Connection with QCD

The Lagrangian of QCD coupled to external fields is given by :

$$\mathcal{L}_{QCD} = \frac{1}{4}\left(F_{\mu\nu}\right)^2 + (\bar{u}\ \bar{d}\ \bar{s})\, i\gamma^\mu \left(\partial_\mu - i\frac{g_s}{2}G_\mu^a\lambda^a - i\ell_\mu\gamma_- - ir_\mu\gamma_+\right) \times \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$$- \bar{q}\,\tilde{M}\,\gamma_-\,q - \bar{q}\,\tilde{M}^\dagger\,\gamma_+\,q \,. \qquad (1)$$

$\bar{q} = (\bar{u}\ \bar{d}\ \bar{s})$; λ^a the Gell-Mann matrices; G_μ the gluon field ; ℓ_μ, r_μ the left handed and right handed external vector field ; $\tilde{M} = s - ip$; s, p scalar and pseudoscalar external field ; $\gamma_\pm = \frac{1}{2}(1 \pm \gamma_5)$

$\mathcal{L}_{QCD}$ is invariant under the following $SU(3)_L \times SU(3)_R$ transformation

$$q \to (g_L\gamma_- + g_R\gamma_+)\,q$$

$$\ell_\mu \to g_L\,\ell_\mu\,g_L^\dagger + i\,g_L\,\partial_\mu g_L \qquad r_\mu \to g_R\,r_\mu\,g_R^\dagger + i\,g_R\,\partial_\mu g_R$$

$$\tilde{M} \to g_R\,\tilde{M}\,g_L^\dagger \qquad (2)$$

We now want to calculate matrix elements of pions etc... . This can be done by evaluating the effective action Γ as a function of the external fields :

$$Z(\ell_\mu, r_\mu, s, p) = e^{i\Gamma} = \int [dq][dG]\ e^{i\int d^4x\,\mathcal{L}_{QCD}} \qquad (3)$$

and determining from Γ the Green's functions.
S-matrix elements are then given by the residues of the poles of the Greens' functions.
As an example, to calculate π-π scattering:
set

$$l_\mu = v_\mu - a_\mu \quad ; \quad r_\mu = v_\mu + a_\mu \tag{4}$$

and look at the $\langle a_\mu(x)\, a_\mu(0)\rangle$ two point function. This will have a pole at $p^2 = m_\pi^2$ and be of the form $\frac{Z_\pi}{p^2 - m_\pi^2}$ near the pole.
The calculate the four point function $\langle a_\mu(x_1)\, a_\nu(x_2)\, a_\lambda(x_3)\, a_\beta(x_4)\rangle$ and look for the poles in all 4 legs. Then

$$A(\pi\pi\to\pi\pi) = \frac{(p_1^2 - m_\pi^2)(p_2^2 - m_\pi^2)(p_3^2 - m_\pi^2)(p_4^2 - m_\pi^2)}{Z_\pi^2} \langle (a_\mu)^4 \rangle \tag{5}$$

Note: This is the general procedure. For the amplitudes up to one loop as we will calculate them we will use a simplified procedure that is correct to that order.

Using our general assumptions at low energies everything is now dominated by the Goldstone Bosons of $SU(3)_L \times SU(3)_R$. We believe for several reasons that this symmetry group is broken by quark VEV (vacuum expectation values) down to $SU(3)_{V=L+R}$.

a) lattice QCD calculations give $\langle \bar{q} q \rangle \neq 0$
b) $SU(3)_L \times SU(3)_R$ is broken in the spectrum: no parity doublets
c) 't Hooft's anomaly conditions require massless fermions for unbroken $SU(3)_L \times SU(3)_R$
d) Vafa and Witten have shown that $SU(3)_V$ remains unbroken
e) large N_c: either unbroken $SU(3) \times SU(3)$ or broken to $SU(3)_V$
f) There is set of light pseudoscalars to play the Goldstone Bosons.

So taking that believe and our general assumptions we rewrite (3) as

$$Z(v_\mu, a_\mu, s, p) = \int [dU]\; e^{i\int d^4x\, \mathcal{L}_{eff}(U)} \tag{6}$$

where $U = \Sigma^2 \to g_R U g_L^\dagger$ and a parametrization

$$U = \exp \frac{\sqrt{2}\, i}{F} \begin{pmatrix} +\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \tag{7}$$

Where we can now again in general evaluate Γ and by taking the parts with external poles determine the S-matrix elements.

This is what was done in the original papers by Gasser and Leutwyler where (6) was evaluated using a background field gauge and the heat kernel expansion. The reason for doing it that way is that that procedure keeps the chiral symmetry exact everywhere.
We will use the simplified procedure of calculating by putting (7) into $\mathcal{L}_{eff}$ and introducing source terms directly for $\pi^\pm, \pi^0, \ldots$ to calculate S-matrix elements. This procedure is consistent to one loop. It breaks the symmetry explicitly so at higher loops compensating terms need to be introduced and there the power of the general procedure becomes more apparent.

