A side note on symmetries:

- Coleman Mandula theorem (S matrix) (1967)
  - interactions
  - Lie algebra (commutators)
  - mass gap

  $\Rightarrow$ Poincaré $\times$ internal symmetries

- Haag – Lopuszanski – Sohnius theorem (1975)
  - allowed for anticommutators
  - fermionic generators can only have spin 1/2
  - supersymmetry algebra

$$Q \,,\, Q^{+} \quad \text{then}$$

$$\{Q, Q\} = \{Q^{+}, Q^{+}\} = 0$$

$$\{Q, Q^{+}\} \propto P^{\mu}$$

$$[P^{\mu}, Q] = [P^{\mu}, Q^{+}] = 0$$

- Consequences: $Q|\text{Boson}\rangle = |\text{Fermion}\rangle$

  $Q|\text{Fermion}\rangle = |\text{Boson}\rangle$

$\Rightarrow$ P^{0} is the Hamiltonian ! the operators Q etc can be quite nontrivial

$$\begin{aligned}
\leadsto \quad P^{\mu}|\text{Fermion}\rangle &= P^{\mu} Q\, |\text{Boson}\rangle \\
&= Q\, P^{\mu}\, |\text{Boson}\rangle \\
&= Q\, p^{\mu}\, |\text{Boson}\rangle \qquad (\text{operator} \rightarrow \text{eigenvalue}) \\
&= p^{\mu}\, Q\, |\text{Boson}\rangle \\
&= p^{\mu}\, |\text{Fermion}\rangle
\end{aligned}$$

$p^{2} = -m^{2} \Rightarrow$ mass the same

- For a given multiplet with $p_\mu \neq 0$ $n_B = n_F$

$\rightarrow$ so it does not have to be true for the vacuum !

for s the spin

$$\mathrm{Tr}\left((-1)^{2s}\right) = n_B - n_F \quad \text{in the multiplet}$$

$$\sum_i \langle i | (-1)^{2s} P^\mu | i \rangle \quad \propto$$

$$\sum_i \langle i | (-1)^{2s} (QQ^\dagger + Q^\dagger Q) | i \rangle =$$

$$\sum_i \langle i | (-1)^{2s} QQ^\dagger | i \rangle + \underbrace{\sum_i \langle i | (-1)^{2s} Q^\dagger \, Q | i \rangle}_{A}$$

$\mathbb{1}$ $\qquad \underbrace{\sum_j |j\rangle\langle j| = \mathbb{1}}_{\text{full multiplet}}$

$$A = \sum_i \sum_j \langle i | (-1)^{2s} Q^\dagger | j \rangle \langle j | Q | i \rangle$$

$$= \sum_i \sum_j \langle j | Q | i \rangle \langle i | (-1)^{2s} Q^\dagger | j \rangle$$

is $\mathbb{1}$

$$= \sum_j \langle j | Q (-1)^{2s} Q^\dagger | j \rangle$$

$$= - \sum_j \langle j | (-1)^{2s} Q Q^\dagger | j \rangle$$

[Q changes boson to fermion + vice versa hence

$$Q(-1)^{2s} = - (-1)^{2s} Q$$

so the 2 terms cancel.

Now all states have the same $P^\mu |i\rangle = p^\mu |i\rangle$

So if $p^\mu \neq 0 \Rightarrow \sum_i \langle i | (-1)^{2s} | i \rangle = 0$

Some more comments:

- We want a renormalisable theory
  - massless vector bosons (spontaneous later) : gauge bosons
  - fermions
  - real/complex scalars

spin $\geq 3/2$ is not renormalisable

- $N=1$ Susy only
  otherwise 2 steps
  Q_1, Q_2 1 Fermion

$$\begin{array}{c} 1 \\ \downarrow \\ 1/2 \\ \downarrow \\ 0 \end{array}$$

$\underline{\underline{\text{real}}}$ adjoint
$\downarrow$ $Q, Q^\dagger$ give left + right $\Rightarrow$ not chiral

$$\left.\begin{array}{ll} 1/2 & \text{left} \\ \downarrow & \\ 0 & \\ \downarrow & \\ 1/2 & \text{right} \end{array}\right\}$$ also not ~~scalar~~ chiral

