Parameters of the standard model.

The standard model has 3 basic building blocks:
i) The gauge groups and the associated gauge bosons
ii) The fermions: the basic constituents of matter
iii) The Higgs sector.

\[ SU(3)_C \times SU(2)_L \times U(1)_Y \]

8 gluons \( W^+, W^-, W^0 \) \( B \)
\( q_5 \) \( q_2 \) \( q_1 \)
\( \frac{1}{4} \left( g_{\mu \nu} F^{\mu \nu} + g_{\mu \nu} F^{\mu \nu} + B^\mu B^\nu \right) + \theta (U(1)_Y) \)

\[ F^{\mu \nu} = \partial_\mu W^\nu - \partial_\nu W^\mu + g_{\mu \nu} \varepsilon^{ijk} W^i_j W^k \]
\[ B^\mu = \partial_\mu B - \partial_\nu B^\nu \]
\[ G^{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g^a \varepsilon^{abc} A^b_\mu A^c_\nu \]

alternative: matrix notation

and if acting on a fermion

\[ D^a_\mu \Psi = \left( \gamma_\mu \frac{i}{2} \lambda^a \right) \Psi \]

\[ \lambda^a \] colour indices
\[ \gamma_\mu \] weak isospin indices
\[ \lambda^a \] hypercharge

so up to here, 3 parameters.

Gauge bosons transform as:

\[ G = G^a_\mu \frac{\lambda^a}{2} \rightarrow UGU^t + \frac{i}{2} g_3 SUU^t \]
ii) Fermions: these are distinguished by their transformations under the 3 gauge groups.

\[
\begin{array}{ccc}
SU(3)_c & SU(2)_L & U(1)_Y \\
q_L & 3 & 2 & 1/3 & \{ \text{quarks} \} \\
u_R & 3 & 1 & 4/3 \\
d_R & 3 & 1 & -2/3 \\
e_L & 1 & 2 & -1 \\
e_R & 1 & 1 & -2 \\
(\nu_R) & 1 & 1 & 0 \\
\end{array}
\]

the gauge symmetries do not allow for mass terms.

The gauge symmetry are:

\[
\bar{\Psi} \rightarrow U \bar{\Psi}
\]

and from the transformation given on the previous page we can check

\[
D_\mu \bar{\Psi} \rightarrow U D_\mu \bar{\Psi}
\]

The fermions do not introduce any new parameters.

\[
q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad e_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}
\]

and each fermion comes in 3 varieties (generations)

\[
\begin{array}{c}
u = \begin{pmatrix} u_{\text{up}}(u) \\ d_{\text{down}}(d) \\ e_{\text{electron}}(e) \\ \nu_{\text{muon}}(\mu) \\ \nu_{\text{tau}}(\tau) \\
L_\nu = & i \bar{\Psi} \gamma_\mu D^\mu \bar{\Psi}
\end{array}
\]

at this level all couplings are given by the gauge symmetries so all interactions are identical: quark-lepton universality.

We still have 3 parameters.

*The exception is that a mass term \[ m_R \bar{\nu} \nu_R \] is not forbidden by any symmetries.
The Higgs sector.

In this sector practically all parameters of the standard model are hidden. I will concentrate on the Higgs in its most simple form. A single doublet with no strong interactions. We believe SU(3)_c to be unbroken

\[ \Phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} \]  

in SU(2)_L doublet and has \( Y_W = +1 \)

The kinetic term is

\[ \frac{1}{2} \left( D\mu \Phi \right)^\dagger D^\mu \Phi \]

Then there are interaction terms

\[ - \frac{i}{2} \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \]

2 new parameters

Now define \( \tilde{\Phi} = i \tau_2 \Phi^\dagger = \begin{pmatrix} \phi^0 & \phi^- \\ \phi^+ & -\phi^0 \end{pmatrix} \) is also a doublet with \( Y_W = -1 \)

\( \tilde{\Phi} \) is the charge conjugate of \( \Phi \)

Most parameters are hidden in

\[ - \bar{q}_L \tilde{\Phi} u_R M_{ij} - \bar{d}_L \tilde{\Phi} d_R M_{ij} - \bar{\ell}_L \tilde{\Phi} e_R M_{ij} \]

+ h.c.

