log \left( \frac{\mu^2}{m^2} \right) . \]

- Pion decay constant, form factors and wave-lengths up to five-loop order (complete automatization limited by the computer calculation time) [Bijnens, Carlone, 2010]
- For the massless pions exact all-order equation (\( \sim 2 - 3 \times 10^2 \) loops within hour) [Kivel, et al, 2008]
**Aim:** to evaluate nucleon physical mass at LLog accuracy (up to possible highest order).

**Pion-nucleon ChPT**

- In order to bypass the counting problem of nucleon-pion ChPT we work in heavy baryon formulation.
- The lowest order Lagrangians are

  \[ \mathcal{L}_{\pi N}^{(0)} = \bar{N} (i\gamma^{\mu} D_{\mu} + g A S^{\mu} u_{\mu}) N, \]

  \[ \mathcal{L}_{\pi N}^{(1)} = \bar{N}_{\nu} \left[ \frac{(v \cdot D)^2 - D \cdot D - ig A \{S \cdot D, v \cdot u\}}{2M} + c_1 \text{tr} (\chi^+) + \left( c_2 - \frac{g_A^2}{8M} \right)(v \cdot u)^2 \right. \]

  \[ + c_3 u \cdot u + \left( c_4 + \frac{1}{4M} \right) i\epsilon_{\mu\nu\rho\sigma} u_\mu u_\nu v_\rho S_\sigma \left. \right] N_{\nu}. \]

- The physical mass of nucleon is known up to two-loop order [Schindler, et al, 07]
Structure of nucleon mass at "LLog" accuracy [Schindler, et al, 07]

\[ M_{\text{phys}} = M - 4c_1 m^2 - \frac{3\pi}{2} g_A^2 \frac{m^3}{(4\pi F)^2} + \frac{3}{4} \left( \frac{g_A^2}{M} + c_2 + 4c_3 - 8c_1 \right) \frac{m^4}{(4\pi F)^2} \ln \left( \frac{\mu^2}{m^2} \right) + \ldots \]

LLog contribution, \( \mathcal{L}^{(0)} + \mathcal{L}^{(1)} \)
Structure of nucleon mass at "LLog" accuracy \cite{Schindler, et al, 07}

\[ \mathcal{L}^{(1)} \]

\[ M_{\text{phys}} = M - 4c_1 m^2 - \frac{3\pi}{2} \frac{g_A^2}{(4\pi F)^2} \frac{m^3}{2} \]  

\[ + \frac{3}{4} \left( \frac{g_A^2}{M} + c_2 + 4c_3 - 8c_1 \right) \frac{m^4}{(4\pi F)^2} \ln \left( \frac{\mu^2}{m^2} \right) + \ldots \]

\[ L(0) + L^{(1)} \]
Structure of nucleon mass at "LLog" accuracy [Schindler, et al., 07]

$$M_{\text{phys}} = M - 4c_1m^2$$

$\sim \sqrt{m_q}$ term, finite part of 1-loop elder then LLog

$$\frac{3\pi}{2} g_A^2 \frac{m^3}{(4\pi F)^2} + \frac{3}{4} \left( \frac{g_A^2}{M} + c_2 + 4c_3 - 8c_1 \right) \frac{m^4}{(4\pi F)^2} \ln \left( \frac{\mu^2}{m^2} \right) + \ldots$$

LLog contribution, $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$

$\sim \sqrt{m_q}$ term, 2-loop NLLog elder then LLog

$$+ \frac{3\pi}{8} g_A^2 (3 - 16g_A^2) \frac{m^5}{(4\pi F)^4} \ln \left( \frac{\mu^2}{m^2} \right) - \frac{3}{4} \left( \frac{g_A^2}{M} + c_2 + 4c_3 - 6c_1 \right) \frac{m^6}{(4\pi F)^4} \ln^2 \left( \frac{\mu^2}{m^2} \right) + \ldots$$

2-loop LLog contribution

- The proper "LLog" accuracy contains the true LLog contribution, as well as, LLog non-analytical in $m_q$ contribution (which is of NLLog nature).
Course of Calculation

LLLog contribution

For \( n \)'th order LLLog coefficient one needs:

1) Lagrangian \( \mathcal{L}_{N\pi}^{(2n(+1))}(\pi^0) \) for tree diagrams

2) 1-loop \( \beta \)-functions for all LECs with \( \chi_{\text{order}} + N_{\text{pions}} = 2n(1) \)
   - Calculation of 1-loop \( \beta \) is performed by FORM
     (typical number of diagrams for 4-loop LLLog \( \sim 10^4 \))
   - The next order Lagrangian is generated as terms necessary for renormalization
     (non-minimal Lagrangian [Bijnens,Carloni,10])
     (typical number of diagrams for 4-loop LLLog \( \sim 5 \times 10^2 \))

\[
\text{LLLog coef.} = \frac{1}{[n/2]!} \left( \sum_k \beta_{1\text{-loop}}^{(k)} \frac{\partial}{\partial c^{(k)}} \right)^{n/2} \text{tree}^{(n)}
\]

Non-analytical in \( m_q \) terms

For \( n \)'th order LLLog non-analytical in \( m_q \) terms one additionally needs:

3) Finite part of one-loop diagrams without external pions

\[
\sim \sqrt{m_q} \text{ LLLog coef.} = \frac{1}{[(n-1)/2]!} \left( \sum_k \beta_{1\text{-loop}}^{(k)} \frac{\partial}{\partial c^{(k)}} \right)^{(n-1)/2} \text{ (finite part 1-loop)}^{(n-1)}
\]
\[ M_{\text{phys}} = M + k_2 \frac{m^2}{M} + k_3 \frac{\pi m^3}{(4\pi F')^2} + k_4 \frac{m^4}{(4\pi F')^2 M} \ln \left( \frac{\mu^2}{m^2} \right) + k_5 \frac{\pi m^5}{(4\pi F')^4} \ln \left( \frac{\mu^2}{m^2} \right) + \cdots \]

**LLLog coefficients**

\[ k_2 = -4c_1 M \]

\[ k_4 = \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 8c_1)M \right) \]

\[ k_6 = -\frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 6c_1)M \right) \]

\[ k_8 = \frac{27}{8} \left( g_A^2 + (c_2 + 4c_3 - \frac{16}{3}c_1)M \right) \]

\[ k_{10} = -\frac{257}{32} \left( g_A^2 + (c_2 + 4c_3 - 5c_1)M \right) \]

**Non-analytical in \( m_q \) LLLog coefficients**

\[ k_3 = -\frac{3}{2} g_A^2 \]

\[ k_5 = \frac{3g_A^2}{8} \left( 3 - 16g_A^2 \right) \]

\[ k_7 = g_A^2 \left( -18g_A^4 + \frac{35g_A^2}{4} - \frac{443}{64} \right) \]

\[ k_9 = \frac{g_A^2}{3} \left( -116g_A^6 + \frac{2537g_A^4}{20} - \frac{3569g_A^2}{24} + \frac{55609}{1280} \right) \]

\[ k_{11} = \frac{g_A^2}{2} \left( -95g_A^8 + \frac{5187407g_A^6}{20160} - \frac{449039g_A^4}{945} + \frac{16733923g_A^2}{60480} - \frac{298785521}{1935360} \right) \]

[Bernard, et al, 92]

[Schindler, et al, 07]

(4-loop)

[McGovern, Birse, 99]

(5-loop)
\[ k_2 = -4c_1 M \]
\[ k_4 = \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 8c_1)M \right) \]
\[ k_6 = -\frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 6c_1)M \right) \]
\[ k_8 = \frac{27}{8} \left( g_A^2 + (c_2 + 4c_3 - \frac{16}{3}c_1)M \right) \]
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Peculiarity 1

No higher powers of \( g_A \), only \( g_A^2 \).

• Consequence of Lorentz invariance (?)
• It implies that: **diagrams with odd-number-of-pion vertices do not contribute to the LLog coefficient of nucleon mass.**
  Note: these diagrams are non-zero, but cancel with each other within the solution of pole-equation.
• **Supposing** that this behavior holds at higher orders we can suppress higher power during calculation (significant speed improvement!)
\[ k_2 = -4c_1 M \]
\[ k_4 = \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 8c_1)M \right) \]
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\[ k_8 = \frac{27}{8} \left( g_A^2 + (c_2 + 4c_3 - \frac{16}{3}c_1)M \right) \]
\[ k_{10} = -\frac{257}{32} \left( g_A^2 + (c_2 + 4c_3 - 5c_1)M \right) \]
\[ k_{12} = \frac{115}{3} \left( g_A^2 + (c_2 + 4c_3 - \frac{24}{5}c_1)M \right) \]

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### Peculiarity 1

- **No higher powers of** \( g_A \), only \( g_A^2 \).