The standard model is chiral $\Rightarrow$ we do not consider it

The simplest Susy model

Complex scalar, Weyl fermion

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi + i \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi$$

try $\delta\phi = \varepsilon\psi$ $\qquad \delta\phi^* = \varepsilon^\dagger \psi^\dagger$ $\qquad \Rightarrow \varepsilon$ has dimension $-\frac{1}{2}$

$$\delta\mathcal{L}_\phi = -\partial^\mu\phi^* \varepsilon \partial_\mu \psi - \varepsilon^\dagger \partial_\mu \psi^\dagger \partial^\mu \phi$$

$\Rightarrow \delta\psi^\dagger \propto \varepsilon \partial_\mu \phi^*$ & $\delta\psi \propto \varepsilon^\dagger \partial^\mu \phi$

$$\delta\psi = \alpha\, \sigma^\mu \varepsilon^\dagger \partial_\mu \phi$$

↳ only option that has a μ and no more derivatives

$$\delta\psi^\dagger = \alpha^* \varepsilon \sigma^\mu \partial_\mu \phi^*$$

Then $\delta\mathcal{L}_\psi = i\alpha^* \varepsilon \sigma^\nu \bar{\sigma}^\mu \partial_\mu \psi \partial_\nu \phi^* + i\alpha \psi^\dagger \bar{\sigma}^\mu \sigma^\nu \varepsilon^\dagger \partial_\mu \partial_\nu \phi$

Allow for partial derivatives for the moment, then both terms are symmetric ~~since~~ in $\mu \leftrightarrow \nu$ since $\partial_\mu \partial_\nu \phi = \partial_\nu \partial_\mu \phi$

$$= -\frac{2}{2} i \alpha^* \varepsilon \partial_\mu \psi \partial^\mu \phi + \frac{2}{2} i \alpha\, \partial_\mu \psi^\dagger \varepsilon^\dagger \partial^\mu \phi$$

so it vanishes for $\alpha = -i$ up to the total derivative parts

Exercise: are the mass terms OK?

$$\mathcal{L} = -m^2 \phi^* \phi - \frac{m}{2}(\psi\psi + \psi^\dagger \psi^\dagger)$$

$$\delta \mathcal{L}_\phi = -m^2 \varepsilon^\dagger \psi^\dagger \phi - m^2 \phi^* \varepsilon \psi$$

$$\delta \mathcal{L}_\psi = \frac{+m}{2} \delta(\psi_\alpha \varepsilon^{\alpha\beta} \psi_\beta) + hc$$

$$= \frac{+m}{2}(-i)(\sigma^\mu \varepsilon^\dagger)_\alpha \partial_\mu \phi \, \varepsilon^{\alpha\beta} \psi_\beta$$

$$+ \frac{+m}{2}(-i)\, \psi_\alpha \varepsilon^{\alpha\beta} (\sigma^\mu \varepsilon^\dagger)_\beta \partial_\mu \phi \qquad + h.c.$$

Now use the ψ equation of motion and $\chi \sigma^\mu \xi^\dagger = -\xi^\dagger \bar{\sigma}^\mu \chi$

$$i \bar{\sigma}^\mu \partial_\mu \psi - m \psi^\dagger = 0$$

$$\mathcal{L}_\psi = -m \, i \partial_\mu \phi \, \varepsilon^\dagger \bar{\sigma}^\mu \psi + h.c$$

using partial integration and the ψ equation this becomes

$$= m^2 \phi \varepsilon^\dagger \psi^\dagger + h.c.$$

so it is possible.