(no \( v_R \) assumed here)

at first sight this implies \( 2 \times 3 \times 3 = 54 \) new parameters but not all of these are observable in the standard model.
\[
\begin{pmatrix}
\lambda_e \\
\lambda_i \\
\lambda_r \\
\lambda_t
\end{pmatrix}
= V_{1e} M^t L V_{2e} \quad + \quad M \quad M^t, \quad M^t M \quad \text{are both Hermitian} \Rightarrow \quad \text{can be diagonalized}
\]

\[
V_{1e} M M^t V_{1e}^t = \begin{pmatrix}
\lambda_e' \\
\lambda_i' \\
\lambda_r' \\
\lambda_t'
\end{pmatrix}
\]

\[
V_{2e} M^t M V_{2e}^t = \begin{pmatrix}
\lambda_e'' \\
\lambda_i'' \\
\lambda_r'' \\
\lambda_t''
\end{pmatrix}
\]

and \(\lambda_i, \lambda_i', \lambda_r, \lambda_t\) are real and solutions of \(\det(M M^t - \lambda I) = 0\) \quad \Rightarrow \quad \det(M^t M - \lambda^* I) = 0\)

\[
\lambda_i = \lambda_i' = \lambda_i'' = \lambda_i^* \quad \text{(not unique)}
\]

you can always change the phase of \(\lambda_i\) by adding \(V_{1e} \rightarrow V_{1e} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i \theta_1} & 0 & 0 \\ 0 & 0 & e^{i \theta_2} & 0 \\ 0 & 0 & 0 & e^{i \theta_3} \end{pmatrix}\)

In case of the top \(\lambda_i\) and \(V_{1e}, V_{2e}\) at the bottom are unique.
Standard form of mixings and masses

Any matrix can be brought into a diagonal form with real elements on the diagonal by 2 unitary matrices.

So we set:

\[
\begin{pmatrix}
\begin{pmatrix}
\lambda_e & \\
\lambda_e & 
\end{pmatrix}
\end{pmatrix} = V_{1e} M_L V_{2e}^T
\]

and similarly for the up and down masses.

But we now have to follow through what happens in the rest of the Lagrangian:

\[
\text{we now define } \quad \begin{array}{c}
\psi^i_{\text{L}} \\
\psi^i_{\text{R}}
\end{array} = \begin{array}{c}
\ell^i_{\text{L}} \\
\ell^i_{\text{R}}
\end{array}
\]

and the mass term becomes

\[
- m_e \bar{e} e - m_\mu \bar{\mu} \mu - m_\tau \bar{\tau} \tau \quad : \text{3 new parameters}
\]

with \( <\phi_0> = \frac{v}{\sqrt{2}} \)

\[
m_e = \lambda_e \frac{v}{\sqrt{2}}
\]

and the effect of \( V_1 \) and \( V_2 \) disappears in the kinetic terms.

For the quarks' life is more difficult since we can in the same way not have 2 different transformations on \( q_{\text{L}} \).

By convention we similarly remove \( M_\nu \) into \( \lambda_u, \lambda_c, \lambda_t \) or equivalently their 3 masses.

We have then as remaining Yukawa couplings

\[
- \bar{q}_L \psi^i_{\text{L}} \bar{\ell}^i_{\text{R}} \lambda_u - \bar{q}_L \psi^i_{\text{L}} \bar{\ell}^i_{\text{R}} \lambda_c - \bar{q}_L \psi^i_{\text{L}} \bar{\ell}^i_{\text{R}} V_{1u} V_{2d} (\lambda_d^T \phi_{\text{R}}^i)_{\text{D}}
\]

so we now have 3 parameters + \( \phi \)
A general 3 x 3 unitary matrix contains $18 - 9 = 3$ angles + 6 phases of these phases 5 can be absorbed by redefining the quark fields so $U$ contains 3 angles + one phase.

We now define

$$d^i_L = U d^i_{\text{man}}$$

so that the man term is diagonal but then it reappears in the couplings to $W$'s and remains in the Yukawa couplings of $\bar{u}^c_L$ to $d^c_R$.

