A short Introduction to Feynman Diagrams

J. Bijnens, November 2008

This assumes knowledge at the level of Chapter two in G. Kane, “Modern Elementary Particle Physics.” This note is more advanced than needed for FYTN04 but hopefully still useful. For more details see any field theory book.

1 Kinetic terms

A quantum field theory is defined by a Lagrangian. The first step is to define the fields. Such that they represent particles. That means that there should be no terms with only one field present and all vacuum expectation values should have been subtracted.

The kinetic terms, those containing exactly two fields, are assumed to be diagonalized. This means that all terms with two fields are of one of the forms, for a field corresponding to a particle with mass $m$

<table>
<thead>
<tr>
<th>Field</th>
<th>Lagrangian</th>
<th>Field strength</th>
</tr>
</thead>
<tbody>
<tr>
<td>real scalar field $\phi$</td>
<td>$\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$</td>
<td></td>
</tr>
<tr>
<td>complex scalar field $\Phi$</td>
<td>$\partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi$</td>
<td></td>
</tr>
<tr>
<td>Dirac fermion $\psi$</td>
<td>$\bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi$</td>
<td></td>
</tr>
<tr>
<td>real vector field $A_\mu$</td>
<td>$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$</td>
<td>$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$</td>
</tr>
<tr>
<td>complex vector field $W_\mu$</td>
<td>$\frac{1}{2} W^<em>_{\mu\nu} W^{\mu\nu} + m^2 W^</em>_{\mu} W^\mu$</td>
<td>$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$</td>
</tr>
</tbody>
</table>

2 An aside on diagonalization

- Diagonal means that there are no terms that mix two different fields but it is only the same field and its complex conjugate that show up in the kinetic terms.

- If mixed terms are present, they can always be removed by redefining fields. This is true for both masses and the terms with partial derivatives as can be seen in the example below.

As the example, take a theory with two real fields, $\phi_1$ and $\phi_2$, that has as Lagrangian, with some constants $a$, $b$, $c$, $\mu_1^2$, $\mu_2^2$ and $\mu_{12}^2$,

$$\mathcal{L} = \frac{a}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + b \partial_\mu \phi_1 \partial^\mu \phi_2 + c \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} \mu_1^2 \phi_1 \phi_1 - \mu_{12}^2 \phi_1 \phi_2 - \frac{1}{2} \mu_2^2 \phi_2 \phi_2$$

$^1$I will sometimes refer to a generic field $\phi$, that means all possibilities then.

$^2$This can always be done by transformations of the type $\phi = v + \phi'$.

1
This has terms that mix \( \phi_1 \) and \( \phi_2 \) and is actually the most general form since \( \phi_1 \phi_2 = \phi_2 \phi_1 \) and the same for the terms with partial derivatives. We can rewrite this in the matrix notation

\[
L = \frac{1}{2} \left( \partial_\mu \phi_1 \partial_\mu \phi_2 \right) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \partial_\mu \phi_1 \\
\partial_\mu \phi_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \phi_1 \\
\phi_2 \end{pmatrix} \begin{pmatrix} \mu_1^2 & \mu_{12}^2 \\
\mu_{12}^2 & \mu_2^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\
\phi_2 \end{pmatrix}
\]

The matrix \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) is a real symmetric matrix and can thus be diagonalized with an orthogonal matrix \( O \) with the result \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} = O^T \begin{pmatrix} \lambda_1 \\
\lambda_2 \end{pmatrix} O \). We now define linear combinations of \( \phi_1 \) and \( \phi_2 \) via

\[
\begin{pmatrix} \phi_3 \\
\phi_4 \end{pmatrix} = O \begin{pmatrix} \phi_1 \\
\phi_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \phi_1 \\
\phi_2 \end{pmatrix} = O^T \begin{pmatrix} \phi_3 \\
\phi_4 \end{pmatrix}
\]

