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Principles of Chiral Perturbation Theory

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Contents

1 Introduction

An elementary discussion of the main concepts used in chiral perturbation theory is given in textbooks [1, 2] and a more detailed picture of the applications may be obtained from the reviews listed in [3, 4, 5]. For an overview of ongoing work, I refer to [6, 7, 8]. Concerning the foundations of the method, however, the literature is comparatively scarce. So, I will concentrate on the basic concepts and explain *why* the method works.

Chiral perturbation theory (χPT) is an *effective field theory*. The main application is QCD, where the method leads to a rather detailed and quantitative understanding of the low energy structure. Despite its name, χPT is a nonperturbative method, because it does not rely on an expansion in powers of the QCD coupling constant. The method invokes a different expansion: it makes use of the fact that, at low energies, the behaviour of scattering amplitudes or current matrix elements can be described in terms of a *Taylor series expansion* in powers of the momenta. The electromagnetic form factor of the pion, e.g., may be exanded in powers of the momentum transfer t. In this case, the first two Taylor coefficients are related to the total charge of the particle and to the mean square radius of the charge distribution, respectively,

$$f_{\pi^+}(t) = 1 + \tfrac{1}{6}\langle r^2\rangle_{\pi^+}\, t + O(t^2) \quad . \tag{1}$$

Scattering lengths and effective ranges are analogous low energy constants occurring in the Taylor series expansion of scattering amplitudes.

For the straightforward expansion in powers of the momenta to hold, it is essential that the theory does not contain massless particles. The exchange of photons, e.g., gives rise to Coulomb scattering, described by an amplitude of the form $e^2/(p' - p)^2$, which does not admit a Taylor series expansion. Now, QCD does not contain massless particles, but it does contain very light ones: pions. The occurrence of light particles gives rise to singularities in the low energy domain, which limit the range of validity of the Taylor series representation. The form factor $f_{\pi^+}(t)$, e.g., contains a branch point singularity at $t = 4M_\pi^2$, such that the formula (1) provides an adequate representation only for $t \ll 4M_\pi^2$. To extend this representation to larger momenta, one needs to account for the singularities generated by the pions. This can be done, because the reason why M_π is so small is understood: the pions are the Goldstone bosons of a hidden, approximate symmetry [9].

The main consequences of this symmetry were derived in the sixties, from a direct analysis of the Ward identities, using current algebra and pion pole dominance. χPT addresses the same problem in a more systematic manner and is considerably more efficient [10, 11, 12, 13]. The corresponding series expansion amounts to a modified Taylor series, which explicitly accounts for the singularities generated by the Goldstone bosons. It provides a solid mathematical basis for what used to be called the "PCAC hypothesis".

2 Goldstone theorem

To start, let me briefly review the notion of a *spontaneously broken symmetry*. Consider any field theory model, for which the Hamiltonian is invariant under some Lie group G. Denote the generators of this group by Q_i, such that

$$[Q_i, \mathbf{H}] = 0 \ .$$

The symmetry is called spontaneously broken if the ground state of the theory is not invariant under G. Suppose, therefore, that, for some of the generators

$$Q_i \left|0\right\rangle \neq 0 \ .$$

This immediately implies that the vacuum is not the only state of zero energy: since $\mathbf{H}$ commutes with Q_i, the vector $Q_i \left|0\right\rangle$ describes a state with the same energy as the vacuum. In a relativistically invariant theory, this can only happen if the spectrum of physical states contains massless particles, Goldstone bosons.

The subset formed by those generators, which do leave the ground state invariant, is a subalgebra: if Q_i and Q_k annihilate the vacuum, then this is also true of the commutator $[Q_i, Q_k]$. These operators therefore generate a subgroup H $\subset$ G. Spontaneous symmetry breakdown thus involves two groups — the symmetry group G of the Hamiltonian and the symmetry group H of the vacuum. Denote the number of parameters required to label the elements of G by n_G such that there are n_G generators and suppose that $n_H < n_G$ is the number of parameters occurring in H. The $n_G - n_H$ generators which belong to the quotient G/H of the two groups do not annihilate the ground state. The corresponding vectors $Q_i \left|0\right\rangle$ are linearly independent, because, otherwise, a suitable linear combination of these generators would leave the vacuum invariant and hence belong to H. Accordingly, spontaneous breakdown of the group G to the subgroup H requires the occurrence of $n_G - n_H$ independent states of zero energy: the spectrum of the theory must contain $n_G - n_H$ different flavours of Goldstone bosons. The Goldstone theorem amounts to a mathematically precise formulation of this statement [14, 15].

3 QCD

In the case of QCD, the relevant spontaneously broken symmetry is an approximate one, related to the occurrence of several quark flavours. I denote the quark field by $q(x)$,

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \\ \vdots \end{pmatrix}$$

The Lagrangian is of the form

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g^2}\mathrm{tr}_c G_{\mu\nu}G^{\mu\nu} + \bar{q}i\gamma^\mu D_\mu q - \bar{q}\,m\,q \quad ,$$

where $G_{\mu\nu}$ is the field strength of the gluon field,

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i[G_\mu, G_\nu] \quad ,$$

D_μ denotes the covariant derivative,

$$D_\mu q(x) = \partial_\mu q(x) - iG_\mu(x)q(x)$$

and m is the quark mass matrix,

$$m = \begin{pmatrix} m_u & & & \\ & m_d & & \\ & & m_s & \\ & & & \ddots \end{pmatrix}$$

Note that the coupling constant g is absorbed in the gluon field: in the notation used here, the covariant derivative involves G_μ rather than gG_μ. In the Lagrangian, the coupling constant then only occurs in front of the term proportional to the square of the field strength. Also, colour indices are suppressed — the symbol tr_c denotes the trace of a colour matrix.

The basic parameters of QCD are the dimensionless bare coupling constant g and the bare quark mass matrix m. Both of these must be tuned to the magnitude of the cutoff μ for the limit $\mu \to \infty$ to make sense: $g = g(\mu)$, $m = m(\mu)$. In order for a change in μ not to modify physical quantities, such as bound state masses, the bare coupling constant must be shifted by an amount determined by the β-function,

$$\mu\frac{dg}{d\mu} = \beta(g) \quad .$$

At small coupling, β is negative, i.e., the theory is asymptotically free. In the minimal subtraction scheme, the β-function is independent of the quark masses. The leading term in the perturbative expansion is of order g^3,

$$\beta(g) = -\beta_0\frac{g^3}{(4\pi)^2} + O(g^5) \quad , \qquad\qquad \beta_0 = 11 - \tfrac{2}{3}\mathrm{N}_f \quad ,$$

where N_f is the number of quark flavours. With this expression for the rhs, the above differential equation is readily integrated. The result for $g(\mu)^2$ is inversely proportional to the logarithm of the cutoff,

$$\frac{g(\mu)^2}{(4\pi)^2} = \frac{1}{\beta_0 \, \ln(\mu^2/\Lambda_{\mathrm{QCD}}{}^2)} \quad ,$$

where Λ_{QCD} is the constant of integration. By definition, this quantity is independent of the cutoff and thus represents a significant parameter, referred to as the renormalization group invariant scale of QCD. The theory is not characterized by the dimensionless coupling constant g occurring in the Lagrangian — the value of this parameter depends on the cutoff — but by the mass scale Λ_{QCD} ("dimensional transmutation").

The tuning of the quark mass matrix is determined by the γ-function,

$$\mu \frac{dm}{d\mu} = -\gamma(g)\, m \quad , \quad \gamma(g) = \frac{g^2}{2\pi^2} + O(g^4)$$

For large values of μ, where it is justified to only retain the leading terms of the perturbative expansion, the solution is of the form

$$m(\mu) = \{\ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}\}^{-\frac{4}{\beta_0}} \; \bar{m} \quad .$$

The constant of integration $\bar{m}$ occurring here is the renormalization group invariant quark mass matrix. When the cutoff is sent to infinity, the bare constants g and m tend to zero, roughly like $g \sim (\ln\mu)^{-1}$, $m \sim (\ln\mu)^{-\frac{1}{2}}$, while the observables of the theory (hadron masses, decay constants, scattering amplitudes etc.) approach finite limits, determined by the renormalization group invariant quantities Λ_{QCD}, $\bar{m}_u$, $\bar{m}_d, \ldots$

The differential equations for the running coupling constant and running quark masses determine the magnitude of these quantities also for values of μ which are not large compared to Λ_{QCD}, but the first one or two terms in the perturbative expansion of the functions β and γ do then not provide an adequate representation. One refers to the functions $g(\mu)$ and $m(\mu)$, defined by these equations, as the running coupling constant and running quark mass, respectively and calls the parameter μ the running scale. The scale may be given any value — the observables are independent thereof. If μ is large, the running coupling constant becomes small, such that the scale dependence is controlled by the perturbation theory formulae given above.

The vector and axial currents are not renormalized. Despite the fact that the dimension of the singlet axial current is anomalous, it does not require wave function renormalization. The scalar and pseudoscalar quark densities, on the other hand, need to be renormalized, in order for their Green functions to approach finite limits when the cutoff is removed. The relevant Z-factor is the inverse of the one occurring in the quark mass matrix — the products $m\,\bar{q}_1 q_2$ and $m\,\bar{q}_1\gamma_5 q_2$ are renormalization group invariant. In the following, I will throughout be working with the running quark masses and densities, without explicitly indicating that these quantitites depend on μ. Equally well, we may use the renormalization group invariant quark masses $\bar{m}$, provided we use the same convention also when normalizing the scalar and pseudoscalar operators.

4 Massless quarks

As far as the strong interactions are concerned, the different quarks u, d, ... have identical properties, except for their mass. From a theoretical point of view, the quark masses represent free parameters of the QCD Lagrangian. The theory makes sense for any value of $m_u, m_d, m_s, \ldots$ We now first consider the fictitious world where all of the quarks are taken massless. This world is a theoretician's paradise: a theory without adjustable dimensionless parameters whatsoever (although the Lagrangian does contain one dimensionless coupling constant g, the value of this constant is without significance, as it merely determines the running scale in units of the renormalization group invariant scale $\Lambda_{\rm QCD}$).

If the quarks are massless, the Lagrangian does not contain any terms which connect the right- and left-handed components of the quark fields,

$$q_{\rm R} = \tfrac{1}{2}(1+\gamma_5)q \;\;,\;\; q_{\rm R} = \tfrac{1}{2}(1-\gamma_5)q \;\;.$$

The Lagrangian of massless QCD, therefore, remains invariant under "chiral" rotations, i.e., under independent transformations of the right- and left-handed quark fields,

$$q_{\rm R} \rightarrow V_{\rm R} q_{\rm R} \;\;,\;\; q_{\rm L} \rightarrow V_{\rm L} q_{\rm L} \qquad\qquad V_{\rm R},\, V_{\rm L} \in {\rm U(N}_f) \;\;.$$

The Noether currents associated with this symmetry of the Lagrangian are given by

$$\begin{array}{rclrcll} V_a^\mu &=& \bar{q}\gamma^\mu \tfrac{1}{2}\lambda_a q \;, & A_a^\mu &=& \bar{q}\gamma^\mu\gamma_5\tfrac{1}{2}\lambda_a q \;, & a = 1,\ldots,{\rm N}_f^2\!-\!1 \\ V_0^\mu &=& \bar{q}\gamma^\mu q \;, & A_0^\mu &=& \bar{q}\gamma^\mu\gamma_5 q \;, & \end{array}$$

where the Gell-Mann matrices λ_1, λ_2, ... form a complete set of traceless, hermitean ${\rm N}_f\!\times\!{\rm N}_f$ matrices.

One of these currents, however, is anomalous: despite the symmetry of the Lagrangian, the singlet axial current A_0^μ fails to be conserved (see section 20),

$$\partial_\mu A_0^\mu = \frac{\mathrm{N}_f}{8\pi^2}\mathrm{tr}_c G_{\mu\nu}\tilde{G}^{\mu\nu} \quad .$$

The actual symmetry group of massless QCD is generated by the charges of the conserved currents V_a^μ, V_0^μ and A_a^μ. It consists of those pairs of elements $V_{\mathrm{R}}, V_{\mathrm{L}} \in \mathrm{U}(\mathrm{N}_f)$ which obey the constraint $\det(V_{\mathrm{R}} V_{\mathrm{L}}^{-1}) = 1$, i.e.,

$$\mathrm{G}_0 = \mathrm{SU}(\mathrm{N}_f)_{\mathrm{R}} \times \mathrm{SU}(\mathrm{N}_f)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{V}} \quad .$$

5 Spontaneous breakdown of chiral symmetry

For QCD to describe the strong interactions observed in nature, it is crucial that chiral symmetry is spontaneously broken, the ground state being invariant only under the subgroup generated by the charges of the vector currents. There are theoretical arguments indicating that chromodynamics indeed leads to the formation of a quark condensate, which is invariant under the subgroup generated by the vector charges, but correlates the right- and lefthanded fields and thus breaks chiral invariance [16]. The available lattice results also support the hypothesis. Taking the generally accepted picture for granted, the vacuum is invariant only under the subgroup

$$\mathrm{H}_0 = \mathrm{SU}(\mathrm{N}_f)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{V}} \quad .$$

The spontaneous symmetry breakdown gives rise to $\mathrm{N}_f^2 - 1$ Goldstone bosons, where N_f is the number of quark flavours. So, the spectrum of QCD with $\mathrm{N}_f > 1$ massless quarks must contain $\mathrm{N}_f^2 - 1$ massless physical states. Their quantum numbers coincide with those of the states obtained by applying the axial charge operators to the vacuum: $J^P = 0^-$.

The factor $\mathrm{U}(1)_{\mathrm{V}}$ which occurs in both G_0 and H_0 is generated by the charge belonging to the singlet vector current V_0^μ. This charge counts the number of quarks minus the number of antiquarks: $3V_0^\mu$ is the current belonging to baryon number.

If the vacuum was symmetric with respect to G_0, only those operators, which are invariant under this group, could pick up a nonzero vacuum expectation value. For a spontaneously broken symmetry, however, this does not hold. One refers to the vacuum expectation values of operators which transform in a nontrivial manner under the symmetry group as *order parameters*.

Since the vacuum is invariant both under Lorentz transformations and under space reflections, only scalar operators can develop nonzero vacuum expectation values. Furthermore, the symmetry of the vacuum insures that only H-invariant operators can give rise to order parameters. In QCD, the scalar operator of lowest dimension which qualifies is $\bar{q}q$. The corresponding order parameter, $\langle 0|\,\bar{q}q\,|0\rangle$, is referred to as the *quark condensate*. The operator $\bar{q}\lambda_a q$ cannot develop a vacuum expectation value, because it is not invariant under H. In the massless theory, the different flavours thus all pick up the same expectation value,

$$\langle 0|\,\bar{u}u\,|0\rangle = \langle 0|\,\bar{d}d\,|0\rangle = \langle 0|\,\bar{s}s\,|0\rangle = \ldots$$

The right-handed quark field $q_{\rm R}$ transforms according to the fundamental representation $\underline{\rm N}_f$ of $\mathrm{SU(N}_f)_{\rm R}$ and is a singlet under $\mathrm{SU(N}_f)_{\rm L}$. The N_f^2 operators $\bar{u}_{\rm R}u_{\rm L}$, $\bar{u}_{\rm R}d_{\rm L}$, ... constitute the irreducible representation $(\underline{\rm N}_f^\star, \underline{\rm N}_f)$ of $\mathrm{SU(N}_f)_{\rm R}\times \mathrm{SU(N}_f)_{\rm L}$, while their hermitean conjugates transform according to $(\underline{\rm N}_f, \underline{\rm N}_f^\star)$. The scalar $\bar{q}q = \bar{q}_{\rm R}q_{\rm L} + \bar{q}_{\rm L}q_{\rm R}$ thus belongs to the direct sum of these two representations.