So in gauging other than unitary needs to taken along

So we have 3 gauge couplings

$\mu$, $\lambda$ or $\nu$, $m_H$ or $m_Z$, $m_H$

8 fermion mass

3 mixing

1 CP violating parameter.

+ $\theta$ + gravitational coupling $= 20$ parameters

Note:

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left( W^\mu_\mu + W^\mu_\mu \right)$$

$$Z_\mu = \cos \theta_W \ W^3_\mu - \sin \theta_W \ B_\mu$$

$$A_\mu = \sin \theta_W \ W^3_\mu + \cos \theta_W \ B_\mu$$

$$\tan \theta_W = \frac{g_Z}{g_W}$$

$$\frac{M_W}{M_Z} = \cos \theta_W$$

$$v = \sqrt{\frac{M_W}{\lambda}}$$

$$q = I_3 + \frac{1}{2} Y_W$$

$$M_W = \frac{v}{\sqrt{g_W}}$$

$$M_Z = \frac{v}{\sqrt{g_Z}} \sqrt{g_W^2 + g_Z^2}$$

$$M_Y = 0$$
For later use:

\[
S = \bar{\psi} \gamma^\mu \psi \\
P = i \bar{\psi} i \gamma^\mu \psi \\
V^\mu = \bar{\psi} \gamma^\mu \psi \\
A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi
\]

<table>
<thead>
<tr>
<th>C</th>
<th>P</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>P</td>
<td>-P</td>
<td>-P</td>
</tr>
<tr>
<td>-V^\mu</td>
<td>V^\mu</td>
<td>V^\mu</td>
</tr>
<tr>
<td>-A^\mu</td>
<td>A^\mu</td>
<td>A^\mu</td>
</tr>
<tr>
<td>x^\mu</td>
<td>x^o, -x^i</td>
<td>-x^o, x^i</td>
</tr>
</tbody>
</table>

\[
P \psi P^{-1} = \gamma^o \psi (x^o, -x^i) \\
T \psi T^{-1} = i \gamma^1 \gamma^3 \psi (-x^o, x^i) \\
C \psi C^{-1} = i \gamma^2 \gamma^3 \bar{\psi} T(x) = i \gamma^2 \gamma^3 \psi^*(x)
\]

\[
U = \begin{pmatrix}
    c_1 & -\lambda_1 c_3 & 0 & 0 \\
    \lambda_1 c_2 & c_1 c_2 c_3 - \lambda_2 \lambda_3 e^{i\varnothing} & c_1 c_2 D_3 + \lambda_2 c_3 e^{i\varnothing} \\
    \lambda_2 D_2 & c_1 \lambda_2 c_3 + c_2 \lambda_3 e^{i\varnothing} & c_1 \lambda_2 D_3 - c_2 c_3 e^{i\varnothing}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    V_{ud} & V_{us} & V_{ub} \\
    V_{cd} & V_{cs} & V_{cb} \\
    V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\]

(\text{general})

\[
= \begin{pmatrix}
    1 - \frac{\lambda^2}{2} & \lambda & \lambda^3 A (i \varnothing) \\
    -\lambda & 1 - \frac{\lambda^2}{2} & \lambda^2 A \\
    \lambda^3 A (i \varnothing - i \varnothing) & -\lambda^3 A & 1
\end{pmatrix} + \mathcal{O} (\lambda^4)
\]

(\text{Wolfenstein})
A Majorana mass term

We have a second Lorentz invariant combination possible for a mass term.
This is using \( \frac{V^c}{R} = i \frac{\gamma^c}{R} \frac{V^*}{R} \)

\[
\mathcal{L}^{\text{Majorana}} = -\frac{m}{2} \bar{V}^c_R \gamma^c_R V_R + h.c.
\]

with \( m \) in general a complex number.

In terms of the Dirac components \( \psi_R \) of \( V_R \) this can be written as

\[
\mathcal{L}^{\text{Majorana}} = -\frac{m}{2} \bar{\psi}_R \gamma^c_\beta \psi_\beta R + h.c.
\]

where \( S_{\beta\beta} \) is a symmetric real 4 by 4 Dirac matrix.

The mass \( m \) is in general a complex number but can always be made real by redefining \( V_R \) by a complex phase.

Now in neutrino physics we can make several different ansatizes.

1) Only \( \nu_L \) exist but they can have a most general Majorana mass.