\( O \) is a constant matrix so the same relation is true for derivatives of the fields. The Lagrangian thus becomes

\[
L = \frac{1}{2} \lambda_1 \partial_\mu \phi_3 \partial^\mu \phi_3 + \frac{1}{2} \lambda_2 \partial_\mu \phi_3 \partial^\mu \phi_3 - \frac{1}{2} \begin{pmatrix} \phi_3 \\
\phi_4 \end{pmatrix} O \begin{pmatrix} \mu_1^2 & \mu_{12}^2 \\
\mu_{12}^2 & \mu_2^2 \end{pmatrix} O^T \begin{pmatrix} \phi_3 \\
\phi_4 \end{pmatrix}
\]

\( \lambda_1 \) and \( \lambda_2 \) must be positive, since kinetic energy must always be positive. We do thus one more redefinition \( \phi_5 = \sqrt{\lambda_1} \phi_3 \) and \( \phi_6 = \sqrt{\lambda_2} \phi_4 \) and obtain

\[
L = \frac{1}{2} \partial_\mu \phi_5 \partial^\mu \phi_5 + \frac{1}{2} \partial_\mu \phi_6 \partial^\mu \phi_6 - \frac{1}{2} \begin{pmatrix} \phi_5 \\
\phi_6 \end{pmatrix} B \begin{pmatrix} \phi_5 \\
\phi_6 \end{pmatrix}
\]

with \( B \) a symmetric matrix

\[
B = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\
0 & \frac{1}{\sqrt{\lambda_2}} \end{pmatrix} O \begin{pmatrix} \mu_1^2 & \mu_{12}^2 \\
\mu_{12}^2 & \mu_2^2 \end{pmatrix} O^T \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} \\
\frac{1}{\sqrt{\lambda_2}} \end{pmatrix}, \quad B^T = B.
\]

\( B \) can thus be diagonalized by an orthogonal matrix \( P \) with \( B = P^T \begin{pmatrix} m_7^2 \\
m_8^2 \end{pmatrix} P \). A final redefinition \( \begin{pmatrix} \phi_7 \\
\phi_8 \end{pmatrix} = P \begin{pmatrix} \phi_5 \\
\phi_6 \end{pmatrix} \) leads now to diagonal kinetic terms

\[
L = \frac{1}{2} \partial_\mu \phi_7 \partial^\mu \phi_7 + \frac{1}{2} \partial_\mu \phi_8 \partial^\mu \phi_8 - \frac{1}{2} m_7^2 \phi_7^2 - \frac{1}{2} m_8^2 \phi_8^2
\]

So you see you can always diagonalize the kinetic terms. If there are interaction terms, you have to rewrite those also in terms of the final diagonalized fields (in this example \( \phi_7 \) and \( \phi_8 \)).
3 Creation and annihilation operators

As explained in Chapter two of Kane we expand the fields in the eigenmodes with creation and annihilation operators

\[ \phi(x) \propto \sum_{\vec{p}} a_{\vec{p}} e^{-i p \cdot x} + a_{\vec{p}}^\dagger e^{i p \cdot x} \]

- I have not written out some normalization factors for the different modes. These are taken care of correctly in the final rules as given.
- For a continuous system the sum over all three momenta \( \vec{p} \) is really an integral.
- The four vectors \( p = (p^0, \vec{p}) \) have \( p^0 = \sqrt{|\vec{p}|^2 + m^2} \)
- \( a_{\vec{p}} \) annihilates or destroys or removes a particle/quantum corresponding to the field \( \phi \) with four momentum \( p \).
- \( a_{\vec{p}}^\dagger \) creates or adds a particle/quantum with four momentum \( p \).
- As the notation indicates, \( a_{\vec{p}}^\dagger \) is the Hermitian conjugate of \( a_{\vec{p}} \) for a real field.
- There is a set of these operators for each field. They can be distinguished by adding indices to indicate which field the operator corresponds to.