Since the dimension four operator $\mathrm{tr}_c G_{\mu\nu}G^{\mu\nu}$ is a singlet under G, its vacuum expectation value, the gluon condensate, does not represent an order parameter. At dimension five or six, however, several H-invariant Lorentz scalars may be built, which transform in a nontrivial manner under G: $\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q$, $(\bar{q}q)^2$, $(\bar{q}\gamma_5 q)^2$, $(\bar{q}\lambda_a q)^2$, $(\bar{q}\gamma_\mu\lambda_a q)^2$, $(\bar{q}\sigma_{\mu\nu}q)^2$, ... Not all of these give rise to independent order parameters. The expectation values $\langle 0|\,(\bar{q}q)^2\,|0\rangle$ and $\langle 0|\,(\bar{q}\gamma_5 q)^2\,|0\rangle$, e.g., are different from zero, even if the state $|0\rangle$ is symmetric with respect to G: these operators belong to a multiplet, whose decomposition into irreducible representations contains a singlet. In the difference $(\bar{q}q)^2 - (\bar{q}\gamma_5 q)^2$, however, the singlet drops out; the expectation value $\langle 0|\,(\bar{q}q)^2 - (\bar{q}\gamma_5 q)^2\,|0\rangle$ does represent an order parameter, which is independent of $\langle 0|\,\bar{q}q\,|0\rangle$ (for a G-invariant state, $\langle 0|\,(\bar{q}q)^2\,|0\rangle = \langle 0|\,(\bar{q}\gamma_5 q)^2\,|0\rangle$).

6 Quark masses

The preceding discussion concerns the fictitious world where all of the quark masses are set equal to zero. In reality, the Lagrangian of QCD contains a quark mass term, which breaks chiral symmetry. The divergence of the currents introduced above is determined by the quark mass matrix,

$$\begin{aligned}
\partial_\mu V_a^\mu &= \tfrac{1}{2} i\bar{q}(m\lambda_a - \lambda_a m)q \;, &\quad \partial_\mu V_0^\mu &= 0 \;,\\
\partial_\mu A_a^\mu &= \tfrac{1}{2} i\bar{q}(m\lambda_a + \lambda_a m)\gamma_5 q \;, &\quad \partial_\mu A_0^\mu &= 2i\bar{q}m\gamma_5 q + \frac{\mathrm{N}_f}{8\pi^2}\mathrm{tr}_c G_{\mu\nu}\tilde{G}^{\mu\nu} \;.
\end{aligned}$$

Since the quark masses are different from one another, only the diagonal vector currents are conserved: $\bar{u}\gamma^\mu u$, $\bar{d}\gamma^\mu d, \ldots$ There is one conserved charge for every one of the quark flavours. Baryon number, electric charge, strangeness, charm, etc. are linear combinations thereof. Accordingly, the symmetry group of real QCD is the subgroup generated by the diagonal vector currens, $\mathrm{G}_1 = \mathrm{U}(1)^{\mathrm{N}_f} \subset \mathrm{G}_0$.

It so happens, however, that some of the quark masses are small. One may treat these as perturbations — QCD possesses an *approximate* chiral symmetry. If only the small quark masses are turned off, the Lagrangian acquires a symmetry group G which is is larger than G_1, but smaller than the group of maximal symmetry arising if all of the quark masses are set equal to zero, $\mathrm{G}_1 \subset \mathrm{G} \subset \mathrm{G}_0$. I now discuss the phenomenological evidence for the occurrence of such an approximate symmetry.

A striking property of the observed pattern of bound states is that they come in nearly degenerate *isospin* multiplets: (π^+, π^0, π^-), (K^+, K^0), $(\bar{K}^0, K^-)$, (P, N), $(\Sigma^+, \Sigma^0, \Sigma^-)$, ... In fact, the splittings within these multiplets are so small that, for a long time, isospin was assumed to represent an *exact* symmetry of the strong interactions; the observed small mass difference between neutron and proton or K^0 and K^+ was blamed on the electromagnetic interaction.

We now know that this picture is incorrect: the bulk of isospin breaking does not originate in the electromagnetic fields, which surround the various particles, but is due to the fact that the d-quark is somewhat heavier than the u-quark: isospin only represents an approximate symmetry of the strong interactions. The symmetry arises in the theoretical limiting case, where $m_u = m_d$. In this limit, the flavours u and d become indistinguishable, as far as QCD is concerned, such that the Lagrangian acquires an exact symmetry with respect to

$$\begin{array}{lll} u & \rightarrow & \alpha u + \beta d \\ d & \rightarrow & \gamma u + \delta d \end{array} \qquad V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SU}(2) \ .$$

The group is generated by the charges of the three vector currents

$$V_+^\mu = \bar{u}\gamma^\mu d \ , \ V_0^\mu = \tfrac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d) \ , \ V_-^\mu = \bar{d}\gamma^\mu u \ .$$

The transformation law states that u and d form an isospin doublet, while the remaining flavours $s, c, \ldots$ are singlets. In addition, the U(1) charges associated with $\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d$, $\bar{s}\gamma^\mu s$, ... are also conserved, such that, in the limit $m_u = m_d$, the full symmetry group of the Lagrangian is given by $\mathrm{G} = \mathrm{SU}(2)_\mathrm{V} \times \mathrm{U}(1)^{\mathrm{N}_f - 1}$.

We may exhibit the piece of the QCD Hamiltonian which breaks isospin symmetry by rewriting the mass term of the u and d quarks in the form

$$m_u \bar{u}u + m_d \bar{d}d = \tfrac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) + \tfrac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d) \ .$$

The remainder of the Hamiltonian is invariant under isospin transformations and the same is true of the operator $\bar{u}u + \bar{d}d$. The QCD Hamiltonian thus consists of an isospin invariant part $\overline{\mathbf{H}}_0$ and a symmetry breaking term, proportional to the mass difference $m_u - m_d$,

$$\mathbf{H}_{\rm QCD} = \overline{\mathbf{H}}_0 + \overline{\mathbf{H}}_1 \;\; , \;\; \overline{\mathbf{H}}_1 = \int d^3x \, \tfrac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d) \;\; .$$

The strength of isospin breaking is controlled by the quantity $m_u - m_d$, which plays the role of a *symmetry breaking parameter*. The fact that the multiplets are nearly degenerate implies that the operator $\overline{\mathbf{H}}_1$ only represents a small perturbation: the mass difference $m_u - m_d$ must be very small.

On the basis of the few strange particles, which had been discovered in the course of the 1950's, Gell-Mann and Ne'eman [17] inferred that the strong interactions exhibit a further approximate symmetry, of the same qualitative nature as isospin, but more strongly broken. The symmetry, termed the *eightfold way*, played a decisive role in unravelling the quark degrees of freedom. By now, it has become evident that the mesonic and baryonic levels are indeed grouped in multiplets of SU(3) — singlets, octets, decuplets — and there is also good phenomenological support for the corresponding symmetry relations among the various observable quantities.

In the framework of QCD, eightfold way symmetry occurs in the theoretical limit, where the three lightest quarks are given the same mass, $m_u = m_d = m_s$. The Hamiltonian then becomes invariant under the transformation

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow V \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad\qquad V \in \mathrm{SU}(3)$$

of the quarks fields. Again, the full symmetry group in addition contains several U(1) factors, $\mathrm{G} = \mathrm{SU}(3)_{\rm V} \times \mathrm{U}(1)^{\mathrm{N}_f - 2}$. In the limit $m_u = m_d = m_s$, the spectrum of the theory consists of degenerate multiplets of this group. The degeneracy is lifted by the mass differences $m_s - m_d$ and $m_d - m_u$, which represent the symmetry breaking parameters in this case. Since the eightfold way does represent an approximate symmetry of the strong interactions, both of these mass differences must be small. Moreover, the observed level pattern requires $|\, m_d - m_u \,| \ll |\, m_s - m_d \,|$.

Formally, the above discussion may be extended to include additional flavours. One may even consider the theoretical limit, where all of the N_f quarks are given the same mass.[1] The Hamiltonian then becomes invariant under $\mathrm{U}(\mathrm{N}_f)_{\rm V}$ and the spectrum consists of degenerate multiplets of this group. The extension, however, does not correspond to an approximate symmetry. The lightest pseudoscalar bound state with the quantum numbers of $\bar{d}c$, e.g., sits at $M_{D^+} \simeq 1.87\,\mathrm{GeV}$. If the mass of the charmed quark is set

[1]The massless theory discussed in section 4 represents a special case.

equal to m_u, this state becomes degenerate with the π^+. Clearly, the mass difference $m_c - m_u$, which plays the role of a symmetry breaking parameter in this case, does not represent a small perturbation. We do not know why the quark masses follow the pattern observed in nature, nor do we understand the equally queer pattern of lepton masses. It so happens that the mass differences between u, d and,s are small, such that the eightfold way represents a decent approximate symmetry. The remaining masses turn out to be quite different, so that the mesons and baryons which contain them do not closely resemble the corresponding light quark bound states. A more promising line of approach is *heavy quark symmetry*, which analyzes the properties of their bound states by treating c, b, t as infinitely heavy [18].

7 Approximate chiral symmetry

The approximate symmetries discussed in the preceding section explain why the bound states of QCD exhibit a multiplet pattern, but they do not account for an observation which is equally striking and which plays a crucial role in strong interaction physics: the mass gap of the theory, M_π, is remarkably small. The approximate symmetry which hides behind this observation was discovered by Nambu [9]. It originates in the fact that m_u and m_d happen to be small.

Consider first the limit, where a single one of the quarks is taken massless, $m_u = 0$, while the remaining masses are kept fixed at their physical values. The QCD Lagrangian then becomes invariant under the chiral transformation $u \rightarrow \exp(i\beta\gamma_5)u$. As mentioned above, this symmetry of the Lagrangian, however, is ruined by the U(1) anomaly — the divergence of the Noether current $\bar{u}\gamma^\mu\gamma_5 u$ is different from zero. So, the limit $m_u \rightarrow 0$ does not give rise to a higher degree of symmetry: the symmetry group is the same as for $m_u \neq 0$.

The theory does acquire more symmetry if two of the quark masses are taken equal, as it then becomes invariant under isospin rotations. If m_u and m_d are not only taken the same, but are put equal to zero, the symmetry group is increased further. The Lagrangian then becomes invariant with respect to a set of *chiral* transformations: independent isospin rotations of the right- and lefthanded components of u and d,

$$\binom{u_{\rm R}}{d_{\rm R}} \rightarrow V_{\rm R}\binom{u_{\rm R}}{d_{\rm R}} \;\; , \;\; \binom{u_{\rm L}}{d_{\rm L}} \rightarrow V_{\rm L}\binom{u_{\rm L}}{d_{\rm L}} \qquad V_{\rm R},\, V_{\rm L} \in {\rm SU}(2) \;\; .$$

In contrast to the chiral U(1) transformation considered above, this symmetry of the classical Lagrangian does survive quantization: in the limit $m_u = m_d = 0$, the theory becomes invariant with respect to the group $\mathrm{G} = \mathrm{SU}(2)_{\rm R} \times \mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)^{\mathrm{N}_f - 1}$. As discussed in section 4, chiral symmetry is broken spontaneously, the ground state being invariant only under the subgroup generated by the charges of the vector currents. For the limit under consideration, where only two of the quark flavours are taken massless, the ground state is symmetric under $\mathrm{H} = \mathrm{SU}(2)_{\mathrm{V}} \times \mathrm{U}(1)^{\mathrm{N}_f - 1}$, such that $\mathrm{G}/\mathrm{H} = \mathrm{SU}(2)$. Accordingly, the spectrum contains three Goldstone bosons, with the quantum numbers of π^+, π^0, π^-.

In reality, chiral symmetry is broken, not only spontaneously, but also explicitly, by the quark masses. As above, we may split the Hamiltonian into a piece which is invariant under the symmetry group of interest and a piece which breaks the symmetry. In the present case, the symmetry breaking part is the mass term of the u and d quarks,

$$\mathbf{H}_{\mathrm{QCD}} = \mathbf{H}'_0 + \mathbf{H}'_{sb} \quad , \quad \mathbf{H}'_{sb} = \int d^3x (m_u \bar{u}u + m_d \bar{d}d) \quad .$$

If the symmetry breaking parameters m_u and m_d are small, the spectrum of $\mathbf{H}_{\mathrm{QCD}}$ must be close to the spectrum of $\mathbf{H}'_0$. In particular, it must contain three one-particle states, whose mass tends to zero when the symmetry breaking is turned off. In fact, we will see that the pion mass tends to zero in proportion to the square root of $m_u + m_d$,

$$M_\pi^2 \propto (m_u + m_d) \quad .$$

The observed spectrum is remarkably close to the one which would result if chiral symmetry was an exact symmetry of the strong interactions: the pions are by far the lightest hadrons. So, the fact that the pions are light can be understood: they are the Goldstone bosons of a spontaneously broken approximate symmetry.

Now, we already noted that the difference between m_s and m_u or m_d must be small, because the eightfold way represents an approximate symmetry of the strong interactions. Since the smallness of the pion mass requires m_u and m_d to be small, we conclude that all three quarks must be light — the mass terms of u, d and s only represent a perturbation. Moreover, the inequality $|\, m_d - m_u \,| \ll |\, m_s - m_d \,|$, which follows from the fact that isospin breaking is much smaller than the breaking of eightfoldway symmetry, implies $m_u,\, m_d \ll m_s$.

The Hamiltonian may be decomposed according to

$$\mathbf{H}_{\mathrm{QCD}} = \mathbf{H}_0 + \mathbf{H}_{sb} \quad , \quad \mathbf{H}_{sb} = \int d^3x (m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s) \quad .$$

The first term, $\mathbf{H}_0$, describes three massless flavours (u, d, s) as well as three massive ones (c, b, t) and is symmetric with respect to the group $\mathrm{G} = \mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{U}(1)^{\mathrm{N}_f - 2}$. The second term, $\mathbf{H}_{sb}$, breaks this symmetry to the subgroup $\mathrm{H} = \mathrm{U}(1)^{\mathrm{N}_f}$. Since the symmetry breaking parameters m_u, m_d and m_s are small, the properties of the theory can be analyzed by treating $\mathbf{H}_{sb}$ as a perturbation. The corresponding perturbation series amounts to an expansion in powers of m_u, m_d and m_s. The point here is that the two groups $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(3)_{\mathrm{V}}$ can be approximate symmetries of the Hamiltonian only if the group $\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$ represents an approximate symmetry, too.

There is an immediate experimental check: the eight lightest bound states, π^+, π^0, π^-, K^+, K^0, $\bar{K}^0$, K^-, η, indeed carry precisely the quantum numbers of the Goldstone bosons generated by the spontaneous breakdown $\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}} \rightarrow \mathrm{SU}(3)_{\mathrm{V}}$. They are not massless, because the quark masses break the symmetry, but the breaking is small enough for these levels to remain lowest.

The above arguments rely on two phenomenological observations:

(a) The pion mass is small compared to the masses of all other hadrons. This indicates that the strong interactions possess an approximate, spontaneously broken symmetry, with the pions as the corresponding Goldstone bosons. Indeed, the Lagrangian of QCD exhibits an approximate symmetry with the proper quantum numbers, provided both m_u and m_d are small.

(b) The multiplet structure seen in the particle data tables indicates that SU(3) is an approximate symmetry of the strong interactions. For QCD to exhibit such a symmetry, the mass differences $m_d - m_u$ and $m_s - m_d$ must be small. Together with (a), this implies that all three quarks u, d and s are light, such that the Lagrangian of QCD exhibits an approximate chiral symmetry, with $\mathrm{G} = \mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$.

The masses of the other quarks occurring in the Standard Model, on the other hand, cannot be treated as a perturbation. Since the corresponding quark fields $c(x)$, $b(x)$ and $t(x)$ are invariant under G, their contribution to the Lagrangian may be included in the $\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$ invariant part of the Hamiltonian, $\mathbf{H}_0$, and does not significantly affect the following discussion. Conversely, that analysis will not shed any light on the properties of bound states involving heavy quarks.

8 Pion pole dominance

The exchange of Goldstone bosons gives rise to singularities in the low energy region, in particular, to poles, connected with one-particle reducible contributions. The analysis of these singularities involves an assumption, referred to as the PCAC or *pion pole dominance hypothesis* : one postulates that, at sufficiently small momenta, the one-particle-singularities dominate over the remainder.

