\[ \text{large } N_c \]

\[ 't \text{ Hooft, Nucl. Phys. B 72 (1974) 461 (large } N_c \) \]
\[ 't \text{ Hooft, Nucl. Phys. B 75 (1974) 461 (20 QCD large } N_c \) \]

\[ \text{Veneziano, Witten} \]
\[ \text{E. Witten, Nucl. Phys. B 160 (1979) 51} \]

\[ \text{Coleman, I/N, Eric Lerner Lecture 1979 } \]
\[ \text{SLAC-PUB-2484 (1980) "classic"} \]

\[ \text{Manohar, large } N_c, \text{ QCD 1997 Les Houches lecture } \]
\[ \text{ hep-ph/9802419 "new view"} \]

Notice I will neglect many small terms in this of course.

Step 1: \[ N_c \to \alpha \]

Step 2: \[ SU(N_c) \to U(N_c) \]

\[ \sum_a \sum_{ij} \frac{T^a}{2} \sum_{kl} \frac{T^a}{2} = \frac{1}{2} \left( \frac{1}{N_c} \sum_{ij} S_{ij} S_{kl} - \frac{1}{N_c} \sum_{ij} S_{ij} S_{kl} \right) \]

The extra \( U(1) \) part is \( 1/N_c \) suppressed.

Step 3: What to keep at \( \mu \) in QCD

\[ \mu \frac{d}{d \mu} = - b_0 \frac{N_c}{16 \pi^2} \frac{3}{3} + \frac{b_0}{3} N_c \frac{2}{3} N \frac{F}{3} \]
So: $N_f$ becomes less important: gluons dominate

So we choose as limit: $g_s = \frac{g}{\sqrt{N_c}}$ and keep $g$ constant

We now formally "add" and "subtract" a $U(1)$ gauge boson.

So $A_\mu$ becomes a full $N\times N$ hermitian matrix.

Note the $U(1)$ does not have self interactions, there always come with commutators.

All self interactions are spinless twas so use a double line notation:

\[ \bar{\psi} \quad A \quad \psi \quad \rightarrow \quad \text{quark} \]

\[ \rightarrow \quad \text{k}(N) \text{ gluon} \]

\[ \rightarrow \quad \text{k}(1) \text{ gluon (but no color sum implied)} \]

3 vector term $A_\mu \left[ A_\mu, A_\nu \right]$
\[ \langle T \left( \bar{q}^A \gamma_\mu q(x) \bar{q}^B \gamma^\mu q(y) \right) \rangle \] and let's do an expansion for this.

\[ \frac{g^2}{N_c^2} \frac{N_c^2}{N_c} = N_c \quad \frac{g^2}{N_c} \]

\[ \frac{g^4}{N_c^2} \frac{N_c^3}{N_c} = N_c \]

\[ \frac{g^4}{N_c^2} \frac{N_c^2}{N_c} = \frac{1}{N_c} \]

\[ \frac{g^4}{N_c^2} \frac{N_c^2}{N_c} = 1 \]

\[ \frac{g^4}{N_c^2} = N_c^2 \]

\[ \Rightarrow \text{Planar graphs are leading} \]
So a graph is

\[ N^a : a \quad _{N_c}^{c=2} - \# \text{ holes} - 2\# \text{ handles} \]

\[ \Rightarrow \text{ similar to string theory / dual model results.} \]

Confinement assumed

\[ \Rightarrow \text{ gluon propagator } (\Gamma) \text{ gluon operator is } \frac{1}{q^2} \frac{1}{\sqrt{N_c}} \]

\[ \Rightarrow \text{ gluon vertex} \quad \Rightarrow \frac{1}{(\sqrt{N_c})^3} N_c \quad \Rightarrow \frac{1}{\sqrt{N_c}} \]

\[ \Rightarrow \text{ goes to zero in large } N_c \text{ limit} \]

\[ \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \]

[Diagrams and equations]

\[ \Rightarrow \text{ number of mesons} \]

\[ \Rightarrow \text{ glueballs - a number} \quad \frac{1}{N_c} \]

\[ \Rightarrow \text{ glueballs / quark-anti-quark} \]