Note: the quark masses can be treated by giving the field s a value

$$\tilde{M} = s = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

even in the absence of extra external fields s and p.

let me explain the last part somewhat better :

what we obtain from our procedure is

$$Z\,(v_\mu, a_\mu, s, p)_{\text{massless QCD}}$$

but in the QCD lagrangian we can absorb the masses formally in s and obtain an expression that is identical to that of massless QCD where the scalar external field with the current quark masses :

$\hat{\ }$ has been modified

$$Z(v_\mu, a_\mu, s, p)_{\text{QCD with } m_q \neq 0} = Z\left(v_\mu, a_\mu, s + \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, p\right)_{\text{massless QCD}}$$

Lowest order

There is no term that only depends on U and is invariant under $U \to g_R U g_L^\dagger$
(remember $\operatorname{tr} U^\dagger U = \operatorname{tr} 1$ is constant)

The lowest order terms are:

$$\mathcal{L}_2 = \frac{F^2}{4} \operatorname{tr} D_\mu U D^\mu U^\dagger \quad ; \quad D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu$$

$$+ \frac{F^2}{4} \operatorname{tr} (\chi U^\dagger + U \chi^\dagger) \quad ; \quad \chi = 2 B_0 \tilde{M}$$

We can derive the equations of motion via $\delta U = e^{i \delta T^a} U$

or $$\operatorname{tr} T^a \{ D^2 U U^\dagger - U D^2 U^\dagger + U \chi^\dagger - \chi U^\dagger \} = 0$$

or $$D^2 U U^\dagger - U D^2 U^\dagger + U \chi^\dagger - \chi U^\dagger - \frac{1}{3} \operatorname{tr} (U \chi^\dagger - \chi U^\dagger) = 0$$

How do we determine B_0 and F?

i) expanding the mass term for the 2nd order in mesons we obtain

$$m_{\pi^+}^2 = (m_u + m_d) B_0$$
$$m_{K^+}^2 = (m_u + m_s) B_0$$

exercise 4: calculate the other masses and check that they obey the Gellmann-Okubo relation

$$m_\eta^2 = \frac{4}{3} m_K^2 - \frac{1}{3} m_\pi^2$$

here set $m_u = m_d$

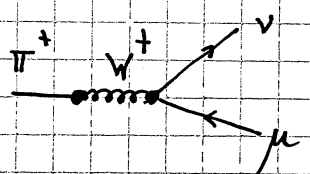
exercise 5: $m_u \neq m_d$ show that $m_{\pi^0}^2 = m_{\pi^+}^2 + O((m_u - m_d)^2)$
(use $m_s \to m_u, m_d$) this explains why isospin is such a good symmetry for the pions; not because $\frac{m_u - m_d}{m_u + m_d}$ is small.

ii) for F : the weak gauge group is $SU(2)_L$ and is embedded in $SU(3)_L$. So the W boson is in the external field l_μ. This can be included simply via

$$l_\mu = \frac{e}{\sqrt{2}\sin\theta_W}\begin{pmatrix} 0 & c_1 W^+ & s_1 W^+ \\ c_1 W^- & 0 & 0 \\ s_1 W^- & 0 & 0 \end{pmatrix}$$

$c_1 = \cos\theta_c$ $\quad s_1 = \sin\theta_c$ the Cabibbo angle.

$c_1 \simeq 1$ So we calculate



this leads to $F \simeq 93.2$ MeV (see exercises)

iii) B_0 and $\bar{q}q$

from QCD (3) we get $\langle \bar{q}^T q^a \rangle = \frac{1}{Z} i \frac{\delta}{\delta s^a} Z$

performing the same operation here leads to

$$\langle \bar{q}^T q^a \rangle = i\, i\, \frac{F^2}{4}\, 2B_0\, \mathrm{tr}(2T^a) + O(\pi^2)$$

or $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle = -B_0 F^2$

and we obtain the Gellman-Oakes-Renner relation

$$m_\pi^2 = -\frac{(m_u + m_d)\langle \bar{q}q \rangle}{F^2}$$

iv) **exercise 6** Calculate ~~$\pi\pi$~~ scattering to lowest order

$$\pi^1\pi^1 \to \pi^2\pi^2$$

Power counting

- Dimensional regularisation : no new scales
- $p^2 = 2\partial$ or one m_a or one s or one p or 2 $v, a, \ldots$
- Diagram has dimension

$$D = -2N_I + 4N_L + \sum_d d\, N_{Vd} \qquad d \geq 2 !$$

N_E external lines

loops = # free momenta after all lines are connected

so $N_L = N_I - \left(\sum_d N_{Vd} - 1\right)$ ← the remaining overall momentum conservation

$$\boxed{D = 2N_L + \sum_d (d-2) N_{Vd} + 2}$$

The basic underlying formula !

- For other EFT's one has to do a similar analysis

N_{V2} is limited by the number of external legs

- Note eg ϕ^4 dimension does not change : λ expansion!