Then the lepton mass term becomes

\[
- \frac{M_{\nu L}}{\bar{\nu}_L} \bar{\nu}_R \nu_R - \frac{i}{2} \frac{M_{\nu L}}{\bar{\nu}_L} \bar{\nu}_L \nu_R
\]
with $M^{ij}$ a general complex matrix and $M^{ij}_{DL}$ a complex symmetric matrix

**Exercise:** Show that in this case we have six parameters

- 3 charged lepton masses
- 3 neutrino masses
- 3 mixing angles
- 3 phases

The matrix with the angles/phase is the "Maki-Nakagawa-Sakata" matrix. Hint: a general complex symmetric matrix can be diagonalized with real elements or be diagonalized by a unitary transformation as

$$M = U^T M_{DL} U$$

(notice the transpose)

2) A certain number of $\nu_R$ exist (sterile or right-handed neutrinos).

In this case the most general lepton mass matrix is given by

$$-M^{ij}_{DL} \bar{\nu}^i_R e^j_R - M^{ij}_{DN} \bar{\nu}^i_L \nu^j_R - \frac{1}{2} M^{ij}_{EI} (\nu^i_R \nu^j_R)$$

+ h.c.

The full analysis in this case is a lot more complicated. I will solve it after explaining the seesaw effect in a one-flavour setting.

Note: case 1 breaks $SU(2) \times U(1)$ explicitly but is a very good approximation to the so-called seesaw mechanism case (see below)
The see-saw effect.

In the case of one generation the mass terms in the presence of a right-handed neutrino allowed by $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge invariance are

$$- \bar{\nu}_L \nu_L e^- - \bar{\nu}_D \nu_D \nu_R - \frac{m}{2} \nu^c_R \nu_R$$

+ h.c.

- $g_L$ can be made real by redefining the phase of $e^-_R$ and $m = \frac{g_L v}{\sqrt{2}}$

- $g_D$ can be made real by redefining the phase of $\nu_R$ (and adjusting the phase of $\nu^c_R$ to keep $g_D$ real)

- $m$ can be made real by adjusting the phase of $\nu_R$ (and $\nu_L, e^-_R$ to keep the others real)

The neutrino part of the mass term now is (with $m_D = \frac{g_D v}{\sqrt{2}}$)

$$- m_{\nu_R} \bar{\nu}_L \nu_L - \frac{m}{2} \nu^c_R \nu_R = - \frac{1}{2} \begin{pmatrix} \bar{\nu}_L \\ \nu_R \end{pmatrix} \begin{pmatrix} m_D & m \\ m & m \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix}$$

where on the r.h.s. I have suppressed all Dirac factors in between.

The latter matrix can be diagonalized by

$$\begin{pmatrix} 0 & m_D \\ m_D & m \end{pmatrix} = U^T \begin{pmatrix} + \frac{m^2}{2} & \sqrt{m^2 + 4m_D^2} \\ \sqrt{m^2 + 4m_D^2} & 0 \end{pmatrix} U$$
\[ U = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \]

\[ a = \frac{1}{2} \left( m^2 + 4 m^2 D \right) - \frac{1}{2} \sqrt{m^2 + 4 m^2 D} \]

\[ b = \frac{1}{2} \left( m^2 + 4 m^2 D \right) + \frac{1}{2} \sqrt{m^2 + 4 m^2 D} \]

In the limit \( m \to m_D \), we get

\[ \lambda_1 = -\frac{m_D}{m} + O\left( \frac{1}{m^3} \right) \]

\[ \lambda_2 = m + \frac{m_D}{m} + O\left( \frac{1}{m^3} \right) \]

and the main eigentols are

\[ \bar{V}_1 = \frac{\bar{Y}}{\sqrt{L}} - \frac{m_D}{m} Y_L + O\left( \frac{1}{m^2} \right) \]

\[ \bar{V}_2 = \frac{m_D}{m} \bar{Y}_L + \bar{Y}_R + O\left( \frac{1}{m} \right) \]

So, for the case \( m \to m_D \), a very good approximation is to treat it as \( \bar{Y}_L \), being the only degree of freedom with a Majorana mass.