For a complex scalar field the two terms have a different operator in the two terms since the total doesn’t have to be real (Hermitian)

\[ \Phi(x) \propto \sum_{\vec{p}} a_{\vec{p}} e^{-i p \cdot x} + b_{\vec{p}}^\dagger e^{i p \cdot x} \]

- \( a_{\vec{p}} \) annihilates or destroys or removes a particle corresponding to the field \( \phi \) with four momentum \( p \).
- \( b_{\vec{p}}^\dagger \) creates or adds an anti-particle with four momentum \( p \).
- As the notation indicates, \( b_{\vec{p}}^\dagger \) is not the Hermitian conjugate of \( a_{\vec{p}} \) for a complex field.

The Hermitian conjugate of the field

\[ \Phi^*(x) \propto \sum_{\vec{p}} b_{\vec{p}} e^{-i p \cdot x} + a_{\vec{p}}^\dagger e^{i p \cdot x} \]

- \( b_{\vec{p}} \) annihilates or destroys or removes an anti-particle corresponding to the field \( \phi \) with four momentum \( p \).
- \( a_{\vec{p}}^\dagger \) creates or adds a particle with four momentum \( p \).
I have used $\vec{p}$ here for both the real and complex scalar fields but remember that they are really distinct operators and with different operators for each field that occurs. This can be taken care of by adding extra indices.

For particles with spin, the different spin modes should be included as well. The sums are thus over $\vec{p}$ and spin $s$.

Dirac fermion  
$$\psi \propto \sum_{\vec{p},s} u_{\vec{p},s} a_{\vec{p},s} e^{-ip \cdot x} + v_{\vec{p},s} b_{\vec{p},s}^\dagger e^{ip \cdot x}$$

Conjugate Dirac fermion  
$$\bar{\psi} \propto \sum_{\vec{p},s} \bar{u}_{\vec{p},s} a_{\vec{p},s} e^{-ip \cdot x} + \bar{v}_{\vec{p},s} b_{\vec{p},s}^\dagger e^{ip \cdot x}$$

The spinors $u_{\vec{p},s}$ and $v_{\vec{p},s}$ are the positive and negative energy solutions of the Dirac equation of spin $s$. $a_{\vec{p},s}$ annihilates or removes a particle with momentum $p$ and spin $s$ while $a_{\vec{p},s}^\dagger$ creates it. $b_{\vec{p},s}$ annihilates or removes an anti-particle with momentum $p$ and spin $s$ while $b_{\vec{p},s}^\dagger$ creates it.

The same principle goes for spin one particles but with polarization vectors $\varepsilon_{\vec{p},i,\mu}$ for a vector with momentum $p$ and spin $s$ instead of spinors for the Dirac case.

Real vector  
$$A_\mu \propto \sum_{\vec{p},s} \varepsilon_{\vec{p},s,\mu} a_{\vec{p},i} e^{-ip \cdot x} + \varepsilon_{\vec{p},s,\mu}^* a_{\vec{p},i}^\dagger e^{ip \cdot x}$$

Complex vector  
$$W_\mu \propto \sum_{\vec{p},s} \varepsilon_{\vec{p},s,\mu} a_{\vec{p},i} e^{-ip \cdot x} + \varepsilon_{\vec{p},s,\mu}^* b_{\vec{p},i}^\dagger e^{ip \cdot x}$$

Conjugate complex vector  
$$W_\mu^* \propto \sum_{\vec{p},s} \varepsilon_{\vec{p},s,\mu} b_{\vec{p},i} e^{-ip \cdot x} + \varepsilon_{\vec{p},s,\mu}^* a_{\vec{p},i}^\dagger e^{ip \cdot x}$$

The same type of comments as above apply here to the creation and annihilation operators.