Why external field method

i) $$Z(v,s,a,p)_{QCD} \simeq Z(v,s,a,p)_{CHPT}$$

$$\Downarrow$$

$$\frac{\delta}{\delta a_\mu}\frac{\delta}{\delta a_\mu} Z_{QCD} \simeq \frac{\delta^2}{\delta a_\mu \delta a_\mu} Z_{CHPT}$$

DIRECT IDENTIFICATION OF CHPT Green functions and QCD Green functions.

Also off-shell well defined!

ii) With the usual choice of π, K fields etc. no simple relationship to all Ward Identities.

Example: $K \to \pi W$

or the πK vector formfactor

For comparison with PCAC theorems:

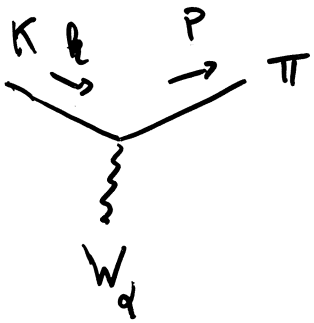
need to use the same pion source

(usually $\partial_\mu a^\mu$)

iii) Then results are obviously independent of pion field parametrization

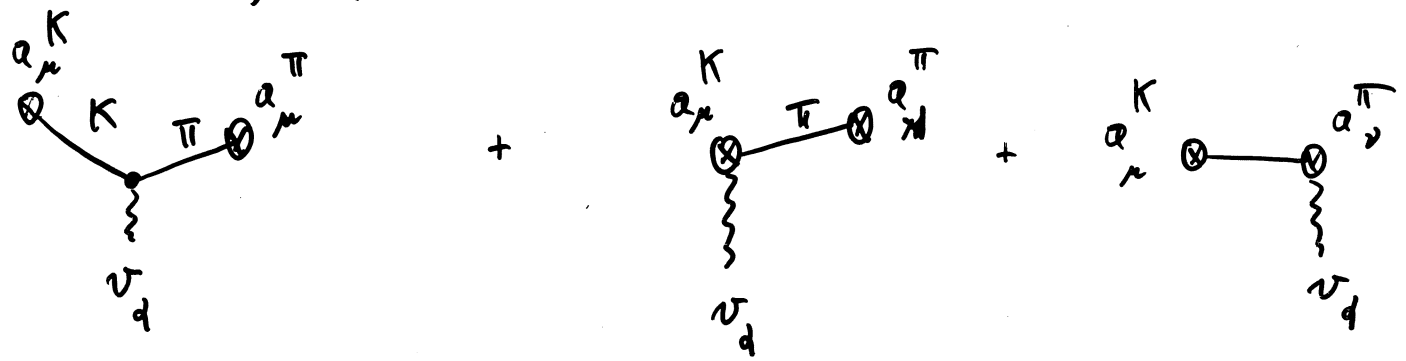
The promised example

$K \to \pi\, W$



$$A = \frac{i}{\sqrt{2}} (p_\pi + p_K)_q \; W^q$$

$$i^3 \langle T(a_\mu^K \; v_q \; a_\nu^\pi) \rangle$$



$$= i\sqrt{2} F^2 \left\{ \frac{(p_\pi + p_K)_q \; p_{\pi\mu} p_{K\nu}}{(p_\pi^2 - m_\pi^2)(p_K^2 - m_K^2)} - \frac{g_{\nu q} p_{\pi\mu}}{p_\pi^2 - m_\pi^2} - \frac{g_{\mu q} p_{K\nu}}{p_K^2 - m_K^2} \right.$$

It is this one that satisfies the Ward Identity for all values of p_π^2 and p_K^2

$$\partial_q^x \langle T(a_\mu^K(y) \; v_q(x) \; a_\nu^\pi(0)) \rangle \quad \propto \quad \langle T(a_\mu^K(y) \; a_\nu^K(x)) \rangle + \langle T(a_\mu^\pi(x) \; a_\nu^\pi(0)) \rangle$$

Similarly: only the Green functions defined with $\partial_\mu a^\mu$ satisfy off-shell current algebra relations

Why dimensional regularization ?

i) Naive powercounting works:

there are no powers of Λ^2 or M^2 to make life difficult

ii) Weinbergs Theorem: divergences are local

and dim. reg. keeps χ symmetry

$\Rightarrow$ Divergences are like local terms in the Lagrangian.

We include all of those anyway!

$\Rightarrow$ No difficulty with extra counterterms to keep the symmetry!

iii) The integrals are often easier than in a cut-off version

(eg Lorentzinvariance true for all $d = 4 - 2\varepsilon$, only true for $\Lambda \to \infty$)

CHPT to $O(p^4)$

So to go to next-to-leading order we need to calculate one-loop diagrams and have all the terms that are compatible with the chiral symmetry included in the Lagrangian of dimension 4. This Lagrangian contains 13 terms for $N_f \geq 4$ flavour, 12 terms for 3 flavours and 10 for 2 flavours.