This is the general case described on page 10 for 3 generations.
The general case

Start with

\[ \ell^i M_{ij} \ell^j = \ell^i M_{ij} \ell^j \]

We can diagonalize \( M_{ij} \) via

\[ U^T M U = \begin{pmatrix} 0 & M_D \\ M_D^T & M_M \end{pmatrix} \]

and the neutrino mass matrix is now

\[ -\frac{1}{2} \left( \begin{array}{cccc} \bar{\nu}_L^1 & \bar{\nu}_L^2 & \bar{\nu}_L^3 & \nu_R^1 \\ \bar{\nu}_L^2 & \bar{\nu}_L^3 & \nu_R^1 & \nu_R^2 \\ \bar{\nu}_L^3 & \nu_R^1 & \nu_R^2 & \nu_R^3 \end{array} \right) \]

and the full 6 by 6 mass matrix can be diagonalized by a 6 by 6 unitary matrix \( U \)

\[ U^T M U = \begin{pmatrix} 0 & M_D \\ M_D^T & M_M \end{pmatrix} \]

\[ U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \]

with \( u_1, u_2, u_3, u_4 \) 3x3 matrices satisfying

\[
\begin{align*}
    u_1 u_1^T + u_2 u_2^T &= 1 \\
    u_3 u_3^T + u_4 u_4^T &= 1 \\
    u_1 u_3^T + u_4 u_2^T &= 0
\end{align*}
\]
The weak interaction states interacting with $\bar{e}, \mu, \tau$ now are

\[
\begin{pmatrix}
\bar{\nu}_L \\
\bar{\nu}_X \\
\bar{\nu}_Z \\
\end{pmatrix} = U_{DL} \begin{pmatrix}
\tilde{\nu}_1 \\
\tilde{\nu}_2 \\
\tilde{\nu}_3 \\
\end{pmatrix} + U_{\tilde{\nu}} \begin{pmatrix}
\tilde{\nu}_R \\
\tilde{\nu}_L \\
\tilde{\nu}_R \\
\end{pmatrix}
\]

The mass eigenstates are

\[
\begin{pmatrix}
\frac{\bar{\nu}_1}{\sqrt{2}} \\
\frac{\bar{\nu}_2}{\sqrt{2}} \\
\bar{\nu}_3 \\
\end{pmatrix} = \begin{pmatrix}
\bar{\nu}_1 \\
\bar{\nu}_2 \\
\bar{\nu}_3 \\
\end{pmatrix}
\]

So we have in principle 6 neutrino masses but the mixing matrix is more complicated than just unitary.

We define

\[
\begin{align*}
\bar{\nu}_1 &= U_{DL} \tilde{\nu}_1 \\
\bar{\nu}_2 &= U_{DL} \tilde{\nu}_2 \\
\end{align*}
\]

and these satisfy \( \bar{\nu}_1^+ \bar{\nu}_1 + \bar{\nu}_2^+ \bar{\nu}_2 = \mathbf{1} \).

\( \bar{\nu}_1, \bar{\nu}_2 \) have 36 free parameters and satisfy 9 real relations:

- 3 phases can be removed by redefining the phases of the charged leptons.

We started with 48 parameters in the matrices \( M^{ij}_{DL}, M^{ij}_{DN}, M^{ij} \)

and now have 9 masses plus 24 parameters in the mixing sector

of these 12 are mixing angles.
Solution of \( \tilde{u}_1^+ \tilde{u}_1 + \tilde{u}_2^+ \tilde{u}_2 = 1 \)

\[
\begin{bmatrix}
\tilde{c}_1 \\
\tilde{c}_2 \\
\tilde{c}_3 \\
\tilde{c}_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{d}_1 \\
\tilde{d}_2 \\
\tilde{d}_3 \\
\tilde{d}_4 \\
\end{bmatrix}
\]

Now the \((1,1)\) element is secure

The first column needs to have square less than 1

6 x 6 unitary matrix has \( \frac{6 \cdot 5}{2} = 15 \) angles

\[
\begin{bmatrix}
\tilde{c}_1 \\
\tilde{c}_2 \\
\tilde{c}_3 \\
\tilde{c}_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{d}_1 \\
\tilde{d}_2 \\
\tilde{d}_3 \\
\tilde{d}_4 \\
\end{bmatrix}
\]

For real matrices: \( \tilde{u}_1 \) and \( \tilde{u}_2 \) we have 12 parameter and 6 constants

So 12 parameter can show up.