$$\{Q, Q^\dagger\} \propto P^\mu$$

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\phi$$

$$= \delta_{\varepsilon_1} \varepsilon_2 \psi - \delta_{\varepsilon_2} \varepsilon_1 \psi$$

$$= -i\, \varepsilon_2 \sigma^\mu \varepsilon_1^\dagger \partial_\mu \phi + i\, \varepsilon_1 \sigma^\mu \varepsilon_2^\dagger \partial_\mu \phi$$

$$= \left(-\varepsilon_2 \sigma^\mu \varepsilon_1^\dagger + \varepsilon_1 \sigma^\mu \varepsilon_2^\dagger\right) i \partial_\mu \phi$$

So on ϕ it goes

What on ψ

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi = \delta_{\varepsilon_1}\left(-i \sigma^\mu \varepsilon_2^\dagger\right)\partial_\mu \phi + \delta_{\varepsilon_2}\, i\left(\sigma^\mu \varepsilon_1^\dagger\right)\partial_\mu \phi$$

$$= -i\, \sigma^\mu \varepsilon_2^\dagger \left(\varepsilon_1 \partial_\mu \psi\right) + i\, \sigma^\mu \varepsilon_1^\dagger \left(\varepsilon_2 \partial_\mu \psi\right)$$

using $\chi(\xi\eta) = -\xi(\eta\chi) - \eta(\xi\chi)$

$$= i\left(\varepsilon_1 \sigma^\mu \varepsilon_2^\dagger\right)\partial_\mu \psi - i\left(\varepsilon_2 \sigma^\mu \varepsilon_1^\dagger\right)\partial_\mu \psi$$

$$+ i\left(\partial_\mu \psi \sigma^\mu \varepsilon_2^\dagger\right)\varepsilon_1 - i\left(\partial_\mu \psi \sigma^\mu \varepsilon_1^\dagger\right)\varepsilon_2$$

$$= \left(-\varepsilon_2 \sigma^\mu \varepsilon_1^\dagger + \varepsilon_1 \sigma^\mu \varepsilon_2^\dagger\right) i \partial_\mu \psi \qquad \text{OK}$$

$$+ \left(-i\, \varepsilon_2^\dagger \bar\sigma^\mu \partial_\mu \psi\right)\varepsilon_1 + i\left(\varepsilon_1^\dagger \bar\sigma^\mu \partial_\mu \psi\right)\varepsilon_2$$

used the $\psi \sigma^\mu \chi^\dagger = -\chi^\dagger \bar\sigma^\mu \psi$ here

on the last term use equations of motion to get them zero (massless case)

~~$-i(\varepsilon_2^\dagger \psi^\dagger)\varepsilon_1$~~

Why do we end up having to use the equations of motion?

on-shelf $\quad \phi$: 2 dof $\quad \psi$: 2 dof

off-shell $\quad \phi$: 2 dof $\quad \psi$: 4 dof (2 complex fields)

The single derivative is responsible for this problem (in time)
eg the Schrödinger wave function is complex but has only one dof

Add 2 degrees of freedom that vanish on-shell so "$F \propto \bar{\sigma}^\mu \partial_\mu \psi$"
via a scalar field (auxiliary) F.

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi + i\, \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^* F$$

$$\delta \phi = \varepsilon \phi \qquad \delta \phi^* = \varepsilon^\dagger \phi^\dagger$$

$$\delta \psi = -i\, \sigma^\mu \varepsilon^\dagger \partial_\mu \phi + \varepsilon F \qquad \delta \psi^\dagger = i\, \bar{\sigma}^\mu \varepsilon\, \partial_\mu \phi^* + \varepsilon^\dagger F^*$$

$$\delta F = -i\, \varepsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \qquad \delta F^* = i\, \sigma^\mu \partial_\mu \psi^\dagger \bar{\sigma}^\mu \varepsilon$$

1) The lagrangian is invariant

2) For all field $\quad [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] X = (\varepsilon_1 \sigma^\mu \varepsilon_2^\dagger - \varepsilon_2 \sigma^\mu \varepsilon_1^\dagger)\, i \partial_\mu X$

for $\quad X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$