Of these 11, 10, 7 are observable with proceses that contain external pions. The relations between the 10 parameters and the 6 that remain in going from $3 \to 2$ parameters can be found in the paper by Gasser and Leutwyler in Nuclear Physics. The terms that appear are:

$$\begin{aligned}
\mathcal{L}_4 = \; & L_0 \; \mathrm{tr}\, D_\mu U D_\nu U^\dagger D^\mu U D^\nu U^\dagger \\
& + L_1 \left(\mathrm{tr}\, D_\mu U D^\mu U\right)^2 + L_2 \left(\mathrm{tr}\, D_\mu U D_\nu U^\dagger\right)^2 \\
& + L_3 \; \mathrm{tr}\left(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\right) \\
& + L_4 \; \mathrm{tr}\left(\chi U^\dagger + U \chi^\dagger\right) \mathrm{tr}\, D_\mu U D^\mu U^\dagger \\
& + L_5 \; \mathrm{tr}\, D_\mu U D^\mu U^\dagger \left(\chi U^\dagger + U \chi^\dagger\right) \\
& + L_6 \left[\mathrm{tr}\left(\chi U^\dagger + \chi^\dagger U\right)\right]^2 + L_7 \left[\mathrm{tr}\left(\chi U^\dagger - U \chi^\dagger\right)\right]^2 \\
& + L_8 \; \mathrm{tr}\, \chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger \\
& + (-i) L_9 \; \mathrm{tr}\left[l_{\mu\nu} D^\mu U^\dagger D^\nu U + r_{\mu\nu} D^\mu U D^\nu U^\dagger\right] \\
& + L_{10} \; \mathrm{tr}\left(l_{\mu\nu} U^\dagger r^{\mu\nu} U\right) \\
& + L_{11} \; \mathrm{tr}\left(l_{\mu\nu} l^{\mu\nu} + r_{\mu\nu} r^{\mu\nu}\right) + L_{12} \; \mathrm{tr}\, \chi \chi^\dagger
\end{aligned}$$

- L_0 can be reabsorbed into L_1 and L_2 for the case of 3 flavours

- There are no terms included with $D^2 U$ or $D_\mu \chi$
  1) to $O(p^4)$ $\mathrm{tr}(A D_\mu \chi)$ can always be related to terms like $\mathrm{tr}\, D_\mu A \chi$ by using partial integration and total derivatives do not contribute to the action
  2) the terms with $D^2 U$ can always be removed by redefining the pion field. As long as the transformation preserves the correct transformation of U under chiral symmetry it is a correct field that can be used to calculate S-matrix elements.

Exercise 7: 2 terms containing $D^2 U$ are

$$L_{13}\, \mathrm{tr}\left(\chi D^2 U^\dagger + D^2 U \chi^\dagger\right) + L_{14}\, \mathrm{tr}\, D^2 U D^2 U^\dagger$$

Show that these can be removed using a transformed field

$$U' = U + a D^2 U + b \chi - c U' \chi^\dagger U'$$

- $$l_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i [l_\mu, l_\nu]$$

  $$r_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i [r_\mu, r_\nu]$$

  are the field strengths of the external vector fields l_μ, r_μ

- The 12 parameters $L_{i=1,12}$ are sufficient to absorb all infinities that appear to this order. If we regulate all diagrams using dimensional regularization (this explicitly conserves all symmetries present avoiding a problem subtracting quadratic divergences) we can analogously to before define an infinite part and a finite part of the L_i.

$$L_i = L_i^R + \frac{\Gamma_i}{16\pi^2 \mu^{2\varepsilon}} \left\{ \frac{-1}{2\varepsilon} - \frac{1}{2} \left[\ln(4\pi) + \Gamma'(1) + 1 \right] \right\}$$

with Γ_i :

i	1	2	3	4	5	6	7	8	9	10	11	12
Γ_i	$\frac{3}{32}$	$\frac{3}{16}$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{11}{144}$	0	$\frac{5}{48}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{-1}{8}$	$\frac{5}{24}$

remember $\mu^\varepsilon = 1 + \varepsilon \ln \mu + O(\varepsilon^2)$

$\Gamma'(1) = \gamma$ the Euler-Lagrange constant

the extra finite terms included in the infinite parts is there for convenience, it corresponds roughly to $\overline{MS}$ used in QCD calculations.

- Only the sum of contributions from $\mathcal{L}_4$ and one loop diagrams is μ independent and finite.
- The L_i^r are just like F and B_0 in principle calculable from QCD. First attempts using lattice gauge theory have been done. We will determine them from phenomenology later.

Exercise 8 : calculate F_π or $W^+ \to W^+ \to \mu^+ \nu$ to $O(p^4)$

This is the most important exercise and is really part of the course

1) Calculate the wave function renormalization : $\mathcal{L}_2 + \mathcal{L}_4 \to m$
2) Calculate the remaining diagrams and vertices.
3) Check that the final result is finite given the $L_i = L_i^r + L_i^{inf}$ above.

1 loop calculation exercise 8

a inverse pion propagator at one loop

We also have to add wave function renormalization to have a properly normalized pion field.