To discuss the content of this hypothesis, I return to the case of a spontaneously broken *exact* symmetry. For definiteness, I consider QCD with two massless flavours, $m_u = m_d = 0$. The spectrum then contains three massless pseudoscalars, $M_{\pi^+} = M_{\pi^0} = M_{\pi^-} = 0$. In the Green functions of the theory, massless one-particle states manifest themselves as poles at $p^2 = 0$. The two-point function of the axial current, e.g., contains such a pole, arising from the exchange of a pion between the two currents. Current conservation and isospin invariance imply that this particular Green function is of the form

$$\int d^4x e^{ipx} \langle 0|\, T A^\mu_a(x) A^\nu_b(0)\, |0\rangle = i\delta_{ab}(p^\mu p^\nu - g^{\mu\nu} p^2)\Pi(p^2) \ .$$

The pole term arises from those matrix elements of the current which connect the vacuum to the one-pion states. On account of Lorentz invariance and isospin symmetry, these are of the form

$$\langle 0|\, A^\mu_a |\pi^b(p)\rangle = i p^\mu \, \delta^b_a \, F \ . \tag{2}$$

The phase of the states $|\pi^a(p)\rangle$ may be chosen such that the constant F is real and positive. The corresponding contribution to the two-point function is proportional to F^2,

$$\Pi(p^2) = \frac{F^2}{-p^2 - i\epsilon} + \overline{\Pi}\,(p^2) \ .$$

In this example, the pion pole dominance hypothesis boils down to the assumption that, at low momenta, the pole term due to one-pion exchange dominates over the remainder, which contains branch points due to multipion exchange, as well as singularities associated with the exchange of massive particles. Accordingly, the low energy behaviour of the two-point function is determined by the constant F; the remainder, $\overline{\Pi}\,(p^2)$, only contributes if the low energy expansion is carried beyond leading order.

The constant F plays a central role in the low energy analysis. It specifies the vacuum-to-pion matrix element of the axial current and represents the overlap of the states $|\pi^a(p)\rangle$ with those obtained by applying the axial charges to the vacuum.[2] The Goldstone theorem asserts that F is different from zero [14].

Since the matrix element $\langle 0|\, A^\mu_a |\pi^b(p)\rangle$ determines the rate of the weak decay $\pi \to \mu\nu$, the quantity F is referred to as the pion decay constant. The measured decay rate implies $F \simeq 93\,\mathrm{MeV}$. Note that the magnitude

[2]Note that the scalar product of the two states is meaningful only in the presence of an infrared cutoff; one may, e.g., replace the charge by $\int d^3x f(x) A^0_a(x)$, where the test function $f(x)$ is equal to one on some finite region of space, but vanishes at large distances.

of the matrix element (2) changes if the quark masses are varied. As we are presently considering the theoretical limit $m_u = m_d = 0$, we cannot use the experimental information as such, but have to distinguish between the physical value of the pion decay constant and the value which results if the quarks are taken massless. The difference is characteristic of the chiral symmetry breaking effects to be discussed later on.

The pole dominance hypothesis may be extended to any other matrix element, which receives one-particle reducible contributions from pion exchange. These are described by a product of pole factors (one for each of the exchanged pions) and a residue, representing a product of one-particle-*irreducible* matrix elements. The pion pole dominance hypothesis is the assumption that, at small momenta, the pole terms dominate over the remainder. The four-point function of the current, e.g., contains a contribution involving the emission or absorption of a pion by each one of the four currents. The residue is the product of a one-particle irreducible amplitude, describing the interaction among the four pions, and four matrix elements of the type $\langle 0| A^\mu |\pi\rangle$, representing the interaction of the pions with the current. Contributions from the exchange of less than four pions also occur. According to pion pole dominance, these pole terms dominate the four-point function at low momenta.

In the case of the function $\Pi(p^2)$, the residue of the pole is a constant, for kinematical reasons. In general, however, the residue still represents a function of the momenta, which contains singularites and does not admit a straightforward Taylor series expansion. In the four-point function, e.g., the residue is proportional to the elastic scattering amplitude, which contains branch point singularities related to rescattering processes with two or more pions as intermediate states. Both in the "Current algebra and PCAC" approach and in the effective Lagrangian method, one assumes that singularites associated with multipion exchange only occur at subleading orders of the low energy expansion; retaining only the leading term of the expansion, the residues reduce to polynomials of the momenta. The coefficients occurring therein play a role analogous to the leading Taylor coefficients of the low energy expansion for theories with an energy gap.

9 Strength of the effective interaction

As an immediate application of the pion pole dominance hypothesis, I now show that, at low energies, the hidden symmetry prevents the Goldstone bosons from interacting with one another. This property is essential for the consistency of χPT.

For simplicity, I again consider two massless flavours, such that the components $a = 1, 2, 3$ of the axial current $A^\mu_a(x)$ are strictly conserved. The argument relies on the properties of the the probability amplitude for the

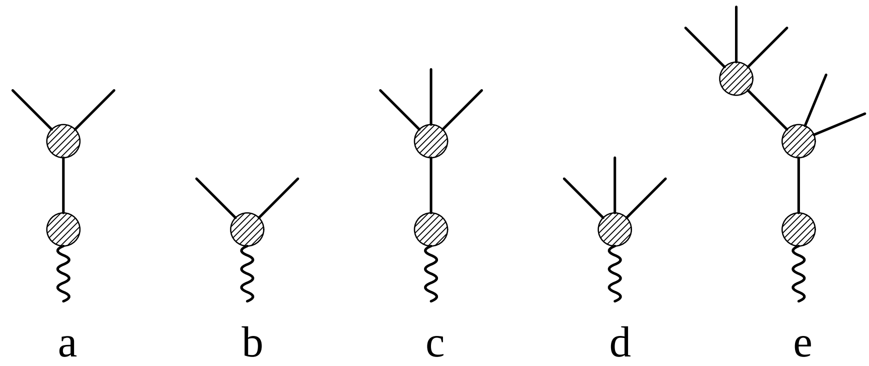

Figure 1: Pion production by an external current. The internal lines represent pion poles, the external ones stand for on-shell pions. The current is denoted by a wiggly line and the shaded circles indicate one-particle irreducible amplitudes.

currents to create pions out of the vacuum. Current conservation requires this amplitude to obey the condition

$$p_\mu \, \langle \pi^{a_1}(p_1), \pi^{a_2}(p_2), \ldots \text{ out}| A^\mu_a \, |0\rangle = 0 \quad , \tag{3}$$

where $p^\mu = p^\mu_1 + p^\mu_2 + \ldots$ is the four-momentum of the final state.

The probability amplitude for the occurrence of a single pion is linear in the momentum,

$$\langle 0| \, A^\mu_b |\pi^a(p)\rangle \;=\; \langle \pi^a(p)| A^\mu_b \, |0\rangle^\star \;=\; i \, p_\mu \, \delta^a_b \, F \tag{4}$$

and obeys the conservation law, provided the pions are massless, $p^2 = 0$. The amplitudes for the creation of several particles, however, may contain singularities due to pion exchange. The graph shown in figure 1a, e.g., represents a one-particle reducible contribution to the production amplitude for two pions, in fact the only such contribution which can occur in an amplitude with three legs. Describing the three-pion-vertex by the function $g_{a_1 a_2 a_3}(p_1, p_2, p_3)$, this graph yields the term

$$\langle \pi^{a_1}(p_1), \pi^{a_2}(p_2) \text{ out}| A^\mu_{a_3} \, |0\rangle = i \frac{p^\mu_3 \, F}{p^2_3 + i\epsilon} \; g_{a_1 a_2 a_3}(p_1, p_2, p_3) + \ldots$$

with $p_1 + p_2 + p_3 = 0$. According to the preceding section, the leading term of the low energy expansion of the residue is a polynomial in the momenta.

Suppose the expansion of the vertex starts with a constant term, such that, at leading order of the expansion, the vertex is replaced by a set of coupling constants $g_{a_1a_2a_3}(0,0,0) \neq 0$. The contribution of graph 1a is of then of order p^{-1}. Since this graph represents the only one-particle reducible contribution, the remainder (fig.1b) is free of poles; at leading order of the low energy expansion, the remainder at most amounts to a constant term and cannot compete with graph 1a. The pion pole dominance hypothesis thus implies that the leading contribution to the conservation law (3) is given by

$$p_\mu \, \langle \pi^{a_1}(p_1), \pi^{a_2}(p_2) \text{ out}|A^\mu_{a_3}\,|0\rangle = iF \; g_{a_1a_2a_3}(0,0,0) + O(p) \quad ,$$

where the symbol $O(p)$ indicates that the terms omitted involve at least one power of momentum and hence vanish if all momenta are sent to zero. We thus arrive at the low energy theorem

$$g_{a_1a_2a_3}(0,0,0) = 0 \quad .$$

Actually, the above argument kills a dead fly: a triple pion vertex does not occur in any case, because it would violate parity as well as G-parity. The reason for considering this vertex, nevertheless, is that the argument given immediately generalizes to vertices involving any number of pions. Consider, e.g., the amplitude for the production of a final state with three pions. In the absence of a three-pion vertex, this amplitude again contains a single one-particle reducible contribution, shown in figure 1c. The contribution is of the form

$$\langle \pi^{a_1}(p_1), \pi^{a_2}(p_2)\pi^{a_3}(p_3) \text{ out }|\,A^\mu_{a_4}\,|0\rangle = i\frac{p_4^\mu\,F}{p_4^2+i\epsilon}\; g_{a_1a_2a_3a_4}(p_1,p_2,p_3,p_4) + \ldots$$

At low momenta, this term again dominates over the remainder (graph 1d), such that current conservation implies

$$g_{a_1a_2a_3a_4}(0,0,0,0) = 0 \quad .$$

In other words, the hidden symmetry prevents pions of zero momentum from scattering elastically.

The production amplitudes for five or more pions contain multiple poles. The one-particle reducible graph shown in figure 1e, e.g., involves a double pole from the two internal lines. Since each of the four-pion-vertices occurring there, however, is suppressed by two powers of momentum, the contribution generated by this graph is only of $O(p)$ and does, therefore, not show up when evaluating the conservation law (3) in the zero momentum limit. Again, current conservation can only be satisfied if the six-pion-vertex (graph 1f) vanishes at zero momentum. By induction, the argument extends to vertices involving any number of pions: current conservation implies that all of these vertices are of order p^2 and disappear if the momenta of the pions tend to zero. Accordingly, the low energy expansion of the production amplitudes $\langle\pi\pi\ldots\ \mathrm{out}|A^\mu\,|0\rangle$ only starts at $O(p)$, irrespective of the number of pions produced. The scattering amplitudes are described by the same graphs, except that some of the pions must be crossed from the final to the initial state and the internal line linking these to the current is to be replaced by an external line. This shows that, independently of the number of pions occurring in the initial and final states, the scattering amplitudes are at most of $O(p^2)$.

At low energies, the interaction among the Goldstone bosons thus becomes weak — pions of zero energy do not interact at all. This is in marked contrast to the interaction among the quarks and gluons, which is strong at low energies, because QCD is an asymptotically free, infrared enslaved theory. The qualitative difference is crucial for chiral perturbation theory to be coherent: in this framework, the interaction among the Goldstone bosons is treated as a perturbation. The opposite behaviour in the underlying theory prevents a perturbative low energy analysis of the interaction among the quarks and gluons.

10 Effective Lagrangian

The effective Lagrangian method is based on the following idea. The graphs shown in figure 1 may be viewed as tree graphs of a field theory, which involves *pion fields* as basic variables. Since the Goldstone bosons do not carry spin, they are described by scalar fields, which I denote by $\pi^a(x)$. The fields are in one-to-one correspondence with the massless one-particle-states $|\,\pi^a(p)\rangle$ occurring in the spectrum of the theory.

In this language, the pole terms generated by pion exchange arise from the propagation of the pion field, described by the correlation function

$$\langle 0|\,T\,\pi^a(x)\pi^b(y)\,|0\rangle = \frac{1}{i}\delta^{ab}\Delta_0(x-y)$$
$$\Delta_0(z) = \int\frac{d^4p}{(2\pi)^4}\,\frac{e^{-ipz}}{-p^2-i\epsilon} = \frac{1}{4\pi^2 i(z^2-i\epsilon)}\quad .$$

The Feynman propagator $\Delta_0(x-y)$ represents the transition amplitude for a pion emitted at the point x to reach the point y, or vice versa: the propagator is an even function, $\Delta_0(x-y)=\Delta_0(y-x)$. Since the Fourier transform thereof is given by $1/(-p^2-i\epsilon)$, the propagation of the field between the various vertices indeed yields the relevant pole terms occurring in one-particle reducible graphs.

The propagator is determined by the kinetic part of the Lagrangian and vice versa. For the effective field theory to yield the massless scalar propagator, the kinetic term must be identified with the standard expression describing scalar free fields,

$$\mathcal{L}_{kin} = \tfrac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a \quad .$$

The vertices, on the other hand, represent interactions of the field. Some of the vertices of figure 1 describe emission or absorption of pions by the currents, others exclusively join pion lines. In the language of the effective field theory, the purely pionic vertices correspond to terms in the interaction Lagrangian,

$$\mathcal{L}_{int} = \mathcal{L}_{int}(\pi, \partial\pi, \partial^2\pi, \ldots) \ \ .$$

A momentum independent vertex joining four pion lines, e.g., corresponds to an interaction term of the form $g_{abcd}\, \pi^a \pi^b \pi^c \pi^d$, while a term of the type $g'_{abcd}\, \partial_\mu \pi^a \partial^\mu \pi^b \pi^c \pi^d$ generates a vertex with two powers of momentum. The translation of the various vertices into corresponding terms of the interaction Lagrangian is trivial: if the vertex in question joins P pion lines and involves a polynomial in the momenta of degree D, the corresponding term in $\mathcal{L}_{int}$ contains P pion fields and D derivatives. Since an interaction involves at least three pions, $P \geq 3$. Moreover, Lorentz invariance implies that D is even, such that the derivative expansion of the interaction Lagrangian starts with

$$\mathcal{L}_{int} = g^0(\pi) + g^1_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b + g^2_a(\pi)\Box\pi^a + O(p^4) \ \ , \tag{5}$$

where the omitted terms involve four or more derivatives. The Taylor series of the function $g^0(\pi)$ in powers of π yields all vertices which are momentum independent:

$$g^0(\pi) = \tfrac{1}{3!}\, g^0_{abc}\pi^a\pi^b\pi^c + \tfrac{1}{4!}\, g^0_{abcd}\pi^a\pi^b\pi^c\pi^d + \ldots$$

Similarly, the expansion of the functions $g^1_{ab}(\pi)$ and $g^2_a(\pi)$ in powers of π generates all those vertices, which contain two powers of momentum, etc. The Lagrangian of the effective field theory merely collects the information about the various vertices — no more, no less.

The virtue of the representation in terms of effective fields is that the tree graphs of a local field theory automatically obey the *cluster decomposition property* : whenever a given number of pions meet, the same vertex occurs, irrespective of the remainder of the diagram. The presence of an interaction among four pions, e.g., also manifests itself in the process $\pi\pi \to \pi\pi\pi\pi$, through a tree graph contribution containing two four-pion-vertices and one internal line, which represents the exchange of a pion between the two vertices (compare figure 1e). The tree graphs generated by the various interaction terms contained in the effective Lagrangian automatically include these contributions. Note that, at this stage, only the tree graphs of the effective field theory are relevant.

As shown in section 9, current conservation not only implies that the Goldstone bosons are massless, but requires all of the vertices to vanish at zero momentum. Hence, the effective Lagrangian does not contain any interaction terms without derivatives, i.e.,

$$g^0(\pi) \equiv 0 \quad . \tag{6}$$

In the language of effective field theory, the Goldstone theorem states that the Lagrangian does not contain a mass term: a contribution $\propto \pi^2$ does not occur. The relation (6) may be viewed as a generalization of the theorem — it states that terms without derivatives are absent altogether.