- **Consequence of Lorentz invariance (?)**

- **It implies that**: *diagrams with odd-number-of-pion vertices do not contribute to the LLog coefficient of nucleon mass.*
  Note: these diagrams are non-zero, but cancel with each other within the solution of pole-equation.

- **Supposing** that this behavior holds at higher orders we can suppress higher power during calculation (significant speed improvement!)
\[ k_2 = -4c_1 M \]
\[ k_4 = \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - 3c_1 M \]
\[ k_6 = -\frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) + \frac{3}{2} c_1 M \]
\[ k_8 = \frac{27}{8} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - \frac{9}{2} c_1 M \]
\[ k_{10} = -\frac{257}{32} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) + \frac{257}{32} c_1 M \]
\[ k_{12} = \frac{115}{3} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - \frac{92}{3} c_1 M \]

**Peculiarity 2**

Universal structure of the expression

\[ k_{2n} = b_n \left( \frac{-3c_1 M}{n - 1} + \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) \right) \]

**Scientific guess:** Coefficients \( b_n \) are related to the pure pion physics.

**Indeed:** They coincide with the LLog expansion of \( m_{\text{phys}}^4 \) (accidentally?)
\[ k_2 = -4c_1 M \]
\[ k_4 = \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - 3c_1 M \]
\[ k_6 = -\frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) + \frac{3}{2} c_1 M \]
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\[ k_{12} = \frac{115}{3} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - \frac{92}{3} c_1 M \]

LLog expression for all orders

We have all-order conjecture for LLog expression for nucleon mass

\[
M = M_{\text{phys}} + \frac{3}{4} m_{\text{phys}}^4 \frac{\log \left( \frac{\mu^2}{m_{\text{phys}}^2} \right)}{\left( 4\pi F \right)^2} \left( \frac{g_A^2}{M_{\text{phys}}} - 4c_1 + c_2 + 4c_3 \right)
\]
\[
- \frac{3c_1}{\left( 4\pi F' \right)^2} \int_{m_{\text{phys}}^2}^{\mu^2} m_{\text{phys}}^4(\mu') \frac{d\mu'^2}{\mu'^2}.
\]

Knowledge of pion mass LLog expansion up to six-order, allows us to guess two more coefficients.
\[ k_2 = -4c_1 M \]
\[ k_4 = \frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - 3c_1 M \]
\[ k_6 = -\frac{3}{4} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) + \frac{3}{2} c_1 M \]
\[ k_8 = \frac{27}{8} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - \frac{9}{2} c_1 M \]
\[ k_{10} = -\frac{257}{32} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) + \frac{257}{32} c_1 M \]
\[ k_{12} = \frac{115}{3} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - \frac{92}{3} c_1 M \]
\[ k_{14} = -\frac{186515}{1536} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) + \frac{186515}{2304} c_1 M \]
\[ k_{16} = \frac{153149887}{259200} \left( g_A^2 + (c_2 + 4c_3 - 4c_1)M \right) - \frac{153149887}{453600} c_1 M \] (7-loop)

LLog expression for all orders

We have all-order conjecture for LLog expression for nucleon mass

\[ M = M_{\text{phys}} + \frac{3}{4} m_{\text{phys}}^4 \log \left( \frac{\mu^2}{m_{\text{phys}}^2} \right) \left( \frac{g_A^2}{M_{\text{phys}}} - 4c_1 + c_2 + 4c_3 \right) \]
\[ -\frac{3c_1}{(4\pi F)^2} \int_{m_{\text{phys}}^2}^{\mu^2} m_{\text{phys}}^4 (\mu') \frac{d\mu'^2}{\mu'^2} \cdot \]
Conclusion

- The RG technique is elaborated for the heavy-baryon nucleon-pion ChPT.
- LLog and (N)LLog non-analytical in $m_q$ coefficients for the nucleon mass up to 4- and 5-loop order.
- Using conjectures which follows from the form of LLog coefficients we have obtained coefficient up to 7-loop order.
- We suggest an all order LLog expression for the nucleon mass.

Nearest future result

- Relativistic calculation (infrared-renormalization scheme)
- LLog coefficients for axial coupling, form factors and nucleon-pion wave-lengths.