This is done by setting $\pi_b = \sqrt{Z_\pi}\,\pi_{phys}$

So the full inverse pion propagator is:

$$+i\left(p^2 - m^2_{phys}\right) = +i\left(p^2 - B_0(m_u + m_d)\right)$$

$$+ \quad \text{(loop)}$$

$$+ \quad \text{(cross)} \qquad \times = \text{vertex from } \mathcal{L}_4$$

$$= \left(Z_\pi - 1\right) \times \left(\text{lowest order result}\right)$$

the left hand side is by definition the correctly normalized field.
so this will be used to determine Z_π

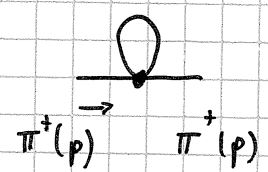
and once we have determined Z_π this will also give us m^2_{phys}

Let us work in the case $m_u = m_d$ to make life easier.

So a physical amplitude is given by: $\dfrac{1}{(\sqrt{Z_\pi})^n} A$ where

A is calculated without 1PI parts on the external legs and the 1PI parts of the propagators are taken into account by using as a propagator $\dfrac{i}{Z_\pi(p^2 - m^2_{phys})}$

So let us calculate the $\pi^+\pi^-$ inverse propagator to one loop:



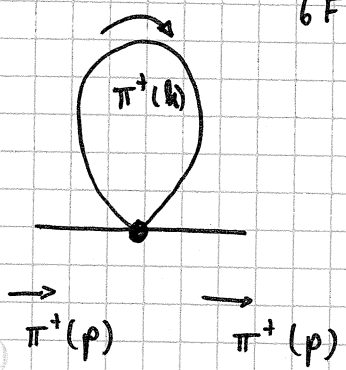
the loop can be π^+, K^+, π^0, η, K^0

so let me calculate the π^+ loop

We know the relevant term in the Lagrangian:

$$\mathcal{L} = \frac{1}{6F^2}\left\{(\partial_\mu \pi^+)^2(\pi^-)^2 + (\partial_\mu \pi^-)^2(\pi^+)^2 - 2\,\partial_\mu \pi^+ \partial_\mu \pi^- \pi^+\pi^-\right\}$$

$$+ \frac{1}{6F^2} B_0 (m_u + m_d)(\pi^+\pi^-)^2$$



$$: \quad i\int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m_\pi^2} \times \frac{1}{6F^2}$$

$$\times \left\{-4p\cdot k - 4p\cdot k - 2\left\{p^2 + p\cdot k + k^2 + k\cdot p\right\}\right.$$
$$\left. + 4B_0(m_u + m_d)\right\}$$

$$= \frac{-1}{6F^2}\int \frac{d^dk}{(2\pi)^d} \frac{-2p^2 - 12\,p\cdot k - 2k^2 + 4B_0(m_u+m_d)}{k^2 - m_\pi^2}$$

at this point we need to know what the relevant integrals are using dimensional regularization.

$$\int \frac{d^dk}{(2\pi)^d} \frac{k_\mu}{k^2 - m_\pi^2} = 0$$ because of reflection symmetry (change variables to $k'_\mu = -k_\mu$)

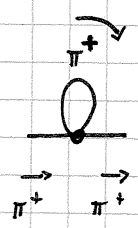
$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_\pi^2} = \frac{i\, m_\pi^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \log(4\pi) + \gamma + 1 - \log\frac{m_\pi^2}{\mu^2}\right)$$

we have set $d = 4 - 2\varepsilon$

and

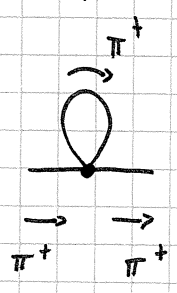
$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{k^2 - m_\pi^2} = m_\pi^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_\pi^2}$$

From here I use $\frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon} + \log(4\pi) + \gamma + 1$

So 

$$= \frac{i}{16\pi^2 F^2}\left[\frac{1}{3}p^2 + \frac{1}{3}m_\pi^2 - \frac{2}{3}B_0(m_u + m_d)\right] m_\pi^2\left(\frac{1}{\bar{\varepsilon}} - \log\frac{m_\pi^2}{\mu^2}\right)$$

This is a one-loop diagram so I can use $B_0(m_u + m_d) = m_\pi^2$ in this part

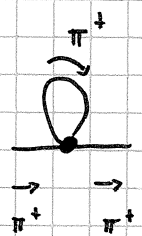


$$= \frac{i}{16\pi^2 F^2}\left(\frac{1}{3}p^2 - \frac{1}{3}m_\pi^2\right) m_\pi^2\left(\frac{1}{\bar{\varepsilon}} - \log\frac{m_\pi^2}{\mu^2}\right)$$

Now we can do the same for K^+, $\bar{K}^0$, π^0 and η loop;

remember to divide the π^0 loop by $\frac{1}{2}$ and all the other symmetry-factors.