The derivative expansion of the effective Lagrangian only starts at $O(p^2)$ and contains terms with $2, 4, 6, \ldots$ derivatives,

$$\mathcal{L}_{\mathit{eff}} = \mathcal{L}^2_{\mathit{eff}} + \mathcal{L}^4_{\mathit{eff}} + \mathcal{L}^6_{\mathit{eff}} + \ldots$$

As indicated in (5), Lorentz invariance permits two different interaction terms of second order in the momenta. The second one may, however, be rewritten as $\partial_\mu\{g^2_a(\pi)\partial^\mu\pi^a\} - \partial_b g^2_a(\pi)\partial_\mu\pi^a\partial^\mu\pi^b$. Conservation of energy and momentum at each one of the vertices implies that total derivatives do not contribute and the remainder may be absorbed in $g^1_{ab}(\pi)$. Without loss of generality, we may therefore set $g^2_a(\pi) = 0$. The leading contribution in the derivative expansion of the effective Lagrangian then takes the form

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{2} g_{ab}(\pi)\partial_\mu\pi^a\partial^\mu\pi^b \quad ,$$

where I have amalgamated the kinetic term with the interaction, setting $g_{ab}(\pi) \equiv \delta_{ab} + 2g^1_{ab}(\pi)$. This shows that, at leading order of the low energy expansion, the properties of the interaction are characterized by the function $g_{ab}(\pi)$. The expansion of this function in powers of π,

$$g_{ab}(\pi) = \delta_{ab} + \partial_c g_{ab}(0)\pi^c + \tfrac{1}{2}\partial_{cd} g_{ab}(0)\pi^c\pi^d + \ldots$$

generates all of the vertices of order p^2. The first term represents the kinetic energy, the second generates the vertices with three pion legs, the third specifies the interactions among four pions etc.

11 Symmetries of the effective theory

The vertex shown in fig.1a links the pion field to the axial current. In the famework of the effective description, this vertex corresponds to a term linear in the field,

$$A^\mu_a = -F\partial^\mu\pi^a + \dots \tag{7}$$

while a term of the form $\partial_\mu\pi\pi\pi$, e.g., corresponds to a vertex, where the axial current emits three pion lines. The full current $A^\mu_a = A^\mu_a(\pi, \partial\pi, \dots)$ consists of an infinite string of terms (G-parity allows an arbitrary odd number of pion fields and Lorentz invariance implies that the number of derivatives is odd). The representation of the vector current in terms of effective fields consists of an analogous string (except that G-parity now only permits terms built with an even number of fields).

The most important property of the currents is that they are conserved, $\partial_\mu V^\mu_a = \partial_\mu A^\mu_a = 0$. The conservation law expresses the fact that QCD with two massless flavours is invariant under G = SU(2) $\times$ SU(2). The effective representation of the currents must obey the same conservation law, i.e., the effective theory must inherit the symmetries of the underlying one. Indeed, if the effective Lagrangian is invariant under G, the Noether theorem automatically provides a representation for the vector and axial currents in terms of the pion field and insures that these currents are conserved. Note, however, that we are using the Noether theorem in the wrong direction here: an invariant Lagrangian leads to conserved currents, but the converse is not true. What counts is the *action*; under the transformations generated by the symmetry group, the Lagrangian may pick up a total derivative, such that the action is not affected. This is precisely what happens in the presence of anomalies. A similar phenomenon also arises in the nonrelativistic domain: the effective Lagrangian describing the magnons of a ferromagnet is invariant under rotations of the spin directions only up to a total derivative [19]. These examples demonstrate that the effective Lagrangian is not necessarily invariant under the global symmetry group G. In fact, global symmetry does not fully determine the structure of the effective Lagrangian.

A critical reader may also have doubts about the validity of the Noether theorem in the present context, because its derivation makes use of the equation of motion for the field. In the Feynman path integral representation of the effective field theory, the pion field freely fluctuates — it is the variable of integration and does not obey an equation of motion. In the tree approximation, the theorem does hold, because the tree graphs describe the classical limit. Beyond the tree approximation, however, the equation of motion is modified by the quantum fluctuations of the field and a proper formulation of the Noether theorem then becomes are rather subtle affair.

The symmetry properties of relativistic effective Lagrangians are analyzed in detail in [13]. While the above discussion only concerns *global* (i.e. space-time independent) symmetry operations, that analysis relies on the Ward identities, which express the symmetry properties of the underlying theory in *local* form (see section 19). The main result established there is an *invariance theorem*, valid for theories with a Lorentz invariant ground state. The theorem states that

(i) If the Ward identities do not contain anomalous contributions, the effective Lagrangian is invariant under G.

(ii) In the presence of anomalies, the effective Lagrangian contains a Wess-Zumino term at order p^4, whose form is known explicitly; the remainder is invariant under G.

For the proof, I refer to [13]. The invariance theorem puts the effective Lagrangian method on firm footing. It demonstrates that this method is strictly equivalent to the current algebra plus PCAC approach. In either case, the essential ingredient is the pion pole dominance hypothesis formulated in section 8. As discussed there, this hypothesis fixes the leading terms in the low energy expansion of the various Green functions up to the Taylor coefficients. The Ward identities imply that the Taylor coefficients occurring in different Green functions are related to one another. In the current algebra plus PCAC analysis, one explicitly works out these relations, by investigating individual Green functions. The simplicity of the effective Lagrangian method derives from the fact that the Ward identities are equivalent to the simple statement that $\mathcal{L}_{\mathit{eff}}$ possesses the same local symmetry properties as the Lagrangian of the underlying theory. This property allows one to explicitly solve the constraints among the Taylor coefficients at a given order of the low energy expansion, for all of the Green functions at once.

In the preceding section, we focussed on the leading term in the derivative expansion of the effective Lagrangian, $\mathcal{L}^2_{\mathit{eff}}$. The invariance theorem asserts that this term is invariant under G, irrespective of anomalies — the Wess-Zumino term represents a specific contribution to $\mathcal{L}^4_{\mathit{eff}}$ and thus only matters if the low energy expansion is investigated beyond leading order. Let us now work out the consequences for the form of $\mathcal{L}^2_{\mathit{eff}}$.

12 Transformation law of the pion field

In the underlying theory, the group acts on the quark fields, according to

$$q_{\mathrm{R}} \xrightarrow{g} V_{\mathrm{R}}\, q_{\mathrm{R}} \qquad q_{\mathrm{L}} \xrightarrow{g} V_{\mathrm{L}}\, q_{\mathrm{L}} \quad . \tag{8}$$

In the framework of the effective field theory, G instead acts on the pion fields, through a representation of the form

$$\pi^a \xrightarrow{g} \varphi^a(g,\pi) \quad .$$

The explicit expression for the corresponding Noether currents is determined by the form of the function $\varphi^a(g,\pi)$. Remarkably, the representation property

$$\varphi(g_1,\varphi(g_2,\pi)) = \varphi(g_1 g_2,\pi) \tag{9}$$

fixes the mapping essentially uniquely [20].

To verify this claim, let us first consider the image of the origin, $\varphi(g,0)$. The composition law shows that the set of elements h which map the origin onto itself forms a subgroup $\mathrm{H} \subset \mathrm{G}$. Moreover, $\varphi(gh,0)$ coincides with $\varphi(g,0)$ for any $g \in \mathrm{G}$, $h \in \mathrm{H}$. Hence the function $\varphi(g,0)$ lives on the space G/H, obtained from G by identifying elements g, g' differing only by right multiplication with a member of H, $g' = gh$. The function $\varphi(g,0)$ thus maps the elements of G/H into the space of pion field variables. The mapping is invertible, because $\varphi(g_1,0) = \varphi(g_2,0)$ implies $g_1^{-1}g_2 \in \mathrm{H}$. Geometically, the pion field variables π^a may therefore be viewed as the coordinates of the quotient space G/H: the Goldstone bosons live in this space.

Next, choose a representative element n in each one of the equivalence classes $\{gh, h \in \mathrm{H}\}$, such that every group element may uniquely be decomposed as $g = nh$. The composition law (9) then shows that the image n' of the element n under the action of $g \in \mathrm{G}$ is obtained by decomposing the product gn into $n'h$ — the standard action of G on the space G/H. This implies that the geometry fully fixes the transformation law of the pion field, except for the freedom in the choice of coordinates on the manifold G/H.

In the case of $\mathrm{G} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{H} = \mathrm{SU}(2)$, the quotient G/H is the group SU(2). The pion field may be represented as an element of this group, i.e., as a 2×2 matrix field $U(x) \in \mathrm{SU}(2)$. Alternatively, we may identify the pion field with the the three coordinates π^1, π^2, π^3, needed to parametrize the group SU(2). The choice of coordinates is not unique. Using canonical coordinates, the relation between the matrix field $U(x)$ and the scalar fields $\pi^a(x)$ takes the form

$$U(x) = \exp i\pi(x) \;\; , \qquad \pi(x) = \sum_{a=1}^{3} \pi^a(x)\tau_a$$

where τ_1, τ_2, τ_3 are the Pauli matrices. The transformation law of these fields may be worked out as follows.

As noted above, the action of $g \in \mathrm{G}$ on the element $n \in \mathrm{G/H}$ is given by $gn = n'h$. In the case under consideration, G consists of pairs of elements $g = (V_{\mathrm{R}}, V_{\mathrm{L}})$, while H contains the equal pairs, $V_{\mathrm{R}} = V_{\mathrm{L}}$. As representative elements of the equivalence classes, we may choose $n = (U, \mathbf{1})$. The transformation law then amounts to

$$gn = (V_{\mathrm{R}}, V_{\mathrm{L}})(U, \mathbf{1}) = (V_{\mathrm{R}}U, V_{\mathrm{L}}) = (V_{\mathrm{R}}UV_{\mathrm{L}}^\dagger, \mathbf{1})(V_{\mathrm{L}}, V_{\mathrm{L}}) = n'h \;\; .$$

Hence the transformation law of the pion field reads

$$U'(x) = V_{\mathrm{R}}U(x)V_{\mathrm{L}}^\dagger \;\; .$$

The matrix $U(x)$ thus transforms linearly. Note, however, that the corresponding transformation law for the pion field $\pi^a(x)$ is nonlinear — the matrix $i\pi^a\tau_a$ is the logarithm of the matrix $U(x)$. As indicated by the above general discussion, the occurrence of nonlinear realizations of the symmetry group is a characteristic feature of the effective Lagrangian technique.

13 Form of the effective Lagrangian

Expressed in terms of the field $U(x)$, the effective Lagrangian is of the form $\mathcal{L}_{eff}(U, \partial U, \partial^2 U, \ldots)$. Lorentz invariance implies that the leading terms in the expansion of this function in powers of derivatives are of the form

$$\mathcal{L}_{eff} = g_0(U) + g_1(U) \times \Box U + g_2(U) \times \partial_\mu U \times \partial^\mu U + O(p^4) \ .$$

The crosses indicate that the coefficients $g_n(U)$ carry indices, which are to be contracted against those of $\Box U$ and $\partial_\mu U$. The remainder contains four or more derivatives of the pion field.

The first term does not contain derivatives. The corresponding action is invariant under $U \rightarrow V_{\mathrm{R}} U V_{\mathrm{L}}^\dagger$ if and only if $g_0(U)$ is independent of U. Hence the first term is an irrelevant cosmological constant and may be dropped. In fact, we arrived at the conclusion that the effective Lagrangian does not contain interaction terms without derivatives, already in section 10; the above rederivation of this result merely illustrates the efficiency of the effective Lagrangian technique.

Integrating by parts, the second term can be transformed into the third one, so that we may drop $g_1(U)$, too. Without loss of generality we can then write the Lagrangian in the form $g_2(U) \times \Delta_\mu \times \Delta^\mu$ where Δ_μ stands for $-iU^{-1}\partial_\mu U$. The advantage of the manipulation is that Δ_μ is invariant under $U \rightarrow V_{\mathrm{R}} U$, such that only $g_2(U)$ is affected by this operation. The requirement that the action must remain invariant, therefore, implies that $g_2(U)$ is independent of U. Finally, under the transformation $U \rightarrow U V_{\mathrm{L}}^\dagger$, the traceless quantity Δ_μ transforms according to the representation $D^{(1)}$. Since the product $D^{(1)} \times D^{(1)}$ contains the identity only once, there is a single invariant of order p^2,

$$\mathcal{L}^2_{eff} = g \, \mathrm{tr} \Delta_\mu \Delta^\mu = g \, \mathrm{tr} \, (\partial_\mu U \partial^\mu U^\dagger) \ . \tag{10}$$

This shows that the leading term in the derivative expansion of the effective Lagrangian contains only one free coupling constant, g.

Expanding the matrix $U(x) = \exp i\pi(x)$ in powers of the pion field, we obtain

$$\mathcal{L}^2_{eff} = 2g \partial_\mu \pi^a \partial^\mu \pi^a + \tfrac{1}{12} g \, \mathrm{tr} \, ([\partial_\mu \pi, \pi][\partial^\mu \pi, \pi]) + \ldots \ ,$$

where interactions involving six or more pion fields are omitted. The first term represents the kinetic energy of the pion field. To arrive at the standard normalization of this term, i.e., to insure that the pion propagator agrees with the one introduced above, the pion field must be scaled: the canonical coordinates π^a are to be replaced by $\pi^a/2\sqrt{g}$, such that the expression for the matrix U takes the form

$$U = \exp\left(\frac{i\pi^a\tau_a}{2\sqrt{g}}\right) \ .$$

The Noether currents associated with the SU(2)×SU(2) symmetry of the Lagrangian in equation (10) are

$$V_a^\mu = ig\,\mathrm{tr}\,(\tau_a[\partial^\mu U, U^\dagger]) \ , \qquad A_a^\mu = ig\,\mathrm{tr}\,(\tau_a\{\partial^\mu U, U^\dagger\}) \ .$$

Comparing the expression for the axial current with equation (4) or (7), we see that the coupling constant g is related to the pion decay constant F by $g = F^2/4$. At leading order in the derivative expansion, the effective Lagrangian, therefore, only involves the pion decay constant,

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{4}F^2\,\mathrm{tr}\,(\partial_\mu U \partial^\mu U^\dagger) \ , \qquad U = \exp\,(i\pi^a\tau_a/F) \ . \tag{11}$$

The field theory characterized by this Lagrangian is referred to as the nonlinear σ-model. It is well-known that, for $d > 2$, this model is not renormalizable — taken by itself, it is not a decent theory. Actually, in the above analysis, only the tree graphs of the effective Lagrangian played a role. Renormalizability is not an issue which concerns the tree graphs. We will have occasion to discuss the significance of loop graphs later on, when we consider the low energy expansion beyond leading order. Clearly, the effective Lagrangian must then also be worked out beyond the leading term in the derivative expansion. In the framework of the effective Lagrangian, the nonlinear σ-model only represents one building block of the construction — it does not occur by itself. As we will see, the effective theory as a whole is a perfectly renormalizable scheme.

14 Geometry and universality

The above explicit result for the effective Lagrangian involves the matrix representation for the pion field. The expression may be rewritten in terms of the variables π^1, π^2, π^3 as follows. The properties of the Pauli matrices imply

$$U = \mathbf{1}\,\cos\alpha + i\,\frac{\vec{\tau}\cdot\vec{\pi}}{|\vec{\pi}|}\,\sin\alpha \ , \qquad \alpha \equiv \frac{|\vec{\pi}|}{F} \ .$$

Inserting this in (11), the Lagrangian takes the form

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{2} g_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b \;\; ,$$
$$g_{ab}(\pi) = \delta_{ab}\,(\sin\alpha/\alpha)^2 + \frac{\pi^a\pi^b}{\vec{\pi}^{\,2}}\{1-(\sin\alpha/\alpha)^2\} \;\; .$$

The expression for $g_{ab}(\pi)$ represents the metric of a sphere of radius F. The effective Lagrangian thus admits the following simple geometric interpretation. The pion field variables are the coordinates needed to label the points of the quotient space G/H = SU(2). Quotient spaces always possess an intrinsic metric. In the present case, the quotient space even represents a group. The intrinsic geometry of the group SU(2) is the one of the three-dimensional unit sphere. The geometry relevant for the effective Lagrangian coincides with the intrinsic geometry of the quotient space, except for an overall normalization factor — the radius of the relevant sphere is given by the pion decay constant. As discussed in section 10, the expansion of the metric in powers of the coordinates determines the interaction vertices. In particular, the four-pion interaction is determined by the second derivatives $\partial_{cd}g_{ab}(0)$, i.e., by the curvature of the manifold at the origin.