4 What are now Feynman diagrams?

Feynman diagrams are a technique to solve quantum field theory. Their main use is to calculate the amplitude (or rather $i$ times the amplitude) for a state with specified incoming particles with momenta and spins specified to evolve to a different state with specified particles and their momenta and spins.\footnote{I have been a little bit sloppy here with the signs on the spins so be careful when comparing with field theory books.}

We divide the Lagrangian into

- Kinetic terms: those with two fields as described in section 1. These terms produce the propagators and give the lines that connect different points of a Feynman diagram. Internal lines must be summed over all momenta and spins.

- Potential terms: those with one field and one charge. These terms produce vertexes and give the interaction between particles. The number of vertices is equal to the number of external lines.

- Source terms: those with one field and no charge. These terms produce source and sink lines and give the external particle lines. The number of source and sink lines is equal to the number of external lines.

\footnote{It is sufficient if all momenta are different, this requirement is there to have contributions from connected diagrams only.}
• Interaction terms: those terms with three or more fields. This part is usually called $\mathcal{L}_I$. It provides connection points called vertices where three or more lines meet.

• At the vertices momentum is conserved: the sum over all incoming momenta must be equal to the sum over all outgoing momenta at each vertex. You can check that for tree level diagrams, i.e. no closed loops, this means that all occurring momenta are specified in terms of the incoming and outgoing momenta.

• The previous point also leads to momentum conservation for the full diagram. The sum over all outgoing momenta is equal to the sum over all incoming momenta.

To construct the amplitudes for a process one now does the following:

1. Connect via vertices and internal lines all incoming particles to all outgoing particles.

2. Internal lines are propagators. These can be seen as a particle traveling in one direction and as an anti-particle traveling in the opposite direction. Both terms contribute but the simplified arguments given below then overestimate the result. Simply take one of the two cases, you can check that both give the same result. The propagator takes this correctly into account. For an internal line corresponding to a scalar particle with mass $m$ and momentum $k$ the propagator is $i/(k^2 - m^2)$.

3. Each vertex gives a factor $i$ and all extra parts that appear in the term in $\mathcal{L}_I$ after the creation operators have been used to create the particles corresponding to the lines leaving the vertex and the annihilation operators have been used remove the particles from the lines coming in to the vertex. The exponential factors are responsible for the momentum conservation and are take into account that way.

In the next section, an example is worked out in more detail to make it somewhat clearer.

5 An example worked out in detail

We look at a Lagrangian with a real scalar field $\phi$ and a complex scalar field $\Psi$ with Lagrangian\textsuperscript{5}

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \partial_\mu \Psi^* \partial^\mu \Psi - M^2 \Psi^* \Psi + g \partial_\mu \phi \partial^\mu \Psi^* \Psi + \lambda \phi^3$$

The first four terms are the kinetic terms and give:

• internal $\phi$ line with momentum $k$: propagator $i/(k^2 - m^2)$.

• internal $\Psi$ line with momentum $k$: propagator $i/(k^2 - M^2)$.

\textsuperscript{5}For being a good Lagrangian there should also be a term with the complex conjugate of $g \partial_\mu \phi \partial^\mu \Psi^* \Psi$. We do not include it, but you can work out its effects yourself.
Let us now look at the next-to-last term. First, the derivative of a field can also be expanded in terms of creation and annihilation operators. Taking the derivative of the expansion of \( \phi \) gives

\[
\partial_\mu \phi(x) \propto \sum \left( -ip_\mu \right) a_\phi e^{-ip \cdot x} + \left( ip_\mu \right) a_\phi^\dagger e^{ip \cdot x}
\]

and we also obtain

\[
\partial_\mu \Psi^\dagger(x) \propto \sum \left( -ip_\mu \right) b_\phi e^{-ip \cdot x} + \left( ip_\mu \right) b_\phi^\dagger e^{ip \cdot x}
\]

I have replaced the \( a \) of section 3 here with \( d \) for the complex scalar field \( \Psi \).