[Diagram showing the relationship between glueballs and quark-anti-quark mesons]
Weig's rule automatically follows

\[ S \sim \frac{1}{\sqrt{N_c}} \]

but

\[ N \sim \frac{g^2 N_c^2}{N_c^2} \frac{1}{\sqrt{N_c^{1.3}}} \sim \frac{1}{N_c^{1.1}} \]
Baryons are more complicated

\[ N_c \text{ quarks} \rightarrow e^{q_1 \ldots q_{N_c}} \]

\[ m_B \sim N_c m_0 + \sum_{i \neq j} V(x_i - x_j) + \ldots \]

\[ N_c^2 \times \frac{g^2}{N_c} \text{ also order } N_c \]

\( \sim \) Hartree approximation is exact in large \( N \) limit (Witten)

Baryon Mass is \( \sqrt{N_c} \)

Baryon, Baryon is \( N \)

\( BMB^x \) is \( \sqrt{N'} \)

18 ad
121
1221
\[ \langle B | \bar{q} \gamma^{\mu} \gamma_5 T^a q | B \rangle = g N \langle B | x^{\mu a} \bar{l}B \rangle \]

\[ \frac{\partial^{\mu a}}{\partial x^{\mu}} \langle B | \bar{q} \gamma^{\nu} \gamma_5 T^a q | B \rangle \quad \text{in} \quad \frac{1}{N} \text{ large coupling} \]

\[ \pi B \to \pi B \quad \text{in} \quad \text{Order} \quad N \]

\[ \text{But} \quad \nu \equiv \text{O}(N^2) \]

\[ \Rightarrow \quad [X^{\mu a}, X^{\nu b}] = 0 \]

\[ [J^a, X^{\nu b}] = \frac{i}{2} \epsilon_{ijk} x^{\nu b} \text{ unrem. spin} \]

\[ [T^a, X^{\nu b}] = \frac{i}{f} \epsilon_{abc} x^{\nu c} \text{ unrem. flavour} \]

\[ \Rightarrow \text{almost on} \quad SU(2N_F) : \quad J^a, T^a, G^{\mu a} \]

\[ [G^{\mu a}, G^{\nu b}] = \frac{i}{2N_F} \epsilon_{ijkl} s_{\mu \nu} J^b + \frac{i}{2} \epsilon_{ijkl} \epsilon_{abc} G^{\mu c} \]

\[ x^{\mu a} = \lim_{N \to \infty} \left( \frac{G^{\mu a}}{N} \right) \]

Lie algebra contraction - Giromini, Seoul
The unitarity triangle

This is an overview of where the different elements of the Kobayashi-Maskawa mixing matrix are determined.

Dimensions of the difficulties that appear in each of the various processes and the errors appear later when we talk about semi-leptonic and non-leptonic decays.

As shown in section 1 we have a mixing matrix with a non-trivial phase. This can be seen in a parametrization invariant way by looking at the Jarlskog determinant

$$\det [M_u, M_D] = \Delta_1^2 \Delta_2 \Delta_3^2 \prod_{i,j} (m_{ui} - m_{uj})(m_{ji} - m_{ij})$$

so no CP violation if

$$\delta = 0, \quad \delta_i = 0 \text{ or } \frac{\pi}{2} \text{ for one } i$$

or 2 up or 2 down masses equal (in the latter case we cannot determine one of the mixing angles, so can be set to 0 or \(\frac{\pi}{2}\)).

For more than 3 generations it is sufficient if this condition is satisfied for any one of the 3x3 submixings.
$V_{ub} : \pi^+ \rightarrow \pi^0 e^+ \nu$  
$\beta$ decay of nuclei \( x \)  
0.3736 $\pm$ 0.0010

$V_{us} : \text{hyperon } \beta\text{ decay }$  
$K \rightarrow \pi^1 \nu^x$  
0.2205 $\pm$ 0.0018

This gives a first constraint from Unitarity:

\[ |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \]

\[ \Rightarrow |V_{ub}| \lesssim 0.055 \]

$V_{cd} : \text{Charm production in neutrino scattering}$  
0.224 $\pm$ 0.014

Unitarity $\Rightarrow |V_{us}|$

$V_{cs} : c \rightarrow s$ decays or Charm production of the strange sea in neutrino scattering (4.01 $\pm$ 0.18)

$V_{ub} : b \rightarrow u$ charmless $B$ decays

\[ |V_{ub}/V_{cb}| \approx 0.08 \pm 0.02 \]

$V_{cb} : B$ lifetime similar unitarity bound to $|V_{ub}|$; notice that in this case the bound is almost saturated.

\[ |V_{cb}| \approx 0.04 \pm 0.003 \]

$V_{tb} : \text{in principle in top decay } t \rightarrow b W^+$

On $V_{cd}$ and $V_{cs}$ in the near future there will only be indirect constraints.