The final result is ($m_u = m_d$!)



$$= \frac{i}{16\pi^2 F^2} \; p^2 \left\{ \frac{2}{3} m_\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_\pi^2}{\mu^2} \right) + \frac{1}{3} m_K^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_K^2}{\mu^2} \right) \right\}$$

$$+ \text{ parts that do not depend on } p^2$$

I now have to calculate the part coming from the Lagrangian $\mathcal{L}_4$.
I will only calculate the term proportional to p^2 and give the result for the other one: so we want from $\mathcal{L}_4$ the term $\partial_\mu \pi^+ \partial^\mu \pi^-$

$$= ?$$

Using $\partial_\mu U = \frac{\sqrt{2}\, i}{F} \partial_\mu M + \ldots\ldots$

we can see that only the terms proportional to L_4 and L_5 have such terms.

$$\mathcal{L}_4 = L_4 \frac{\sqrt{2}\, i}{F} \frac{-\sqrt{2}\, i}{F} \operatorname{tr} \partial_\mu M \partial^\mu M \operatorname{tr}(2\chi)$$

$$+ L_5 \frac{\sqrt{2}\, i}{F} \frac{-\sqrt{2}\, i}{F} \operatorname{tr} \partial_\mu M \partial^\mu M \, 2\chi$$

$$= \left(4(m_u + m_d) B_0 + 4 m_s B_0 \right) \frac{2 L_4}{F^2} \; 2 \partial_\mu \pi^+ \partial^\mu \pi^-$$

$$+ \; 4 (m_u + m_d) B_0 \; \frac{2 L_5}{F^2} \; \partial_\mu \pi^+ \partial^\mu \pi^-$$

$$\mathcal{L}_4 = \partial_\mu \pi^+ \partial^\mu \pi^- \left\{ \frac{16 L_4}{F^2} (m_u + m_d + m_s) B_0 + \frac{8 L_5}{F^2} (m_u + m_d) B_0 \right\}$$

So our full result for the (inverse) propagator; the part proportional to p^2 is:

$$i p^2 = i p^2$$

$$+ i p^2 \frac{1}{16\pi^2 F^2} \frac{2}{3} m_\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_\pi^2}{\mu^2} \right) + \frac{1}{3} m_K^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_K^2}{\mu^2} \right)$$

$$+ i p^2 \left(\frac{8 L_4}{F^2} (m_\pi^2 + 2 m_K^2) + \frac{8 L_5}{F^2} m_\pi^2 \right)$$

$$= i p^2 (Z_\pi - 1)$$

So

$$Z_\pi = 1 + \frac{1}{16\pi^2 F^2} \left\{ \frac{2}{3} m_\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_\pi^2}{\mu^2} \right) + \frac{1}{3} m_K^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_K^2}{\mu^2} \right) \right\}$$

$$+ \frac{8 L_4}{F^2} (m_\pi^2 + 2 m_K^2) + \frac{8 L_5}{F^2} m_\pi^2$$

We can now in principle also calculate the rest of the diagrams, the part independent of p^2 and then for $p^2 = m_{phys}^2$ the full right-hand side vanishes. This way we can obtain m_π^2 to p^4 order.
The result is ($m_u = m_d = \hat{m}$)

$$m_\pi^2 = 2 \hat{m} B_0 \left\{ 1 + \frac{1}{32\pi^2 F^2} m_\pi^2 \log \frac{m_\pi^2}{\mu^2} - \frac{1}{96\pi^2 F^2} m_\eta^2 \log \frac{m_\eta^2}{\mu^2} \right.$$

$$\left. + \frac{16 B_0}{F^2} (2 L_6^r - L_4^r)(m_u + m_d + m_s) + \frac{16 \hat{m} B_0}{F^2} (2 L_8^r - L_5^r) \right\}$$

So we can see here:

- the result is fully finite
- L_i^r and the logarithms are dependent on μ but in the sum the μ-dependence cancels.
- The result depends on the unknown parameters $L^r_{4,5,6,8}$
- The result contains $m_\pi^2 \log m_\pi^2$. This is not analytic in m_π^2 and the logarithms are large

The decay constants

remember $$l_\mu = -\frac{e}{\sqrt{2}\sin\theta_W}\begin{pmatrix} 0 & W^+\cos\theta_c & W^+\sin\theta_c \\ W^-\cos\theta_c & 0 & 0 \\ W^-\sin\theta_c & 0 & 0 \end{pmatrix}$$

so we want to get F_π now; this is proportional to the $\pi^+ W^-$ coupling. So let us calculate it to p^4 order.

