FYTN15 / FMFN25
December 17, 2019

Statistical Mechanics — Homework Assignment 3

Due: noon Wednesday, 8 January.

1. (a) (1 p) A paramagnet in one dimension can be modeled as a linear chain of $N+1$ spins. Each spin interacts with its neighbors in such a way that the energy is $E = n\epsilon$, where n is the number of domain walls separating regions of up spins from down spins. Calculate first the entropy $S(E)$ of the system, and then the energy E as a function of the temperature T (i.e., find $E(T)$).

   (b) (1 p) A rubber band at temperature T is fastened at one end to a peg, and supports from its other end a mass m. Model the rubber band as a linked chain of N segments joined end to end. Each segment has a length a and can be oriented either parallel or antiparallel to the vertical direction. Find an expression for the mean length $\langle l \rangle$ of the rubber band as a function of m. Neglect the kinetic energy and mass of the segments themselves, and any interaction between the segments.

2. (2 p) Consider a one-dimensional grand-canonical lattice gas with occupation variables $n_i \in \{0, 1\}$ ($i = 1, \ldots, L$) and periodic boundary conditions ($n_{L+1} = n_1$). The energy, E, and the number of particles, N, are given by

$$E = -\epsilon \sum_{i=1}^{L} n_i n_{i+1}, \qquad N = \sum_{i=1}^{L} n_i.$$

   Show that the partition function $\Xi(\beta, \mu, L)$ satisfies

$$\frac{\ln \Xi}{L} \to \ln \left[\frac{1}{2}(1 + xy) + \sqrt{y + \frac{1}{4}(1 - xy)^2} \right] \quad \text{as} \quad L \to \infty,$$

   where $x = \mathrm{e}^{\beta\epsilon}$ and $y = \mathrm{e}^{\beta\mu}$, and explain how this result can be used to compute $\langle E \rangle / L$ and $\langle N \rangle / L$ (as functions of β and μ).

3. Consider a three-state "clock" model, where each spin $\mathbf{s}_i = (s_{xi}, s_{yi})$ is a unit vector in two dimensions with three possible orientations: $(1, 0)$ and $(-1/2, \pm\sqrt{3}/2)$. The energy function is given by

$$E = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij}\, \mathbf{s}_i \cdot \mathbf{s}_j$$

   where $J_{ij} = J$ if i and j are nearest neighbors and $J_{ij} = 0$ otherwise ($J > 0$). Suppose each lattice site has z nearest neighbors. The mean-field approximation can be defined by the ansatz $E_{\mathrm{MF}} = -\sum_{i=1}^{N} h\, s_{xi}$.

(a) (0.5 p) Show that $s = \langle s_{xi} \rangle_{\mathrm{MF}}$ is given by (1 p)

$$s = \frac{\mathrm{e}^{\beta h} - \mathrm{e}^{-\beta h/2}}{\mathrm{e}^{\beta h} + 2\mathrm{e}^{-\beta h/2}} \qquad (\beta = 1/k_{\mathrm{B}}T)$$

(b) (1.5 p) Show that the mean-field free energy A_{MF} can be written as (2 p)

$$\frac{\beta A_{\mathrm{MF}}}{N} = -\ln 3 + \frac{2}{3}(1-s)\ln(1-s) + \frac{1}{3}(1+2s)\ln(1+2s) - \frac{zJ\beta}{2}s^2$$

(c) (1 p) This expression predicts that a phase transition occurs at some $\beta = \beta_{\mathrm{c}}$. Estimate $zJ\beta_{\mathrm{c}}$ and draw $A_{\mathrm{MF}}(s)$ as function of s for a few representive values of $zJ\beta$.

4. Consider the zero-field Ising model, with nearest-neighbor interactions, on a square lattice. The partition function can be written as

$$Q(K) = \sum_{\{s_i = \pm 1\}} \exp\left(K \sum_{\langle ij \rangle} s_i s_j\right)$$

(a) (1.5 p) Apply the Migdal-Kadanoff bond moving scheme to this system. Show that the coupling parameter K transforms as

$$K \to K'(K) = \frac{1}{2} \ln \cosh 4K, \tag{1}$$

and that this mapping has a finite, unstable fixed point K^* $(0 < K^* < \infty)$.

(b) (1.5 p) The correlation length $\xi(K)$ (in lattice spacings) behaves as $\xi(K) \sim |K - K_{\mathrm{c}}|^{-\nu}$ near the critical point K_{c}. Use Eq. 1 to estimate the exponent ν.