It is not difficult to understand why the relevant geometry is one of constant curvature. For the effective Lagrangian to remain invariant under G = SU(2)×SU(2), the metric occurring therein must admit G as a group of isometries. In the present case, the transformations of the pion field represent right- and left-translations on the group SU(2), $U \rightarrow V_{\mathrm{R}} U V_{\mathrm{L}}^\dagger$. For compact groups, this property fixes the metric uniquely, up to normalization — the sphere is the only compact manifold with a six-parameter group of isometries.

The coordinates are a matter of choice. Replacing the above canonical coordinates by stereographic ones, the effective Lagrangian simplifies to

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{2}\,\frac{\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}}{(1+\frac{1}{4}\,\vec{\pi}^{\,2}/F^2)^2} = \tfrac{1}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}\,\left(1-\tfrac{1}{2}\vec{\pi}^{\,2}/F^2+\ldots\right)$$

I recommend it as an exercise to work out the $\pi\pi$ scattering amplitude with the above two explicit expressions, to check that, on the mass shell, the amplitude does not depend on the coordinates used and to verify that the result agrees with Weinberg's formula [21]. Off the mass shell, the amplitudes, however, differ: for the Green functions of the pion field, the choice of field variables does matter. The Green functions of the pion field do not have physical significance. The effective theory is of interest only as a vehicle, which allows one to analyze the low energy properties of QCD in an efficient manner. In QCD, scattering amplitudes and Green functions formed with the currents or with the operators $\bar{q}\lambda q$, $\bar{q}\gamma_5\lambda q$ are meaningful quantities, but a pion field operator only occurs in the effective theory. For a given choice of the effective field, the effective theory does give rise to perfectly unambiguous Green functions also for this field. In contrast to the results for the scattering amplitudes or for the Green functions formed with the vector, axial, scalar and pseudoscalar currents, those of the pion field, however, depend on the choice of the effective field.

Remarkably, the specific properties of the underlying theory did not matter in the construction of the effective Lagrangian. The analysis applies to any theory for which SU(2) × SU(2) is spontaneously broken to SU(2). The explicit expression found for the effective Lagrangian is valid for any model with these symmetry properties — the low energy structure is universal.

The extension to three massless flavours is straightforward. In this case, the Goldstone bosons live in G/H = SU(3). Accordingly, there are eight pion fields, which may be identified with the canonical coordinates on SU(3),

$$U = \exp\left(i\pi^a\lambda_a/F\right) \ ,$$

where $\lambda_1, \ldots, \lambda_8$ are the Gell-Mann matrices. There is again only one invariant at order p^2,

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{4}\,F^2\,\mathrm{tr}\,(\partial_\mu U \partial^\mu U^\dagger) \ . \qquad (12)$$

In the case of the linear σ-model with N scalar fields, the relevant symmetry groups are G= O(N), H = O(N−1), such that the quotient G/H is the (N−1)-dimensional sphere. Accordingly, the pion field is a unit vector $U^A(x)$ with N components. The derivative expansion of the effective Lagrangian again starts with a term of order p^2 and involves a single coupling constant,

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{2}\,F^2 \partial_\mu U^A \partial^\mu U^A \ . \qquad (13)$$

The Higgs sector of the Standard Model corresponds to N = 4. The corresponding "pion" field lives in the three-sphere. This manifold may be mapped one-to-one onto the group SU(2), setting

$$U = \mathbf{1} U^0 + i\tau_a U^a \ .$$

One readily checks that the map takes the Lagrangian in equation (12) into the one in equation (13). This does not represent a great surprise, because the groups G = O(4) and H = O(3), occurring in the spontaneous breakdown of the Higgs model, are locally isomorphic to SU(2) × SU(2) and to SU(2), respectively. The equivalence of the two effective theories implies that, at low energies, the Green functions of the Higgs model and of QCD with two massless flavours are the same, except for the magnitude of the constant F. For QCD, $F \simeq 93$ MeV, while in the case of the Higgs model, $F \simeq 245$ GeV.

15 Symmetry breaking

In the preceding sections, we have dropped the masses of the u- and d-quarks. In their presence, the Lagrangian of the theory is not invariant under $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}}$, because the mass term

$$\mathcal{L} = \mathcal{L}_0 - \bar{q} m q$$

connects the right- and left-handed components of the quark fields,

$$\bar{q} m q = \bar{q}_{\mathrm{R}} m q_{\mathrm{L}} + \mathrm{h.c.} \quad . \tag{14}$$

In the notation used here, the quark field q only contains u and d and m is the matrix

$$m = \begin{pmatrix} m_u & \\ & m_d \end{pmatrix} \quad .$$

We may allow for the presence of other quarks, $s, c, \ldots$ They only appear in $\mathcal{L}_0$. The only property of $\mathcal{L}_0$ we will make use of is that this part of the Lagrangian is invariant under $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}}$.

It is instructive to compare the QCD Lagrangian with the Hamiltonian of a Heisenberg ferromagnet,

$$\mathbf{H} = \mathbf{H}_0 - \sum_a \mu\, \vec{\mathbf{s}}_a \cdot \vec{\mathrm{H}} \quad .$$

Here, $\vec{\mathbf{s}}_a$ is the spin associated with lattice site a, μ is the magnetic moment and $\vec{\mathrm{H}}$ is an external magnetic field. The term $\mathbf{H}_0$ is invariant under simultaneous O(3) rotations of all spin variables, while the term which involves the external magnetic field breaks this symmetry. Clearly, the quark masses (m_u, m_d) play a role analogous to the external magnetic field and the quark condensate $\langle 0|\, \bar{u}u\, |0\rangle$, $\langle 0|\, \bar{d}d\, |0\rangle$ is analogous to the magnetization. In particular, spontaneous magnetization at zero external field corresponds to a nonzero value of the quark condensate in the chiral limit $m_u, m_d \to 0$.

In the case of the magnet, the symmetry breaking term transforms according to the spin 1 representation of O(3). The decomposition of the quark mass term given in equation (14) shows that this term transforms according to the representation $D^{(\frac{1}{2},\frac{1}{2})}$ of $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}}$. Equivalently, we may say that the QCD Lagrangian is invariant under the transformation (8) of the quark fields, provided the mass matrix is transformed accordingly,

$$m \to V_{\mathrm{R}} m V_{\mathrm{L}}^{\dagger} \quad .$$

In this interpretation, the mass matrix plays the role of a "spurion".

The occurrence of a mass term, of course, modifies the form of the effective Lagrangian,

$$\mathcal{L}_{\mathit{eff}} = \mathcal{L}_{\mathit{eff}}(U, \partial U, \partial^2 U, \ldots, m) \ \ ,$$

which now remains invariant under the transformation $U(x) \rightarrow V_{\mathrm{R}} U(x) V_{\mathrm{L}}^{\dagger}$ of the pion field only if one simultaneously also transforms the quark mass matrix in the same manner. The modification of the Lagrangian generated by the quark masses may be analyzed by expanding in powers of m. The first term in this expansion is the effective Lagrangian of the massless theory, which we have considered in the preceding sections. The term linear in m is of the form

$$\mathcal{L}_{sb} = f(U, \partial U, \ldots) \times m \ \ .$$

Next, we observe that derivatives of the pion field are suppressed by powers of the momenta. At leading order in an expansion in both, powers of m and powers of derivatives, the symmetry breaking term in the effective Lagrangian reduces to an expression of the form $f(U) \times m$. Moreover, this expression must be invariant under simultaneous chiral transformations of the matrices U and m. There are only two independent invariants: $\mathrm{tr}(mU^{\dagger})$ and its complex conjugate. Hence the leading symmetry breaking contribution is of the form

$$\mathcal{L}_{sb} = \tfrac{1}{2} F^2 \{ B \, \mathrm{tr}(mU^{\dagger}) + B^{\star} \, \mathrm{tr}(m^{\dagger} U) \} \ \ ,$$

where I have extracted a factor F^2 for later convenience. The symmetry breaking involves a new low energy constant, B. Since only the product Bm matters, the phase of B occurs together with the phase of the quark mass matrix and is related to the possible occurrence of a parity violating term of the form $\theta \, G_{\mu\nu} \tilde{G}^{\mu\nu}$ in the Lagrangian of QCD. The fact that the neutron dipole moment is very small implies that the strong interactions conserve parity to a very high degree of occuracy. Let us therefore require that the effective Lagrangian is parity invariant. Using the standard basis, where the quark mass matrix is diagonal and real, this requirement implies that B is real (the parity operation sends π into $-\pi$ and hence interchanges U with $U^{\dagger}$),

$$\mathcal{L}_{sb} = \tfrac{1}{2} F^2 B \, \mathrm{tr}\{ m (U + U^{\dagger}) \} \ \ . \tag{15}$$

Since U is an element of SU(2), the sum $U + U^{\dagger}$ is proportional to the unit matrix. Accordingly, the leading contribution to the symmetry breaking part of the effective Lagrangian only involves the sum $m_u + m_d$ of the two quark masses and, therefore, conserves isospin — the breaking of isospin symmetry generated by the mass difference $m_u - m_d$ only shows up if the low energy expansion is carried beyond leading order.

Expanding $U = \exp\,(i \, \vec{\pi} \vec{\tau} / F)$ in powers of the pion field $\vec{\pi}$, the Lagrangian (15) gives rise to the following contributions:

$$\mathcal{L}_{sb} = (m_u + m_d) B \left\{ F^2 - \tfrac{1}{2} \vec{\pi}^{\,2} + \tfrac{1}{24} \, \vec{\pi}^{\,4} F^{-2} + \ldots \right\} \ \ . \tag{16}$$

Up to a sign, the first term represents the vacuum energy generated by the symmetry breaking. The second is quadratic in the pion field and, therefore, amounts to a pion mass term. The remaining contributions show that the symmetry breaking necessarily also modifies the interaction among the Goldstone bosons.

The derivative of the QCD Hamiltonian with respect to m_u is the operator $\bar{u}u$. Accordingly, the corresponding derivative of the vacuum energy represents the vacuum expectation value of $\bar{u}u$. Evaluating this derivative with the first term in equation (16), we obtain

$$\langle 0|\,\bar{u}u\,|0\rangle = \langle 0|\,\bar{d}d\,|0\rangle = -F^2 B\{1 + O(m)\} \ . \tag{17}$$

This shows that the low energy constant B is related to the value of the quark condensate. In analyzing the form of the effective Lagrangian, we have retained only terms linear in the quark masses. The curly bracket indicates that, in this relation, the higher order terms generate corrections of $O(m)$.

16 Mass of the Goldstone bosons

According to equation (16), the pion mass is given by

$$M_\pi^2 = (m_u + m_d)B\{1 + O(m)\} \ . \tag{18}$$

If m_u and m_d are set equal to zero, the pion mass vanishes, as it should: SU(2)$\times$SU(2) is then an exact symmetry, such that the Goldstone bosons are strictly massless. As long as the symmetry breaking is small, the Goldstone bosons only pick up a small mass, proportional to the square root of the symmetry breaking parameter $m_u + m_d$. In accord with the remarks made above, isospin breaking does not manifest itself at this order of the expansion — the masses of π^+, π^0 and π^- are the same.

Eliminating the low energy constant B, the relations (17) and (18) lead to the well-known result of Gell-Mann, Oakes and Renner [22],

$$F^2 M_\pi^2 = (m_u + m_d)|\langle 0|\,\bar{u}u\,|0\rangle| + O(m^2) \ .$$

The relation shows that the pion mass is determined by the product of the sum $m_u + m_d$ with the quark condensate. The first factor is a measure of the *explicit* symmetry breaking (which occurs in the Lagrangian of the theory), while the second is a measure of *spontaneous* symmetry breaking (for massless quarks, a nonzero value of the order parameter $\langle 0|\,\bar{u}u\,|0\rangle$ can only arise if the ground state of the theory does not possess the same symmetries as the Lagrangian).

The extension from two to three quark flavours is straightforward. The above analysis goes through without any essential modifications and leads to an effective Lagrangian of the same form,

$$\mathcal{L}_{\mathit{eff}} = \tfrac{1}{4}F^2\,\mathrm{tr}\,\{\partial_\mu U\partial^\mu U^+ + 2Bm(U+U^+)\}\ \ . \tag{19}$$

The field $U(x)$ is now an element of SU(3) and describes eight Goldstone bosons; m is the diagonal 3×3 matrix formed with m_u, m_d and m_s.

The kinetic part of the Lagrangian (19) is given by the terms quadratic in the field $\pi^a(x)$, which now carries eight components,

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{2}\{\partial_\mu\pi^a\partial^\mu\pi^a - B\mathrm{tr}(\lambda_a\lambda_b m)\pi^a\pi^b\} + \ldots \tag{20}$$

The evaluation of the trace shows that the masses of those mesons, which carry charge or strangeness, are given by[3]

$$\begin{aligned} M^2_{\pi^+} &= (m_u+m_d)B + O(m^2)\\ M^2_{K^+} &= (m_u+m_s)B + O(m^2)\\ M^2_{K^0} &= (m_d+m_s)B + O(m^2)\ \ . \end{aligned} \tag{21}$$

Ignoring the higher order contributions as well as electromagnetic effects, the above relations may be used to estimate the quark mass ratios from the observed pion and kaon masses,

$$\begin{aligned} \frac{m_u}{m_d} &\simeq \frac{M^2_{K^+} - M^2_{K^0} + M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 0.66\\ \frac{m_s}{m_d} &\simeq \frac{M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 20.1\ \ . \end{aligned}$$

The mass pattern of the Goldstone bosons breaks SU(3) symmetry — the absence of isospin breaking at leading order, observed in the case of SU(2)×SU(2), does not repeat itself here. In fact, none of the other multiplets shows SU(3) breaking effects comparable to those seen in the masses of the pseudoscalar octet. At first sight, the fact that M^2_K is 13 times larger than M^2_π even appears to indicate that a framework, which assumes the group G = $\mathrm{SU}(3)_{\scriptscriptstyle R}\times\mathrm{SU}(3)_{\scriptscriptstyle \mathrm{R}}$ to represent a decent approximate symmetry, is doomed to failure. Note, however, that in the mass pattern of the Goldstone bosons, an apparently very strong breaking of SU(3) would occur even if the quark

[3] Note that these formulae concern pure QCD — the electromagnetic interaction generates corrections of order e^2.

masses m_u, m_d, m_s where tiny, such that the group G would represent an almost perfect symmetry of the QCD Lagrangian: unless the ratios $m_u : m_d : m_s$ are close to one, the Goldstone bosons pick up very different masses also in that case. The point here is that for SU(3) to be a decent approximate symmetry of the strong interactions, it is *not* necessary that the differences $m_u - m_d$ and $m_d - m_s$ are small compared to the mean mass $\frac{1}{3}(m_u + m_d + m_s)$. What counts is the magnitude of the differences in comparison to the scale of the theory. If $m_u - m_d$ and $m_d - m_s$ are small in this sense, the symmetry breaking part of the Hamiltonian only generates small corrections. The quantity to compare $M_K^2 - M_\pi^2$ with is not $M_K^2 + M_\pi^2$, but the square of the mass of one of those states, which remain massive when the symmetry breaking is turned off. A quantitative estimate of the magnitude of symmetry breaking will be given in section 24.

What surprised us even more, when we realized that the light quark masses must be very different [23], is that m_u turns out to be quite different from m_d, despite the fact that isospin is an almost perfect symmetry of the strong interactions. The explanation is the same as for the case of SU(3): for isospin to be a decent approximate symmetry, it is not necessary that the difference $m_u - m_d$ is small compared to the sum $m_u + m_d$. It suffices that the difference is small compared to the scale of the theory. In the case of the nucleon, e.g., the mass difference $m_u - m_d$ makes the neutron heavier than the proton by merely 2‰ (the electromagnetic self-energies are of opposite sign). In the ratio $(M_{K^0}^2 - M_{K^+}^2)/(M_{K^0}^2 + M_{K^+}^2)$, the isospin breaking effects are enhanced, because the denominator is a small quantitiy of order m, but even so, the result is only of the order of 1%.