So how does unitarity work:

$V_{ud}, V_{us}$ & $|V_{ub}|, V_{cb}$ experimentally well bound  
$\Rightarrow$ this determines essentially $V_{cd}$ and $V_{cs}$ and $V_{tb}$ $|V_{cd}|$ and $V_{tb}$ also constrain experimentally only be phase $S$ is little determined; need CP violating observables.
in Wolfenstein: $\lambda$ from $K \to \pi \ell \nu$

$A$ from $b \to c$

$\frac{1}{\sqrt{s^2 + \eta^2}}$ from $b \to u$

$$V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} = 0$$

in the unitarity triangle.

$$1 - \lambda$$

$-\lambda$

in a sum of 3 complex numbers vanishes $\Rightarrow$ the form a closed triangle in the complex plane.

$$R_b = \left| \frac{V_{ub}^* V_{cd}}{V_{cb}^* V_{td}} \right| = \sqrt{s^2 + \eta^2}$$

$$R_t = \left| \frac{V_{td}^* V_{cb}}{V_{cd}^* V_{td}} \right| = \sqrt{(1-s)^2 + \eta^2}$$

Notice that all terms in (c) are order $\lambda^3$.

The length of the sides can in principle be measured from non CP-violating quantities.

A 3rd constraint comes from any CP-violating quantity (so far only $\xi_{4024}$)

$$\text{Im} \left| V_{td} \right|^2 \propto \eta (1-s)$$

gives a hyperbolic constraint.

$$m_\xi = 124 GeV$$

$R_b : B_0 \to \bar{B}_0$ mixing.
Effective field theory / Matching

A fundamental field theory:
- renormalizable: to have predictions at all orders
- well defined
- expansions are typically in small coupling constants

An effective field theory:
- can be nonrenormalizable
- calculational tool and/or because we cannot do better
- degrees of freedom can be very different from underlying theory
- Goldstone theorem by Weinberg
- field redefinitions can make theories look very different
- expansions can be in other quantities than coupling constants
- power counting
- using dimensions in constructing effective Lagrangians

Matching: integrating out (often in principle, not in practice)
- best way: calculate observables in both theories
  put equal \( \Rightarrow \) match parameters in both theories
  (remember limits here!)
- examples:
  - QCD \( \eta \) \( \Rightarrow \) \( \eta' \) quark: thresholds
  - W exchange versus fermi interaction
    (\( \eta \) vs at one loop)
  - linear versus nonlinear sigma model
  - QED \( \Rightarrow \) 1/QET
The meaning of loops and renormalization in effective theories

In this part I will first give a heuristic description of the meaning of loops in nonrenormalisable theories. I will explain this difference using the example of an effective theory with quarks only and an underlying theory with quarks and W-bosons. Since here I want to show how to deal with the infinite parts I will not explicitly calculate diagrams nor introduce complications like GIM-mechanism, weak mixing angles, QCD corrections,...

The decay of an s-quark is given by

\[ \frac{1}{8} \left( \bar{u} \gamma \left( 1 - Y_5 \right) u \right) \left( \bar{u} \gamma \left( 1 - Y_5 \right) d \right) \frac{i}{2} \frac{g_w^2}{\sin^2 \theta_W - m_W^2} \]  

(1)

This is given by the amplitude:

\[ \left( \bar{u} \gamma \left( 1 - Y_5 \right) u \right) \left( \bar{u} \gamma \left( 1 - Y_5 \right) d \right) \frac{i}{2} \frac{g_w^2}{\sin^2 \theta_W - m_W^2} \]  

(2)