a) lowest order: $$D_\mu U = \partial_\mu U + i U l_\mu - i r_\mu U$$

$$\text{So} \quad \mathcal{L}_2 = \frac{-ie}{\sin\theta_W}\frac{F^2}{4}\,\mathrm{tr}\, U \not{W}_\mu D^\mu U^+ + D_\mu U(-\not{W}_\mu U^+)$$

$$W_\mu = \begin{pmatrix} 0 & W_\mu^+\cos\theta_c & W_\mu^+\sin\theta_c \\ W^-\cos\theta_c & 0 & 0 \\ W^-\sin\theta_c & 0 & 0 \end{pmatrix} = \frac{ieF^2}{2\sqrt{2}\sin\theta_W}\,\mathrm{tr}\left(\not{W}_\mu U^\dagger D^\mu U\right) + \dots$$

$$= \frac{-eF}{2\sin\theta_W}\,\partial_\mu \pi^+ W_\mu^- \cos\theta_c + \dots$$

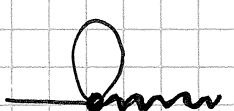
$\pi^+(p)$ ⟶ W^+ $= \dfrac{-eF}{2\sin\theta_W}\cos\theta_c \; p_\mu \times W^{-\mu}$

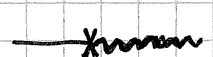
Note: the full amplitude for $\pi^+ \to e^+ \nu$ is $\dfrac{-e^2F}{4\sqrt{2}\sin^2\theta_W}\cos\theta_c \; p_\mu \; \dfrac{i}{p^2 - m_W^2}\,\bar{\nu}\gamma_\mu(1-\gamma_5)e$ by adding the W-propagator and Wνe vertex.

b) p^4 order; there are 3 types of contributions.

1) We are calculating with bare fields; so we have to adjust for that:

$$\left(\frac{1}{\sqrt{Z_\pi}} - 1\right) \times \text{lowest order} \simeq -\frac{1}{2}(Z_\pi - 1) \times \text{lowest order}$$

2) 

3)  where X is a vertex from $\mathcal{L}_4$

- Z_π we already calculated before, so that contribution we have
- ⟶✕ we need the parts of $\mathcal{L}_4$ that contain a $\partial_\mu \pi^+ l^\mu$ vertex.

L_1, L_2, L_3: term : l_μ + at least 3 $\partial_\mu M$: no

L_6, L_7, L_8: no l_μ

L_9 : $l_\mu + 2\,\partial_\mu M$: no

L_{10} : $l_\mu + r_\mu$: no

so only the L_4 and L_5 term can contribute

again $D_\mu U = i l_\mu + \dfrac{\sqrt{2}\,i}{F}\partial_\mu M + \ldots$

So

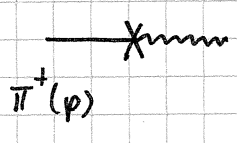
$$\mathcal{L}_4 = L_4 (-i) \frac{\sqrt{2}\, i}{F} \quad \mathrm{tr}\; 2\chi \;\; \mathrm{tr}\left(l_\mu \partial^\mu M + \partial^\mu M l_\mu \right)$$

$$+ L_5 (-i) \frac{\sqrt{2}\, i}{F} \quad \mathrm{tr}\left[\left(l_\mu \partial^\mu M + \partial^\mu M l_\mu \right) 2\chi \right]$$

$$= \frac{\sqrt{2}\, L_4}{F} \quad 2 (m_u + m_d + m_s) 2 B_0 \quad 2 \quad l_\mu \partial^\mu \pi^+$$

$$+ \frac{\sqrt{2}\, L_5}{F} \quad 2 \; 2 B_0 \quad (m_u + m_d) \; l_\mu \partial^\mu \pi^+$$

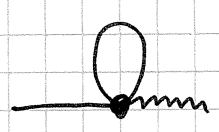
or



$$= i(-i p_\mu) \frac{-e \cos\theta_c}{\sqrt{2} \; \sin\theta_W} \quad W_-^\mu$$

$$\times \left\{ \frac{8\sqrt{2}\, B_0 L_4}{F} (m_u + m_d + m_s) + \frac{4\sqrt{2}\, B_0 L_5}{F} (m_u + m_d) \right\}$$

This leaves to be calculated the loop contribution :



for this we need the $l_\mu M^3$ term of $\mathcal{L}_2$

$$\mathcal{L}_2 = \frac{i F^2}{4} \mathrm{tr}\left(l_\mu D_\mu U^\dagger U - l_\mu U^\dagger D_\mu U \right) = \frac{i F^2}{2} \mathrm{tr}\, l_\mu \left(D_\mu U^\dagger U \right)$$

here we need M^3 so

$$D_\mu U^\dagger U = \frac{-\sqrt{2}\, i}{F} \frac{-1}{F^2} \left(\partial_\mu M M^2 - \partial_\mu (M^2) M \right) + \frac{(-i)^3 2\sqrt{2}}{6} \partial_\mu (M^3)$$

$$= \frac{-i\sqrt{2}}{3 F^3} \left(- \partial_\mu M M^2 + 2 M \partial_\mu M M - M^2 \partial_\mu M \right)$$

So $\mathcal{L}_2 = \frac{-1}{3\sqrt{2}F} \operatorname{tr} l_\mu \left(\partial_\mu M M^2 - 2M \partial_\mu M M + M^2 \partial_\mu M \right)$

At this level we could calculate the full vertices but we can make life simpler by the following observation:

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2} = 0$$

so any diagram where the ∂_μ is associated with a particle in the loop vanishes.