We can now write out the interaction term \( g \partial_\mu \phi \partial^\mu \Psi^\dagger \Psi \) in glorious detail

\[
g \partial_\mu \phi \partial^\mu \Psi^\dagger \Psi = \sum_{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3} \left\{ g(-ip_{1\mu})(-ip_{3\mu})a_\phi \tilde{p}_1 b_\phi \tilde{p}_2 d_\phi \tilde{p}_3 e^{i(-p_1-p_2+p_3) \cdot x} + g(-ip_{1\mu})(-ip_{2\mu})a_\phi \tilde{p}_1 b_\phi \tilde{p}_2 b_\phi^\dagger \tilde{p}_3 e^{i(-p_1+p_2-p_3) \cdot x} + g(-ip_{1\mu})(ip_{3\mu})a_\phi \tilde{p}_1 b_\phi \tilde{p}_2 d_\phi \tilde{p}_3 e^{i(-p_1+p_2+p_3) \cdot x} + g(ip_{1\mu})(-ip_{2\mu})a_\phi \tilde{p}_1 b_\phi \tilde{p}_2 b_\phi^\dagger \tilde{p}_3 e^{i(p_1-p_2-p_3) \cdot x} + g(ip_{1\mu})(ip_{3\mu})a_\phi \tilde{p}_1 b_\phi \tilde{p}_2 d_\phi \tilde{p}_3 e^{i(p_1+p_2-p_3) \cdot x} + g(ip_{1\mu})(-ip_{2\mu})a_\phi \tilde{p}_1 b_\phi \tilde{p}_2 b_\phi^\dagger \tilde{p}_3 e^{i(p_1+p_2+p_3) \cdot x} \right\}. \tag{1}
\]

The sum over the three momenta is there for all eight terms. Let us now analyze the contents of these eight terms and what they do. I label the terms case 1, \ldots, 8. A \( \phi \)-particle with momentum \( p_1 \) is denoted by \( \phi(p_1) \), a \( \Psi \)-particle with momentum \( p_2 \) is denoted by \( \Psi(p_2) \), and a \( \Psi \)-anti-particle with momentum \( p_3 \) is denoted by \( \Psi^\dagger(p_3) \). The eight terms give different in/out combinations at the vertex with somewhat different factors and momentum conservation. The rules of the previous section lead to the eight cases (one from each term)

<table>
<thead>
<tr>
<th>Case</th>
<th>in/out</th>
<th>factor in diagram</th>
<th>momentum conservation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \phi(p_1) ) in, ( \Psi^\dagger(p_2) ) in, ( \Psi(p_3) ) in</td>
<td>(-igp_1 \cdot p_2)</td>
<td>( p_1 + p_2 + p_3 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( \phi(p_1) ) in, ( \Psi^\dagger(p_2) ) in, ( \Psi(p_3) ) out</td>
<td>(-igp_1 \cdot p_2)</td>
<td>( p_1 + p_2 = p_3 )</td>
</tr>
<tr>
<td>3</td>
<td>( \phi(p_1) ) in, ( \Psi(p_2) ) out, ( \Psi(p_3) ) in</td>
<td>( igp_1 \cdot p_2)</td>
<td>( p_1 + p_3 = p_2 )</td>
</tr>
<tr>
<td>4</td>
<td>( \phi(p_1) ) in, ( \Psi(p_2) ) out, ( \Psi^\dagger(p_3) ) out</td>
<td>( igp_1 \cdot p_2)</td>
<td>( p_1 = p_2 + p_3 )</td>
</tr>
<tr>
<td>5</td>
<td>( \phi(p_1) ) out, ( \Psi^\dagger(p_2) ) in, ( \Psi(p_3) ) in</td>
<td>( igp_1 \cdot p_2)</td>
<td>( p_2 + p_3 = p_1 )</td>
</tr>
<tr>
<td>6</td>
<td>( \phi(p_1) ) out, ( \Psi^\dagger(p_2) ) in, ( \Psi(p_3) ) out</td>
<td>( igp_1 \cdot p_2)</td>
<td>( p_2 = p_1 + p_3 )</td>
</tr>
<tr>
<td>7</td>
<td>( \phi(p_1) ) out, ( \Psi(p_2) ) out, ( \Psi(p_3) ) in</td>
<td>(-igp_1 \cdot p_2)</td>
<td>( p_3 = p_1 + p_2 )</td>
</tr>
<tr>
<td>8</td>
<td>( \phi(p_1) ) out, ( \Psi(p_2) ) out, ( \Psi^\dagger(p_3) ) out</td>
<td>(-igp_1 \cdot p_2)</td>
<td>( 0 = p_1 + p_2 + p_3 )</td>
</tr>
</tbody>
</table>
The other interaction term can be expanded analogously