At low energies we describe this by a Lagrangian which only has quarks. This we do via the Fermi interaction of the type

\[ \mathcal{L}_{\text{eff}}^{(2)} = \frac{G_F}{\sqrt{2}} \left( \bar{u} \gamma \left( 1 - Y_5 \right) u \right) \left( \bar{u} \gamma \left( 1 - Y_5 \right) d \right) \]  

(3)

Comparing the amplitudes that follow from (3) with the one that follows from diagram (1) at low energies we obtain

\[ G_F = -\frac{g_w^2}{4 \sqrt{2} m_W^2} \]  

(4)

So: at low energies (1) and (3) are the same and the parameters in \( \mathcal{L}_{\text{eff}}^{(2)} \) can be determined by calculating the process both in the underlying
theory and in the effective theory and setting both results to be equal.
This gave (4) as the result.
Let us now do the same for $K^0\bar{K}^0$ mixing. This is the process where
\[ \bar{s} d \rightarrow s \bar{d} \] and proceeds via the diagram
\[ \begin{array}{c}
\bar{s} \\
\uparrow \\
W^+ \\
\downarrow \\
\bar{d}
\end{array} \quad \begin{array}{c}
\bar{d} \\
\uparrow \\
W^- \\
\downarrow \\
s
\end{array} \quad \text{crossed}
\] (5)
At low momenta, the amplitude is
\[ A = b \int \frac{d^4p}{(2\pi)^d} \frac{p^2}{(p^2)^2 (p^2 - m_w^2)^2} + \text{cround} = i \epsilon \frac{G_F^2}{\Lambda^2} \] (6)
so the result is finite and well defined. Now what happens if we try to calculate this process using our effective theory (3). We then have the Feynman diagrams
\[ \begin{array}{c}
\bar{s} \\
\uparrow \\
\bar{d} \\
\downarrow \\
s
\end{array} \quad \begin{array}{c}
\bar{d} \\
\uparrow \\
\bar{s} \\
\downarrow \\
s
\end{array} \quad \text{crossed}
\] (7)
which gives an amplitude:
\[ A = b' \int \frac{d^4p}{(2\pi)^d} \frac{p^2}{(p^2)^2 (p^2 - m_w^2)^2} + \text{cround} = i(\epsilon' \Lambda^2 + \epsilon'') \frac{G_F^2}{\Lambda^2} \] (8)
This integral is quadratically divergent! So how can we describe $K^0\bar{K}^0$ mixing in our effective theory? ($\Lambda$ is the cut-off)
The solution is that we forget the possibility of more terms in our effective Lagrangian. We restrict the Lagrangian to terms with only quarks
so we can add to (3):
\[ L^{(1)}_{\text{eff}} \rightarrow L^{(1)}_{\text{eff}} + \alpha \begin{bmatrix} \bar{s} \gamma_\mu (1 - g_5) s \\ \bar{d} \gamma_\mu (1 - g_5) d \end{bmatrix} \] (9)
So now the amplitude in our effective theory is:

\[ i \left( a + (c' + c'')G^2_F \right) = i \cdot c \cdot \frac{G^2_F}{r} \]  

\[ \text{(10)} \]

The equality follows because our effective theory should reproduce the more fundamental underlying theory.

So we now set \( a = a_{\text{inf}} + a_{\text{r}} \) with \( a_{\text{inf}} = -c' \cdot \lambda^2 \) so we obtain

\[ a_{\text{r}} + c''G^2_F = c \cdot \frac{G^2_F}{r} \]  

\[ \text{(11)} \]

So what have we done:

- we included all allowed terms in our effective Lagrangian
- the infinite parts that occur in the loop diagrams are absorbed into the coefficients of the new terms, these occur at higher orders in \( G^2_F \)
- from our effective theory alone we could only determine \( K^0 - \bar{K}^0 \) mixing up to the free parameter \( a_{\text{r}} \).

In the remainder of this part I will prove that a similar procedure can be carried out for the case of chiral perturbation theory. There we will not do an expansion in \( G^2_F \) but do an expansion in energies, momenta and masses. These we all denote by \( p \).

As shown in the previous section the lowest order is order \( p^2 \). So what is the next order? We only have mesons so Lorentz invariance requires the next order to be \( p^4 \) in the effective Lagrangian. What happens now with one loop diagrams? Take

\[ \text{(12)} \]