So it is sufficient for the present to set

$$\partial_\mu M = \begin{pmatrix} 0 & \partial_\mu \pi^+ & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then life is somewhat easier:

$$\mathcal{L}_2 = \frac{-1}{3\sqrt{2}F} \frac{-e \cos\theta_C}{\sqrt{2}\sin\theta_W} W^-_\mu \partial^\mu \pi^+ \cdot$$

$$\left\{ \pi^+\pi^- + \left(\frac{-\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}\right)^2 + K^0\bar{K}^0 \right.$$

$$-2 \left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}\right)\left(\frac{-\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}\right)$$

$$\left. + \left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}\right)^2 + \pi^+\pi^- + K^+K^- \right\}$$

$$= \frac{+e\cos\theta_C}{2\sin\theta_W} \frac{1}{3F} W^-_\mu \partial^\mu \pi^+ \left\{ 2\pi^+\pi^- + 2\pi^0\pi^0 + K^0\bar{K}^0 + K^+K^- \right\}$$

propagator

$$= i \; i \int \frac{d^d k}{(2\pi)^d} (-i p_\mu) W^-_\mu \frac{+e \cos\theta_c}{2 \sin\theta_W} \frac{1}{3F}$$

π^+ π^0 (symmetry $\times 2$ $\times \frac{1}{2}$ for π^0 loop)

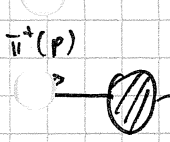
$$\times \frac{1}{p^2 - m_\pi^2} \left\{ 2 + 2 \right\}$$

$$+ \frac{1}{p^2 - m_K^2} \left\{ 1 + 1 \right\}$$

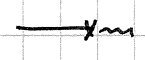
$$= \frac{+i \; e \cos\theta_c}{2 \sin\theta_W} p_\mu W^-_\mu \frac{1}{3F}$$

$$\frac{i}{16\pi^2} \left\{ 4 m_\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_\pi^2}{\mu^2} \right) + 2 m_K^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_K^2}{\mu^2} \right) \right\}$$

So the full result is:



$$= \frac{-e F \cos\theta_c}{2 \sin\theta_W} p_\mu W^{-\mu} \left[1 + \left\{ \right. \right.$$



$$\frac{8 L_4}{F^2} \left(m_\pi^2 + 2 m_K^2 \right) + \frac{8 L_5}{F^2} m_\pi^2$$

$$\left[+ \frac{4}{3} m_\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_\pi^2}{\mu^2} \right) + \frac{2}{3} m_K^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_K^2}{\mu^2} \right) \right] \frac{1}{16\pi^2 F^2}$$

Z_π $\left\{ \right.$

$$- \frac{4 L_4}{F^2} \left(m_\pi^2 + 2 m_K^2 \right) - \frac{4 L_5}{F^2} m_\pi^2$$

$$\left. \left[- \frac{1}{3} m_\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_\pi^2}{\mu^2} \right) - \frac{1}{6} m_K^2 \left(\frac{1}{\bar{\varepsilon}} - \log \frac{m_K^2}{\mu^2} \right) \right] \frac{1}{16\pi^2 F^2} \right\} \left. \right]$$

Now remember the definition of F_π this leads to

$$\frac{F_\pi}{F} = 1 + \frac{m_\pi^2}{16\pi^2 F^2}\left(-\log\frac{m_\pi^2}{\mu^2}\right) + \frac{m_K^2}{16\pi^2 F^2}\left(-\log\frac{m_K^2}{\mu^2}\right) + \frac{4L_4^r}{F^2}\left(m_\pi^2 + 2m_K^2\right) + \frac{4L_5^r}{F^2}m_\pi^2$$

So a good exercise at this point is to verify that the $\frac{1}{\bar{\varepsilon}}$ parts in the loops cancel against those in $L_i = L_i^r + \frac{1}{\bar{\varepsilon}}\times$.

Similarly we can calculate $\frac{F_K}{F}$ and $\frac{F_\eta}{F}$

and from $\frac{F_K}{F_\pi}$ L_5^r can be determined ; that then determines F_η

as an example of a physical prediction at next-to-leading order.

Note: when we do this we obtain $L_5^r(\mu = 0.770\ \text{GeV}) = 1.4 \times 10^{-3}$

(measure F_π from $\pi^+ \to \mu^+ \nu$

F_K from $K^+ \to \mu^+ \nu$

and $\frac{e}{\sin\theta_W}$ from $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$

and $\sin\theta_c$ from $K^+ \to \pi^0 e^+ \nu_e$)

Then we can determine the scale Λ that suppresses higher order contributions :

$$\frac{1}{\Lambda^2} = \frac{\text{coeff of } \mathcal{L}_4}{\text{coeff of } \mathcal{L}_2} = \frac{L_5^r}{(F^2/4)} \Rightarrow \Lambda \simeq 1.2\ \text{GeV}$$

(remember only order of magnitude estimate)