For the pions, where one might have expected the relative magnitude of isospin breaking to be largest, the matrix elements of the symmetry breaking term turn out to vanish; there, isospin breaking only shows up at order $(m_u - m_d)^2$. Indeed, we noted in section 15 that the leading term in the derivative expansion of the effective Lagrangian for two light flavours only involves the sum $m_u + m_d$ and thus hides the isospin breaking part of the QCD Hamiltonian. As demonstrated by the mass difference between the K^0 and the K^+, this is not the case for three light flavours. The mass term in equation (20) also induces mixing between the states π^0 and η, through an angle of order $(m_u - m_d)/(m_s - \hat{m}) \ll 1$. The repulsion of the two levels generates a mass difference between π^0 and π^+,

$$M_{\pi^0}^2 \simeq M_{\pi^+}^2 - \tfrac{1}{4}\left(\frac{m_u - m_d}{m_s - \hat{m}}\right)^2 (M_K^2 - M_\pi^2) \qquad \hat{m} \equiv \tfrac{1}{2}(m_u + m_d) \ .$$

Numerically, this effect is tiny — the observed mass difference mainly originates in the electromagnetic self-energy of the charged pion.

Dropping terms of order $(m_u - m_d)^2/(m_s - \hat{m})$, the mass of the η becomes

$$M_\eta^2 = \tfrac{2}{3}(\hat{m} + 2m_s)B + O(m^2) \ .$$

Accordingly, the squares of the masses obey the Gell-Mann-Okubo-formula,

$$3M_\eta^2 + M_\pi^2 - 2M_{K^+}^2 - 2M_{K^0}^2 = O\left(m^2, (m_u - m_d)^2/(m_s - \hat{m})\right) \ .$$

This relation is satisfied remarkably well, confirming that the group SU(3) does represesent a decent approximate symmetry. A quantitative measure for the magnitude of the symmetry breaking results from a comparison of two independent determinations of the ratio $m_s/\hat{m}$. Using the ratio $(M_{K^0}^2 + M_{K^+}^2)/M_{\pi^+}^2$, one obtains $m_s/\hat{m} = 24.2$, while the ratio $M_\eta^2/M_{\pi^+}^2$ leads to $m_s/\hat{m} = 22.7$. If the Gell-Mann-Okubo formula were exact, the two results would be the same. The symmetry breaking seen here is unusually small. In most cases, one finds departures from SU(3) symmetry at the 20-30% level. A typical example is the ratio of decay constants: the observed values yield $F_K/F_\pi = 1.22$, while exact SU(3) symmetry would require the two constants to be the same.

17 Currents and external fields

In principle, the low energy structure of the current vertices may be investigated by means of the Noether currents. For the case of two flavours, the Noether currents belonging to the leading term in the derivative expansion of the effective Lagrangian were worked out explicitly in section 13. The result specifies the currents as functions of the pion field and its first derivatives. The formulae given there represent the leading term in the derivative expansion of the full effective field theory representation, which is of the form

$$V_a^\mu = V_a^\mu(\pi, \partial\pi, \ldots) \ , \qquad A_a^\mu = A_a^\mu(\pi, \partial\pi, \ldots) \ .$$

For the general analysis, however, this method is not adequate, because the low energy expansion of the Green functions gives rise to one-particle reducible contributions, which involve more than one current at the same vertex. Such vertices cannot be represented in terms of the above functions, which describe emission and absorption of pions by a single current — they require their own effective field representation.

The external field method is considerably more efficient, as it treats all of the vertices on the same footing. In this approach, one studies the response of the system to the perturbations generated by suitable external fields. To analyze the Green functions formed with the vector and axial currents, e.g., one introduces an external field $v_\mu^a(x)$ coupled to the vector currents as well as a field $a_\mu^a(x)$ for the axial currents and perturbs the QCD Lagrangian by a term of the form

$$\mathcal{L}_{\rm QCD} \rightarrow \mathcal{L}_{\rm QCD} + v_\mu^a V_a^\mu + a_\mu^a A_a^\mu \;\; ; \qquad V_a^\mu = \bar{q}\gamma^\mu \tfrac{1}{2}\lambda_a q \; , \;\; A_a^\mu = \bar{q}\gamma^\mu\gamma_5 \tfrac{1}{2}\lambda_a q \;\; .$$

The perturbation generates excitations. Suppose the external fields vanish for $t \rightarrow \pm\infty$ and assume that, in the remote past, the system was in the unperturbed ground state. Denote the vector, which describes this state in the Heisenberg picture, by $|0\,\mathrm{in}\rangle_{v,a}$. The effective action is defined as the logarithm of the probability amplitude for the system to end up in the ground state for $t \rightarrow +\infty$,

$$\exp iS_{\mathit{eff}}\{v,a\} = \langle 0\,\mathrm{out} \mid 0\,\mathrm{in}\rangle_{v,a} \quad .$$

Perturbation theory shows that this amplitude is given by the expectation value (in the unperturbed ground state) of the time-ordered exponential of the perturbation,

$$\exp iS_{\mathit{eff}}\{v,a\} = \langle 0|\, T \exp\, i\!\int\! d^4x \bar{q}\gamma^\mu\{v_\mu(x) + \gamma_5 a_\mu(x)\}q\,|0\rangle \;\; . \tag{22}$$

The matrix fields $v_\mu(x), a_\mu(x)$ occurring here are defined by

$$v_\mu(x) \equiv \tfrac{1}{2}\lambda_a v_\mu^a(x) \qquad\quad a_\mu(x) \equiv \tfrac{1}{2}\lambda_a a_\mu^a(x) \;\; .$$

For definiteness, I consider the case relevant for most of the applications, where m_u, m_d and m_s are treated as perturbations, retaining the masses of the remaining, heavy quarks at their physical values. The fields $v_\mu(x), a_\mu(x)$ then represent hermitean 3×3 matrices. Also, I disregard the singlet currents, taking these matrices to be traceless.

The above formula shows that the expansion of the effective action in powers of the external fields yields the Green functions of the currents. The two-point function of the axial current, e.g., is the coefficient of the term quadratic in $a_\mu(x)$,

$$\exp iS_{\mathit{eff}}\{v,a\} = 1 - \tfrac{1}{2}\!\int\! d^4x d^4y\, a_\mu^a(x) a_\nu^b(y)\, \langle 0|\, T A_a^\mu(x) A_b^\nu(y)\,|0\rangle \, + \ldots$$

The advantage of writing the transition amplitude as an exponential is known from statistical mechanics: the exponent then collects the connected part of the correlation functions. So, *the effective action is the generating functional of the connected parts of all of the Green functions formed with the vector and axial currents.*

The presence of external fields in the Lagrangian of the underlying theory, of course, also shows up in the effective Lagrangian, which now involves the pion field as well as the external ones,

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(\pi, v, a, \partial\pi, \partial v, \partial a, \ldots) \ .$$

The first term in the expansion in powers of v_μ, a_μ is the effective Lagrangian considered previously, while the contributions linear in v_μ, a_μ yield the effective field representations of the currents mentioned above,

$$\begin{gathered}\mathcal{L}_{eff}(\pi, v, a, \partial\pi, \partial v, \partial a, \ldots) = \\ \mathcal{L}_{eff}(\pi, \partial\pi, \ldots) + v_\mu^a V_a^\mu(\pi, \partial\pi, \ldots) + a_\mu^a A_a^\mu(\pi, \partial\pi, \ldots) + O(v^2, va, a^2) \ .\end{gathered}$$

The higher order contributions account for those vertices, which contain more than one vector or axial current.

The Green functions of the scalar and pseudoscalar currents may also be generated from the effective action, if we allow the quark mass matrix to become a space-time dependent field $m(x)$. The QCD Lagrangian then takes the form

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} + \bar{q}\gamma^\mu\{v_\mu(x) + \gamma_5 a_\mu(x)\}q - \bar{q}_{\rm R} m(x) q_{\rm L} - \bar{q}_{\rm L} m^\dagger(x) q_{\rm R} \ ,$$

where $\mathcal{L}^0_{\rm QCD}$ does not contain external fields and describes the light quarks as massless particles. The corresponding effective action now also depends on the external field $m(x)$,

$$\exp iS_{eff}\{v, a, m\} = \langle 0\,\mathrm{out} \mid 0\,\mathrm{in}\rangle_{v,a,m} \ .$$

The expansion of this functional in powers of $v_\mu(x), a_\mu(x)$ and $m(x)$ yields the Green functions of the vector, axial, scalar and pseudoscalar currents, *in the limit* $m_u = m_d = m_s = 0$. The quark condensate of the massless theory, e.g., is given by the term linear in $m(x)$, all other sources being switched off,

$$S_{eff}\{v, a, m\} = -\int d^4x \, \langle 0| \, \bar{q}_{\rm R} m q_{\rm L} + \bar{q}_{\rm L} m^\dagger q_{\rm R} \, |0\rangle + \ldots$$

The same generating functional also contains the Green functions of real QCD. To extract these, one considers the infinitesimal neighbourhood of the physical quark mass matrix m_0 rather than the vicinity of the point $m = 0$: Set $m(x) = m_0 + \tilde{m}(x)$ and treat $\tilde{m}(x)$ as a perturbation. The expansion of the effective action in powers of v_μ, a_μ and $\tilde{m}(x)$ yields the Green functions of the various currents for the case of physical interest, where the quark masses are different from zero.

18 Multipion exchange, unitarity and loops

Until now, we exclusively dealt with the leading terms of the low energy expansion. According to the pion pole dominance hypothesis, these are given by the pole terms due to one-pion exchange. In the language of the effective field theory, the pole terms arise from the tree graphs. The hypothesis does not imply, however, that one-pion exchange dominates to all orders. In fact, clustering requires that processes involving the simultaneous exchange of two or more pions between the same two vertices necessarily also occur. The corresponding contribution is determined by the product of the vertices describing emission and absorption, integrated over the relevant phase space. Since the low energy behaviour of the vertices is fixed by the leading term in the effective Lagrangian, the same is true of the multipion exchange contributions. These processes generate specific low energy singularities, which manifest themselves at nonleading orders of the low energy expansion.

Figure 2 indicates some of the effective field theory graphs contributing to elastic scattering. Note the distinction to the graphs in figure 1: The vertices occurring there represent full one-particle irreducible amplitudes, which include contributions from multipion exchange, while those in fig.2 correspond to interaction terms of the effective Lagrangian. In fact, all of the above graphs depict one-particle irreducible contributions to the scattering amplitude — the four-pion vertex marked with a shaded blob in figs.1c and 1e includes all of these.

At low energies, the scattering amplitude is dominated by the tree graph in fig.2a. Graphs 2b, 2c and 2d represent the exchange of a pair of pions in the s-, t- and u-channel, respectively. The two pions may undergo a collision under way (2e) and contributions involving the exchange of more than two particles also occur (2f, 2g). In the language of the effective field theory, these processes correspond to loop graphs.

Quite generally, graphs involving loops are essential for the transition amplitudes to conserve probability. Since tree graphs generate purely real contributions to the T-matrix, they do not satisfy the unitarity relation $\mathrm{Im}\, T = T^\dagger T$. As pointed out by Lehmann [24], unitarity requires, e.g., that the low energy expansion of the elastic $\pi\pi$ scattering amplitude contains specific contibutions of order p^4, which are not polynomials in the invariants s and t, but contain logarithmic branch points. Within the effective field theory, the branch points arise from one-loop graphs of the type 2b, 2c and 2d. Indeed, general kinematics insures that the perturbative expansion of a local field theory automatically leads to a unitary scattering matrix, provided all of the graphs are taken into account, including those containing loops. The corresponding representation of the effective action is given by the path integral

$$e^{iS_{\!\mathit{eff}}\{v,a,m\}} = \mathcal{Z}^{-1}\int [d\pi]\; e^{i\int d^4x \mathcal{L}_{\mathit{eff}}(\pi,v,a,m,\partial\pi,\partial v,\partial a,\partial m,\dots)} \;\;.$$

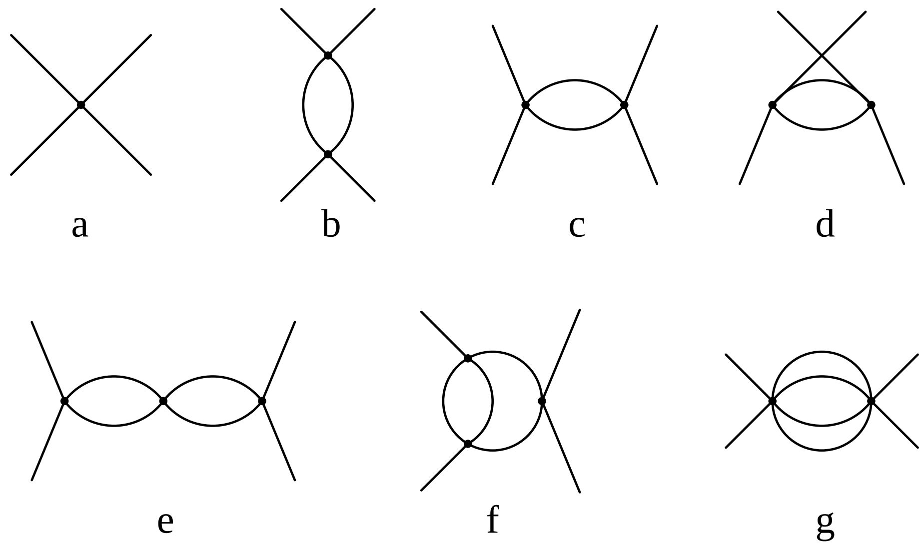

Figure 2: Effective field theory graphs contributing to the elastic $\pi\pi$ scattering amplitude. The dots represent interaction terms of the effective Lagrangian.

The tree graphs represent the classical limit of this path integral. While they yield the correct result for the leading terms of the low energy expansion, the quantum fluctuations described by graphs containing loops do contribute at nonleading order — the pion field of the effective theory is a quantum field, not a classical one.

Here, the effective Lagrangian method shows its full strength: the path integral not only yields all of the pole terms generated by one-pion exchange, but automatically also accounts for all of the singularities due to the multi-pion exchange contributions required by clustering. Because the framework used is a local field theory, clustering and unitarity are incorporated ab initio.

The above formula is a corner stone of chiral perturbation theory. It provides the link between the underlying and effective theories and is exact, to any finite order of the low energy expansion. While the left-hand side represents the generating functional for the Green functions of the underlying theory, the right-hand side only involves the effective Lagrangian.

As pointed out by Weinberg [10], the path integral of the effective theory may be evaluated perturbatively, using the momenta, quark masses and external fields as expansion parameters. The resulting perturbation series is identical with the low energy expansion of the effective action. The higher order terms of the low energy expansion may be worked out explicitly by evaluating the path integral to the required accuracy: the higher orders in the derivative expansion of the effective Lagrangian need to be accounted for, as well as graphs involving loops. It is crucial here that, at low energies, the effective interaction among the Goldstone bosons is weak (compare section 9). This property insures that the interaction may be accounted for perturbatively.

In principle, the cuts generated by multipion exchange may also be analyzed without recourse to an effective Lagrangian, exploiting analyticity and unitarity and evaluating dispersion relations rather than loop graphs. The virtue of the effective Lagrangian method is that it systematically accounts for all of the singularities relevant at a given order of the expansion, is straightforward and free of ambiguities.

19 Ward identities

One of the virtues of the external field method is that, in this framework, the Ward identities take a remarkably simple form. These express the symmetry properties of the underlying theory in terms of the Green functions. The Ward identity obeyed by the two-point function formed with an axial current and a pseudoscalar density, e.g., reads

$$\begin{aligned}&\partial^x_\mu \,\langle 0|\, T\; \bar{q}(x)\gamma^\mu\gamma_5\lambda_a q(x)\; \bar{q}(y)\gamma_5\lambda_b q(y)\,|0\rangle = \\ &i\langle 0|\, T\; \bar{q}(x)\gamma_5\{m,\lambda_a\}q(x)\; \bar{q}(y)\gamma_5\lambda_b q(y)\,|0\rangle - \delta^4(x-y)\langle 0|\,\bar{q}\{\lambda_a,\lambda_b\}q\,|0\rangle \;\; .\end{aligned} \tag{23}$$

The relation may be derived from current conservation and equal time commutation relations. This derivation, however, leaves to be desired, because it involves formal manipulations with products of operators and step functions. A mathematically satisfactory framework — which, moreover, yields all of the Ward identities at once — is the following.