$$\lambda \phi^3 = \sum_{\vec{p}_1, \vec{p}_2, \vec{p}_3} \left\{ a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3} e^{i(-p_1 - p_2 - p_3) \cdot x} + \ldots \right\}$$

and a similar table can be constructed for this term as well, the complication that the same field shows up several times is discussed below.

The factor can be remembered by: an $i$, a derivative $\partial_\mu$ is changed to $-ip_\mu$ for incoming and $ip_\mu$ for outgoing momentum $p$, the remaining factors are what was present as constants in the Lagrangian.

Let us now look at the process $\phi(q_1)$ and $\psi(q_2)$ in, $\phi(q_3)$ and $\psi(q_4)$ out.

We denote a $\phi$ line by a dashed line and $\Psi$ by a full line:

$\phi$: \[\cdots\] $\Psi$: \[\text{full line}\]

The process we want to get at is

\[\begin{array}{ccc}
\phi & q_1 & \rightarrow \\
\Psi & q_2 & \rightarrow \\
\phi & q_3 & \rightarrow \\
\Psi & q_4 & \rightarrow \\
\end{array}\]

We should now replace the $\cdots$ by all combinations of vertices and lines that connect the incoming lines on the left to the outgoing lines on the right.

The Lagrangian has two possible vertices from the two interaction terms in the Lagrangian

\[\begin{array}{c}
\bullet \\
\bullet \quad \bullet \\
\bullet \\
\end{array}\]

After some thinking you should find three ways (without loops) to connect the incoming lines to the outgoing lines

\[\begin{array}{ccc}
(a) & & (b) \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}\]

In diagram (b) note that the dashed lines cross but there is no vertex there.
Let us now work out diagram (a) in detail. Let’s write down the momenta again

\[ q_1 \rightarrow q \rightarrow B \]

or \( \bar{q} \leftarrow \)

or \( q_2 \rightarrow \)

and again \( q_3 \rightarrow q_4 \)

The vertex A is case 3 from the table with \( p_1 = q_1 \), \( p_3 = q_2 \) and \( p_2 = q \) so we get that \( q = q_1 + q_2 \) and a factor \( i g q_1 \cdot q \) = \( i g q_1 \cdot (q_1 + q_2) \). Note that if we had chosen the \( \bar{q} \) option instead we would have used case 1 and \( \bar{q} = -q_1 - q_2 \) and obtained the same factor.

The vertex B is case 7 with \( p_1 = q_3 \), \( p_2 = q_4 \) and \( p_3 = q \) so \( q = q_3 + q_4 \) and we obtain a factor \( (-i) g q_3 \cdot q_4 \). Note that the two requirements on \( q \) require overall momentum conservation \( q_3 + q_4 = q_1 + q_2 \).

The internal line is a \( \Psi \) line with momentum \( q \) contributing the propagator \( i/(q^2 - M^2) \).

The Feynman amplitude \( iA \) for this diagram is thus by putting together all factors

\[
iA^{(a)} = \frac{i g^2 (q_1 \cdot (q_1 + q_2)) (q_3 \cdot q_4)}{(q_1 + q_2)^2 - M^2}.
\]

You can check that if you had chosen \( \bar{q} \) instead for the internal line the final expression would have been identical.