Consider the Dirac operator, which in the presence of the external fields introduced above takes the form

$$D = -i\gamma^\mu\{\partial_\mu - i(G_\mu + v_\mu + a_\mu\gamma_5)\} + m\tfrac{1}{2}(1-\gamma_5) + m^\dagger\tfrac{1}{2}(1+\gamma_5) \ .$$

The colour field G_μ is a dynamical field, which mediates the strong interactions, while the flavour fields v_μ, a_μ are classical auxiliary variables.[4] In the path integral, G_μ is to be integrated over, while v_μ, a_μ are held fixed. The colour group acts on the quark and gluon fields,

$$q(x)' = V_c(x)\, q(x) \ , \ \ G_\mu(x)' = V_c(x) G_\mu(x) V_c^{-1}(x) - i\,\partial_\mu V_c(x) V_c^{-1}(x)$$

but leaves the external fields untouched. The flavour group, on the other hand, acts on the quark fields, through independent rotations of the right- and left-handed components,

$$q_{\rm R}(x)' = V_{\rm R}(x)\, q_{\rm R}(x) \quad , \quad q_{\rm L}(x)' = V_{\rm L}(x)\, q_{\rm L}(x) \ ,$$

leaves the gluons untouched, but affects the external fields. The above expression for the Dirac operator shows that the linear combinations

$$f^{\rm R}_\mu = v_\mu + a_\mu \ , \ \ f^{\rm L}_\mu = v_\mu - a_\mu$$

transform like gauge fields of the two factor groups in $\mathrm{SU}(3)_{\rm R}\times\mathrm{SU}(3)_{\rm L}$,

$$\begin{aligned} f^{\rm R}_\mu(x)' &= V_{\rm R}(x) f^{\rm R}_\mu(x) V_{\rm R}(x)^{-1} - i\partial_\mu V_{\rm R}(x) V_{\rm R}(x)^{-1} \ , \\ f^{\rm L}_\mu(x)' &= V_{\rm L}(x) f^{\rm L}_\mu(x) V_{\rm L}(x)^{-1} - i\partial_\mu V_{\rm L}(x) V_{\rm L}(x)^{-1} \ , \end{aligned} \tag{24}$$

while the mass matrix transforms according to

$$m(x)' = V_{\rm R}(x) m(x) V_{\rm L}^{-1}(x) \ . \tag{25}$$

The external fields $v_\mu(x)$ and $a_\mu(x)$ thus promote the global flavour symmetry of the Lagrangian to a local one, where the group elements $V_{\rm R}, V_{\rm L}$ may depend on space-time — as it is the case with the gauge transformations of colour. This illustrates the ancient observation of Weyl, according

[4]The present discussion concerns QCD; if the electroweak interactions are turned on, some of the flavour fields also acquire physical significance.

to which any continuous symmetry may be extended to a local symmetry by introducing suitable gauge fields: it suffices to replace the ordinary derivatives occurring in the Lagrangian by covariant ones. In the present case, the relevant covariant derivatives of the quark fields are

$$D_\mu q_{\mathrm{R}} = (\partial_\mu - iG_\mu - if^{\mathrm{R}}_\mu)\, q_{\mathrm{R}} \qquad D_\mu q_{\mathrm{L}} = (\partial_\mu - iG_\mu - if^{\mathrm{L}}_\mu)\, q_{\mathrm{L}} \;\; .$$

Since the Lagrangian only involves these covariant derivatives, it is invariant under the gauge transformations of the fields $q(x), G_\mu(x), f^{\mathrm{R}}_\mu(x), f^{\mathrm{L}}_\mu(x)$ and $m(x)$ specified above.

This line of reasoning is formal, because it deals with the fields as if they were classical variables. The classical field theory characterized by a given Lagrangian represents the set of all tree graphs of the corresponding quantum field theory. The argument just given only shows that, *in the tree graph approximation*, the vacuum-to-vacuum transition amplitude is gauge invariant. The full transition amplitude also receives contributions from the quantum fluctuations of the dynamical variables $G_\mu(x), q(x)$, described by graphs containing loops. The divergences occurring in these graphs require the introduction of a cutoff, which modifies the properties of the dynamical variables and may ruin the symmetries of the Lagrangian. The choice of the regularization procedure is irrelevant, in the sense that, for a renormalizable theory, the result is independent thereof. If there is a regularization, which maintains the symmetries of the classical theory, then these symmetries also hold at the level of the quantum theory, but, in general, this is not the case.

The linear σ-model

$$\mathcal{L}_\sigma = \tfrac{1}{2}\partial_\mu\phi^A\partial^\mu\phi^A - \tfrac{1}{4}\lambda\left(\phi^A\phi^A - v^2\right)^2$$

is an example, where a symmetry preserving regularization does exist: dimensional regularization. The dynamical variables of that model are N scalar fields $\phi^A(x)$. One may generate the corresponding Green functions by supplementing the Lagrangian with a term of the form $m^A(x)\phi^A(x)$, where $m^A(x)$ is an external field, analogous to the matrix field $m(x)$, needed to generate the Green functions of the scalar and pseudoscalar currents in QCD. Also, one may introduce external vector fields coupled to the currents, replacing the ordinary derivatives with covariant ones,

$$D_\mu\phi^A = \partial_\mu\phi^A - if^{AB}_\mu\phi^B \;\;, \quad f^{AB}_\mu = -f^{BA}_\mu \;\; .$$

The Lagrangian

$$\mathcal{L}_\sigma = \tfrac{1}{2}D_\mu\phi^A D^\mu\phi^A - \tfrac{1}{4}\lambda\left(\phi^A\phi^A - v^2\right)^2 + m^A\phi^A$$

is gauge invariant and dimensional regularization maintains this symmetry. Accordingly, the corresponding vacuum-to-vacuum transition amplitude is gauge invariant to all orders in the perturbative expansion,

$$S_{\mathit{eff}}\{f', m'\} = S_{\mathit{eff}}\{f, m\} \ . \tag{26}$$

The effective action collects all of the Green functions formed with the currents $\phi^A \partial_\mu \phi^B - \phi^B \partial_\mu \phi^A$ and with the field ϕ^A. The invariance property (26) summarizes all of the Ward identities obeyed by the Green functions. The expansion of this relation in powers of the external fields shows that, in the linear σ-model, the formal derivation of the Ward identities by means of the equal time commutation relations does lead to the correct result.

20 Anomalies

In fermionic theories with chiral couplings of the gauge fields, where some of the vertices involve $\gamma_\mu \gamma_5$ rather than γ_μ, the situation is different. A regularization, which preserves the symmetries of the Lagrangian with respect to chiral gauge transformations, does not exist. In particular, dimensional regularization fails: γ_5 cannot be continued in the dimension. Indeed, the effective action of QCD is not invariant under a gauge transformation of the external fields,

$$S_{\mathit{eff}}\{v', a', m'\} \neq S_{\mathit{eff}}\{v, a, m\} \ ,$$

because the Ward identities pick up extra contributions, generated by loop graphs. The formal derivation of the Ward identities, based on the equal time commutation relations, misses these. The extra terms are referred to as *anomalies*. The problem does not concern the interaction of the quarks with the dynamical field G_μ, which is vector-like: the quantum fluctuations preserve the symmetry with respect to *colour* gauge transformations. The problem is caused by the perturbations generated by the external field a_μ, whose interaction with the quarks distinguishes the right- and left-handed components. The effective action fails to be invariant under the gauge group of *flavour*.

We encountered the phenomenon already in section 4, where we noted that one of the global U(1)-symmetries of the massless theory is ruined by an anomaly: The Ward identities obeyed by the singlet axial current contain an anomalous contribution proportional to $\epsilon^{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma}$. In the presence of external fields, analogous terms built with v_μ and a_μ also occur, such as $\epsilon^{\mu\nu\rho\sigma} \partial_\mu v_\nu \partial_\rho v_\sigma$ or $\epsilon^{\mu\nu\rho\sigma} \partial_\mu a_\nu \partial_\rho a_\sigma$. The problem arises from fermionic one-loop graphs, involving three, four or five vertices, at which the quark emits one of the fields G_μ, v_μ or a_μ.

The external fields v_μ, a_μ entering the generating functional $S_{\mathit{eff}}\{v, a, m\}$ introduced above are traceless; the effective action only collects the Green functions formed with the currents of $SU(3)_R \times SU(3)_L$ and does not contain those involving the singlet currents. The anomaly occurring in the Ward identities for that current does, therefore, not concern us here. It is not difficult to see that, if the flavour fields are traceless, triangle graphs involving gluons as well as v_μ or a_μ vanish upon performing the traces over colour and flavour. More generally, anomalous contributions involving gluons do then not arise. This is the basis of the nonrenormalization theorem of Adler and Bardeen [25], which states that the change in the effective action, produced by a chiral rotation of the flavour fields, can be given explicitly, to all orders. Under an infinitesimal chiral rotation,

$$V_R(x) = 1 + i\alpha(x) + i\beta(x) \quad , \quad V_L(x) = 1 + i\alpha(x) - i\beta(x) \quad ,$$

the external fields undergo the gauge transformation

$$\begin{aligned} \delta v_\mu &= \partial_\mu \alpha + i[\alpha, v_\mu] + i[\beta, a_\mu] \quad , \quad \delta a_\mu = \partial_\mu \beta + i[\alpha, a_\mu] + i[\beta, v_\mu] \quad , \\ \delta m &= i(\alpha + \beta)m - im(\alpha - \beta) \quad . \end{aligned}$$

The corresponding change in the effective action involves the difference β between V_R and V_L,

$$S_{eff}\{v', a', m'\} = S_{eff}\{v, a, m\} - \int d^4x \, \mathrm{tr}\{\beta(x)\Omega(x)\} + O(\beta^2) \quad . \tag{27}$$

The explicit expression for Ω is proportional to the number of colours and exclusively contains the external vector and axial vector fields,

$$\Omega = \frac{N_c}{4\pi^2} \varepsilon^{\mu\nu\rho\sigma} \{\partial_\mu v_\nu \partial_\rho v_\sigma + \tfrac{1}{3} \partial_\mu a_\nu \partial_\rho a_\sigma + \ldots\} \quad . \tag{28}$$

The terms listed are those responsible for the Adler-Bell-Jackiw anomaly of the triangle graphs. In addition, Ω also contains terms involving three or four external fields, describing the anomalies in quark loops with 4 or 5 external field vertices. The specific form of these is not relevant here — the main point is that the change in the effective action is an explicitly known expression, which is of geometrical nature and does not depend on the gluon field G_μ, nor on the QCD coupling constant, nor on the masses of the heavy quarks, nor on the external field $m(x)$, which contains the masses of the light quarks.

The transformation law (27) represents a compact summary of all of the Ward identities obeyed by the Green functions of QCD. It states, in particular, that anomalies only occur in the 3-, 4- and 5-point functions, formed exclusively with the currents. Moreover, since the quantity Ω is proportional to $\epsilon^{\mu\nu\rho\sigma}$, the Green function in question must contain an odd number of axial currents. The three-point function $\langle 0 | \, T A_\mu V_\rho V_\sigma \, | 0 \rangle$ is the most prominent example; the anomalous contribution in the Ward identities obeyed by this quantity plays a central role in the decay $\pi^0 \to \gamma\gamma$. The two-point function formed with an axial current and a pseudoscalar density, on the other hand, obeys an anomaly free Ward identity: the one written down in equation (23).

21 Higher orders in $\mathcal{L}_{\mathit{eff}}$

The effective Lagrangian collects all of the vertices, purely pion interactions, as well as those describing the interaction with the external fields $v_\mu(x), a_\mu(x), m(x)$. To order the various vertices, we first observe that the quark masses generate a pion mass proportional to the square root of $m_u + m_d$. When analyzing on-shell matrix elements such as scattering amplitudes, the momenta obey the condition $p^2 = M_\pi^2 \propto (m_u + m_d)$. A coherent bookkeeping of the powers of momenta thus requires that the quark masses are counted as perturbations of order p^2. We will use this counting of powers also for the corresponding external field, treating the matrix $m(x)$ as a quantity of $O(p^2)$

$$m(x) \sim p^2 \quad .$$

Concerning $v_\mu(x)$ and $a_\mu(x)$, we note that gauge transformations modify the combinations $v_\mu \pm a_\mu$ by a term involving the first derivative of the matrices $V_{\mathrm{R}}(x), V_{\mathrm{L}}(x)$. It is therefore convenient to count these fields as quantities of the same order as the derivative,

$$v_\mu(x), a_\mu(x) \sim p \quad .$$

This bookkeeping insures that the Ward identities relate terms of the same order in the low energy expansion.

Lorentz invariance then implies that, as before, the expansion of the effective Lagrangian in the number of derivatives and external fields only involves even powers,

$$\mathcal{L}_{\mathit{eff}} = \mathcal{L}^2_{\mathit{eff}} + \mathcal{L}^4_{\mathit{eff}} + \mathcal{L}^6_{\mathit{eff}} + \ldots \tag{29}$$

If the fields v_μ, a_μ are turned off, the leading term in this expansion is given by the two pieces worked out in the preceding sections,

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{4} F^2 \,\mathrm{tr}\,(\partial_\mu U \partial^\mu U^\dagger) + \tfrac{1}{2} F^2 B \,\mathrm{tr}(m U^\dagger + U m^\dagger) \quad .$$

The first one is of order p^2, because it involves two derivatives of the pion field. The contribution generated by the symmetry breaking counts as a term of the same order, because it is linear in $m = O(p^2)$. The next term in the expansion, $\mathcal{L}^4_{\mathit{eff}}$, contains contributions with four derivatives, terms with two derivatives and one factor of m, as well as contributions which are quadratic in m, etc.

If the fields v_μ, a_μ are turned on, each one of the terms $\mathcal{L}^2_{\mathit{eff}}, \mathcal{L}^4_{\mathit{eff}}, \ldots$ picks up additional contributions. Their form is very strongly constrained by the Ward identities. Suppose for a moment that we are dealing with the effective Lagrangian of an anomaly free theory, such as the linear σ-model. As discussed above, the Ward identitites are then equivalent to the statement that the generating functional $S_{\mathit{eff}}\{v, a, m\}$ is invariant under a gauge transformation of the fields v_μ, a_μ, m. The implications for the form of the effective Lagrangian are remarkably simple: The invariance theorem asserts that $S_{\mathit{eff}}\{v, a, m\}$ is gauge invariant if and only if the effective Lagrangian is (for a proof, I again refer to [13]).

The invariance theorem also covers the case of theories like QCD, where the Ward identities contain anomalous contributions. In that case, the effective Lagrangian consists of two parts,

$$\mathcal{L}_{\mathit{eff}} = \overline{\mathcal{L}}_{\mathit{eff}} + \mathcal{L}_{WZ} \quad .$$

The quantity $\mathcal{L}_{WZ}$ is the famous Wess-Zumino term, an explicitly known expression, involving the pion field $U(x)$ as well as the external fields $v_\mu(x)$ and $a_\mu(x)$. In the bookkeeping introduced above, the Wess-Zumino Term is a term of order p^4. The theorem asserts that, once this contribution is removed, the remainder, $\overline{\mathcal{L}}_{\mathit{eff}}$, is gauge invariant, i.e., has the same properties as the effective Lagrangian of an anomaly free theory. This implies, that, in the derivative expansion (29), all of the terms except $\mathcal{L}^4_{\mathit{eff}}$ are gauge invariant.

If the Wess-Zumino term is dropped, the path integral of the effective theory yields a gauge invariant effective action, $\delta S_{\mathit{eff}} = 0$. By construction, the term $\mathcal{L}_{WZ}$ modifies the effective action in such a manner that it instead obeys the transformation law $\delta S_{\mathit{eff}} = -\int d^4x \mathrm{tr}\{\beta\Omega\}$. Accordingly, the Green functions generated by this functional automatically obey modified Ward identities, which contain the relevant anomalous contributions.