Diagram (b) can be treated in the same way with the internal line having momentum \( k = q_2 - q_3 \) from the left vertex and \( k = q_4 - q_1 \) from the right vertex, we thus again get overall momentum conservation. You can check that the full expression for this diagram is

\[
iA^{(b)} = \frac{i g^2 (q_3 \cdot (q_2 - q_3)) (q_1 \cdot q_4)}{(q_2 - q_3)^2 - M^2},
\]

when working out all the factors similar to what was done for diagram (a).

Diagram (c) has an additional complication, since the \( \phi \) field appears several times in the vertex A on top of the diagram.

\[ q_1 \rightarrow \quad \bullet \quad \rightarrow q_3 \]

\[ q_2 \rightarrow \quad \bullet \quad \rightarrow q_4 \]

This leads to extra terms in the expression. Let us write the term \( \lambda \phi^3 = \lambda \phi_1 \phi_2 \phi_3 \) and do the expansion in creation and annihilation operators with \( \phi_1 \) having momentum \( p_1 \), \( \phi_2 \) \( p_2 \) and \( \phi_3 \) \( p_3 \). Then of the eight cases you get when writing out the vertex three cases can contribute but each of them does it twice. Thus we get
The factor is always the same, this would not have been the case if there had been derivatives. If you check how the momentum conservation goes for all the options, you will see they always lead to \( Q = q_1 - q_3 \). The upper vertex is the sum of all the six contributions in the table and gives thus a factor 6\( i\lambda \).

The propagator is \( \phi \) propagator and gives \( i/(Q^2 - m^2) \) and the bottom vertex gives, using the same reasonings as used for diagram (a), \( ig(q_1 - q_3) \cdot q_4 \). The final expression for the amplitude from diagram (c) is thus

\[
iA^{(c)} = \frac{-i6g\lambda(q_1 - q_3) \cdot q_4}{(q_1 - q_3)^2 - m^2}
\]

The full tree level Feynman amplitude for this process is the sum of the three terms

\[
iA^{\text{tree}} = iA^{(a)} + iA^{(b)} + iA^{(c)}.
\]

### 6 Extra complications for spins, fermions and other degrees of freedom

- For spins, these are dealt with by spinors and polarization vectors.
- The expressions for the vertices get extra factors of \( u_{\vec{p},s}, v_{\vec{p},s}, \bar{\nu}_{\vec{p},s}, \bar{\nu}_{\vec{p},s}, \varepsilon_{\vec{p},s \mu} \) and/or \( \varepsilon^*_{\vec{p},s \mu} \).
- Remember also to keep in the expressions for the vertex all the other parts, e.g. gamma matrices, and the indices that need to be summed over.
- In internal lines always sums appear over the spinors or polarization vectors that appear on both sides.
- For vectors this leads to the polarization sum \( \sum_s \varepsilon^*_{\vec{p},s \mu} \varepsilon_{\vec{p},s \nu} = -g_{\mu\nu} \)

In general there are extra terms but these complications do not appear at this level.
- For fermions in internal lines, always the combination (because both particles and antiparticles contribute)

\[
\sum_s \left( u_{\vec{p},s} \bar{\nu}_{\vec{p},s} + v_{\vec{p},s} \bar{\nu}_{\vec{p},s} \right) = p\gamma^\mu + m,
\]
appears and the equal sign follows from the properties of solutions of the Dirac equation. This way, you will always connect all spinors together in consistent combinations when you connect all vertices, keeping track of Dirac indices.

- One more complication appears for fermions. If the vertices are written in the form $\bar{\psi}_1 \Gamma \psi_2$ then the rules are
  
  - an extra $-1$ for each closed fermion loop
  - an extra $(−1)$ for each crossing of fermion lines without a vertex.

It is the last sign that corresponds to the minus sign in the wave function when interchanging two fermions.

- If additional indices appear in the vertices, they can be dealt with similarly by keeping track of them and having the propagator diagonal in these indices. Colour connections through diagrams can be dealt with in this way.