It is not difficult to convert the above explicit expression for $\mathcal{L}^2_{\mathit{eff}}$ into a gauge invariant one: it suffices to replace the ordinary derivatives of the pion field by covariant derivatives. The form of the covariant derivative immediately follows from the transformation law for the pion field, derived in section 12,

$$U'(x) = V_{\mathrm{R}}(x)U(x)V^\dagger_{\mathrm{L}}(x) \quad .$$

The transformation law (24) of the external fields shows that the quantity

$$D_\mu U = \partial_\mu U - i f^{\mathrm{R}}_\mu U + i\, U f^{\mathrm{L}}_\mu \tag{30}$$

transforms in the same manner as the field U. Hence the expression

$$\mathcal{L}^2_{\mathit{eff}} = \tfrac{1}{4}\, F^2\, \mathrm{tr}\,(D_\mu U D^\mu U^\dagger) + \tfrac{1}{2} F^2 B\, \mathrm{tr}(m U^\dagger + U m^\dagger)\} \tag{31}$$

is invariant under a gauge transformation of the fields. Indeed, it represents the most general expression of order p^2 with this property.

At order p^4, there are quite a few independent invariants, e.g.

$$\begin{aligned}\mathcal{L}^4_{\mathit{eff}} = & L_1\langle D_\mu U D^\mu U^\dagger\rangle^2 + L_5\langle D_\mu U D^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger)\rangle + L_7\langle \chi U^\dagger - U\chi^\dagger\rangle^2 \\ & - iL_9\langle f^{\mathrm{R}}_{\mu\nu} D_\mu U D^\nu U^\dagger + f^{\mathrm{L}}_{\mu\nu} D_\mu U^\dagger D^\nu U\rangle + \ldots + \mathcal{L}_{WZ} \quad ,\end{aligned}$$

where the external field $m(x)$ has been replaced by $\chi(x) \equiv 2Bm(x)$. The symbol $\langle A\rangle$ denotes the trace of the 3×3 matrix A and $f^{\mathrm{R}}_{\mu\nu}, f^{\mathrm{L}}_{\mu\nu}$ stand for the field strengths belonging to $f^{\mathrm{R}}_\mu, f^{\mathrm{L}}_\mu$, respectively. All of the above terms (for a full list, see [12]) are manifestly gauge invariant, except for $\mathcal{L}_{WZ}$. The effective coupling constants $L_1, L_2, \ldots$ are the analogues of the two quantities F, B, which specify the effective Lagrangian at leading order of the derivative expansion.

22 Renormalizability

As mentioned in section 18, the low energy expansion of the path integral over the pion field may be worked out by means of perturbation theory. The leading term of the expansion is given by the tree graphs. In the case of the $\pi\pi$ scattering amplitude, e.g., the tree graph contribution, shown in fig.2a, is of order p^2. The one loop graphs of fig.2b,2c and 2d generate a contribution of order p^4, while graphs with two loops only contribute at order p^6. More generally, graphs containing a different number of loops occur at different orders of the low energy expansion: in d dimensions, graphs with ℓ loops are suppressed compared to the tree graphs by the power $[p^{d-2}]^\ell$. The rule is readily checked for individual graphs such as those shown in fig.2. The loop integrals are homogeneous functions of the external momenta and of the pion mass, which enters through the propagators. The degree of homogeneity is determined by the dimension of the integral, which in turn is fixed by the overall power of the pion decay constant, arising from the various vertices. A more thorough discussion of the issue can be found in ref.[10].

As discussed above, the Lagrangian $\mathcal{L}^2_{\mathit{eff}}$ is not the full story — graphs involving vertices of $\mathcal{L}^4_{\mathit{eff}}, \mathcal{L}^6_{\mathit{eff}}, \ldots$ also need to be taken into account. In the case of the $\pi\pi$ scattering amplitude, graphs containing ℓ loops are of order $p^{2+\ell(d-2)}$, provided they exclusively involve vertices of $\mathcal{L}^2_{\mathit{eff}}$. Graphs containing one vertex of $\mathcal{L}^4_{\mathit{eff}}$ ($\mathcal{L}^6_{\mathit{eff}}$) are smaller by one (two) powers of p^2. Hence, to evaluate the scattering amplitude in four dimensions to order p^4, we need to work out the tree and one-loop graphs of $\mathcal{L}^2_{\mathit{eff}}$ and add the tree graphs with one vertex from $\mathcal{L}^4_{\mathit{eff}}$. Higher orders in the derivative expansion of the effective Lagrangian and two-loop graphs only start contributing at order p^6.

Note that the graphs can be ordered by counting powers of the momentum only if $d > 2$. In two dimensions, the constant F is dimensionless and the degree of homogeneity is therefore independent of the number of loops. In $d = 2$, the Lagrangian $\mathcal{L}^2_{eff}$ taken by itself specifies a decent, renormalizable theory, which moreover is asymptotically free and thus shares the qualitative properties of four-dimensional nonabelian gauge theories. In particular, the low energy structure of the theory cannot be analyzed perturbatively. (Incidentally, supplementing $\mathcal{L}^2_{eff}$ by the Wess-Zumino term, one arrives at a two-dimensional field theory with very peculiar properties: the Wess-Zumino-Novikov-Witten model. In this model, the coupling constant F can be tuned in such a fashion that the β-function vanishes - the theory becomes conformally invariant.)

In $d = 4$, the Lagrangian $\mathcal{L}^2_{eff}$ by itself is meaningless, but taken together with the infinite string of higher order terms $\mathcal{L}^4_{eff}, \mathcal{L}^6_{eff}, \ldots$, it does specify a renormalizable framework. If one disregards from those vertices, which involve the tensor $\epsilon^{\mu\nu\rho\sigma}$, one may regularize the loop integrals by means of dimensional regularization, which preserves the symmetries of the Lagrangian. The poles occurring at $d = 4$ then only require counter terms, which are Lorentz invariant and symmetric under $\mathrm{SU(3)}_{\mathrm{R}} \times \mathrm{SU(3)}_{\mathrm{L}}$. By construction, the full effective Lagrangian contains all terms permitted by this symmetry. The divergences may, therefore, be absorbed in a renormalization of the coupling constants occurring in the Lagrangian. In particular, the divergences contained in the one-loop graphs are absorbed in a renormalization of the coupling constants $L_1, L_2, \ldots$, occurring in $\mathcal{L}^4_{eff}$. Dimensional regularization also takes care of a technical complication, connected with the fact that the effective Lagrangian contains derivative couplings. This property implies that the measure occurring in the functional integral does not coincide with the standard translation invariant measure on the space of the pion fields. In general, the measure generates additional contributions involving power divergences, such as $\delta(0) \sim \Lambda^4$. In dimensional regularization, however, power divergences do not occur (in particular, $\delta(0)$ vanishes) and the complications associated with the measure can simply be ignored.[5]

23 Illustration: electromagnetic form factor

As an illustration of the method, let us return to the e.m. form factor of the pion, considered in section 1. With the machinery developed above, it is straightforward to calculate this quantity to first nonleading order in the low

[5]For a more detailed discussion and references to the literature, see, e.g. [12].

energy expansion. To keep these notes within bounds, I do not describe the calculation here, but refer to the literature [12]. Other sample calculations may be found in the reviews quoted in the introduction. The result is of the form

$$f_{\pi^+}(t) = 1 + \frac{t}{F^2}\{2L_9 + 2\phi(t, M_\pi) + \phi(t, M_K)\} + O(t^2, tm) \qquad (32)$$

In this case, the leading term of the expansion is trivial, as it represents the charge of the particle, $f_{\pi^+}(0) = 1$. At first nonleading order, there are two types of contributions: (i) The term proportional to L_9 evidently comes from a tree graph containing one vertex from $\mathcal{L}^4_{\mathit{eff}}$; it is linear in the momentum transfer t. (ii) The functions $\phi(t, M_\pi)$ and $\phi(t, M_K)$ are generated by one-loop graphs, which exclusively involve vertices from $\mathcal{L}^2_{\mathit{eff}}$; they are nontrivial functions of t, containing branch cuts, starting at $t = 4M_\pi^2$ and $t = 4M_K^2$. In dispersive language, the cuts are generated by $\pi\pi$ and $K\bar{K}$ intermediate states.

The loop integrals diverge. In dimensional regularization, the function $\phi(t, M)$ contains a pole at $d = 4$. The residue of the pole is momentum independent — the quantity $\phi(t, M) - \phi(0, M)$ tends to a finite limit when $d \to 4$. Accordingly, the divergence may be absorbed in a suitable renormalization of the coupling constant L_9. The result for the form factor is a finite expression, which is independent of the regularization used, but involves an effective coupling constant.

The result shows that chiral symmetry does not determine the pion charge radius: its magnitude depends on the value of the coupling constant L_9 — the effective Lagrangian is consistent with chiral symmetry for any value of the coupling constants. The symmetry, however, *relates* different observables. The slope of the K_{l_3} form factor $f_+(t)$, e.g., is also fixed by L_9. The experimental value of this slope, $\lambda_+ = 0.030$, can therefore be used to first determine the magnitude of L_9 and then to calculate the pion charge radius. This gives $\langle r^2 \rangle_{\pi^+} = 0.42$ fm^2, to be compared with the experimental result, 0.44 fm^2.

In the case of the neutral kaon, the analogous representation reads

$$f_{K^0}(t) = \frac{t}{F^2}\{-\phi_\pi(t) + \phi_K(t)\} + O(t^2, tm). \qquad (33)$$

A term of order one does not occur here, because the charge vanishes and there is no contribution from $\mathcal{L}^4_{\mathit{eff}}$, either. Chiral perturbation theory thus provides a parameter free prediction in terms of the one loop integrals $\phi_\pi(t), \phi_K(t)$. In particular, the slope of the form factor is given by [12]

$$\langle r^2 \rangle_{K^0} = -\frac{1}{16\pi^2 F^2} \ln \frac{M_K}{M_\pi} = -0.04 \text{ fm}^2 \quad , \qquad (34)$$

to be compared with the experimental value -0.054 ± 0.026 fm^2. This result represents the first prediction of this type — in the meantime, similar parameter free one-loop predictions have been discovered for quite a few other observables.

24 Magnitude of the coupling constants

One of the main problems encountered in the effective Lagrangian approach is the occurrence of an entire fauna of effective coupling constants. If these constants are treated as totally arbitrary parameters, the predictive power of the method is equal to zero — as a bare minimum, an estimate of their order of magnitude is needed.

In principle, the effective coupling constants $F, B, L_1, L_2, \ldots$ are calculable. They do not depend on the light quark masses, but are determined by the scale Λ_{QCD} and by the masses of the heavy quarks. The available, admittedly crude evaluations of F and B on the lattice demonstrate that the calculation is even feasible in practice. As discussed above, the coupling constants $L_1, L_2, \ldots$ are renormalized by the logarithmic divergences occurring in the one loop graphs. This property sheds considerable light on the structure of the chiral expansion and provides a rough estimate for the order of magnitude of the effective coupling constants [26]. The point is that the contributions generated by the loop graphs are smaller than the leading (tree graph) contribution only for momenta in the range $\mid p \mid \lesssim \Lambda_\chi$, where

$$\Lambda_\chi \equiv 4\pi F/\sqrt{N_f} \tag{35}$$

is the scale occurring in the coefficient of the logarithmic divergence (N_f is the number of light quark flavours). This indicates that the derivative expansion is an expansion in powers of $(p/\Lambda_\chi)^2$, with coefficients of order one. The stability argument also applies to the expansion in powers of m_u, m_d and m_s, indicating that the relevant expansion parameter is given by $(M_\pi/\Lambda_\chi)^2$ and $(M_K/\Lambda_\chi)^2$, respectively.

A more quantitative picture may be obtained along the following lines. Consider again the e.m. form factor of the pion and compare the chiral representation (32) with the dispersion relation

$$f_{\pi^+}(t) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'-t}\mathrm{Im} f_{\pi^+}(t') \ .$$

In this relation, the contributions ϕ_π, ϕ_K from the one loop graphs of χPT correspond to $\pi\pi$ and $K\bar{K}$ intermediate states. To leading order in the chiral expansion, the corresponding imaginary parts are slowly rising functions of

t. The most prominent contribution on the rhs, however, stems from the region of the ρ-resonance which nearly saturates the integral: the vector meson dominance formula, $f_{\pi^+}(t) = (1 - t/M_\rho^2)^{-1}$, which results if all other contributions are dropped, provides a perfectly decent representation of the form factor for small values of t. In particular, this formula predicts $\langle r^2 \rangle_{\pi^+} = 0.39\ \mathrm{fm}^2$, in satisfactory agreement with observation ($0.44\ \mathrm{fm}^2$). This implies that the effective coupling constant L_9 is approximately given by [12]

$$L_9 = \frac{F^2}{2M_\rho^2} \quad . \tag{36}$$

In the channel under consideration, the pole due to ρ exchange thus represents the dominating low energy singularity — the $\pi\pi$ and $K\bar{K}$ cuts merely generate a small correction. More generally, the validity of the vector meson dominance formula shows that, for the e.m. form factor, the scale of the derivative expansion is set by $M_\rho = 770$ MeV.

Analogous estimates may be given for all effective coupling constants at order p^4, saturating suitable dispersion relations with contributions from resonances [27, 28], e.g.

$$L_5 = \frac{F^2}{4M_S^2} \ , \qquad\qquad L_7 = -\frac{F^2}{48M_{\eta'}^2} \ ,$$

where $M_S \simeq 980$ MeV and $M_{\eta'} = 958$ MeV are the masses of the scalar octet and pseudoscalar singlet, respectively. In all those cases, where direct phenomenological information is available, these estimates do remarkably well. I conclude that the observed low energy structure is dominated by the poles and cuts generated by the lightest particles — hardly a surprise.

The effective theory is constructed on the asymptotic states of QCD. In the sector with zero baryon number, charm, beauty, ... , the Goldstone bosons form a complete set of such states, all other mesons being unstable against decay into these (strictly speaking, the η occurs among the asymptotic states only for $m_d = m_u$; it must be included among the degrees of freedom of the effective theory, nevertheless, because the masses of the light quarks are treated as a perturbation — in massless QCD, the poles generated by the exchange of this particle occur at $p = 0$). The Goldstone degrees of freedom are explicitly accounted for in the effective theory — they represent the dynamical variables. All other levels manifest themselves only indirectly, through the values of the effective coupling constants. In particular, low lying levels such as the ρ generate relatively small energy denominators, giving rise to relatively large contributions to some of these coupling constants.

In some channels, the scale of the chiral expansion is set by M_ρ, in others by the masses of the scalar or pseudoscalar resonances occurring around 1

GeV. This confirms the rough estimate (35). The cuts generated by Goldstone pairs are significant in some cases and are negligible in others, depending on the numerical value of the relevant Clebsch-Gordan coefficient. If this coefficient turns out to be large, the coupling constant in question is sensitive to the renormalization scale used in the loop graphs. The corresponding pole dominance formula is then somewhat fuzzy, because the prediction depends on how the resonance is split from the continuum underneath it.

The quantitative estimates of the effective couplings given above explain why it is justified to treat m_s as a perturbation. At order p^4, the symmetry breaking part of the effective Lagrangian is determined by the constants $L_4, \ldots, L_8$. These constants are immune to the low energy singularities generated by spin 1 resonances, but are affected by the exchange of scalar or pseudoscalar particles. Their magnitude is, therefore, determined by the scale $M_S \simeq M_{\eta'} \simeq 1$ GeV. Accordingly, the expansion in powers of m_s is controlled by the parameter $(M_K/M_S)^2 \simeq \frac{1}{4}$. The asymmetry in the decay constants, e.g., is determined by L_5. The estimate of this coupling constant given above yields

$$\frac{F_K}{F_\pi} = 1 + \frac{M_K^2 - M_\pi^2}{M_S^2} + \chi\text{logs} + O(m^2) \;\; ,$$

where the term "χlogs" stands for the chiral logarithms generated by the one loop graphs. This shows that the breaking of the chiral and eightfold way symmetries is controlled by the mass ratio of the Goldstone bosons to the non-Goldstone states of spin zero. In χPT, the observation that the Goldstones are the lightest hadrons thus acquires quantitative significance: For momentum independent quantities such as masses, decay constants, charge radii or scattering lengths, the magnitude of consecutive orders in the chiral perturbation series is determined by the square of the above mass ratio.

With this remark, I close the present lecture notes, which concern the foundations of the method. Plenty of applications are described in the literature and several different directions of research are currently under active investigation — the references quoted in the introduction provide a rough orientation.

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