Groups and representations

The material here is partly in Appendix A and B of the book.

1 Introduction

The concept of symmetry, and especially gauge symmetry, is central to this course. Now what is a symmetry: you have something, e.g. a vase, and you do something to it, e.g. turn it by 29 degrees, and it still looks the same then we call the operation performed, i.e. the rotation, a symmetryoperation on the object.

But if we first rotate it by 29 degrees and then by 13 degrees and it still looks the same then also the combined operation of a rotation by 42 degrees is a symmetry operation as well.

The mathematics that is involved here is that of groups and their representations. The symmetry operations form the group and the objects the operations work on are in representations of the group.

2 Groups

A group is a set of elements g where there exist an operation $*$ that combines two group elements and the results is a third:

$$\exists * : \forall g_1, g_2 \in G : g_1 * g_2 = g_3 \in G \tag{1}$$

This operation must be associative:

$$\forall g_1, g_2, g_3 \in G : (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3) \tag{2}$$

and there exists an element unity:

$$\exists \mathbf{1} \in G : \forall g \in G : g * \mathbf{1} = \mathbf{1} * g = g \tag{3}$$

and for every element in G there exists an inverse:

$$\forall g \in G : \exists g^{-1} \in G : g * g^{-1} = g^{-1} * g = \mathbf{1} \tag{4}$$

This is the general definition of a group. Now if all elements of a group commute, i.e. the order in which they are standing is not important,

$$\forall g_1, g_2 \in G : g_1 * g_2 = g_2 * g_1 \Rightarrow G \text{ is Abelian} \tag{5}$$

and conversely, if it is not true for some elements G is non-Abelian.

2.1 Examples of $G, *$

1. $\mathbb{R}, +$ All real numbers with as operation addition, the unit element is zero, the inverse is $-$ the number.

2. $\mathbb{Z}, +$ positive and negative integers with addition.

3. $\mathbb{R}_0, \cdot$ The set of all positive real numbers, i.e. without zero, with multiplication as the operation. the unit element is 1, inverse is $1/x$.

4. $\{e^{i\alpha}\}, \cdot$ the set of all complex numbers with unit magnitude and multiplication. This group is also known as $U(1)$.

5. $SU(n), \cdot$. The set of all complex $n \times n$ matrices that are unitary and have determinant one. The operation is matrix multiplication. Unitary means $U^\dagger U = \mathbf{1}$.

6. $SO(n), \cdot$. The set of all real $n \times n$ matrices that are orthogonal and have determinant one. The operation is matrix multiplication. Orthogonality means $O^T O = \mathbf{1}$.

7. S_3 the group of all permutations of 3 elements.

The first 4 groups are Abelian, the last three are non-Abelian.

2.2 Lie Groups

The groups above consists of continuous groups and discrete groups. Discrete means that the elements are not continuously connected; $\mathbb{Z}$ and S_3 fall in this class. The others are continuous. This means that the group space can be described by a set of coordinates that are real numbers. When this can be done by a finite number of coordinates these are known as Lie groups.

The Lie groups are special in that there is only a limited class of them, known since long; the full set of finite groups has only been recently classified. It is also possible to describe the structure of a Lie group in a more simple fashion, known as a Lie algebra, all[1] elements of a Lie group can be written as

$$g = \exp\left\{ i \sum_i x_i T_i \right\} \tag{6}$$

where the T_i are hermitian matrices and the x_i real numbers. The space of $\sum_i x_i T_i$ is referred to as the Lie algebra. The T_i are referred to as generators of the Lie group. The satisfy the commutator relations $[T_i, T_j] = i \sum_k f_{ijk} T_k$. The f_{ijk} are referred to as the structure constants of the Lie group.

[1] well not quite, for the mathematicians: all compact Lie groups in the area continuously connected with the unit element.

3 Representations

A representation is a set of objects that the group acts on that is complete in the sense that no action of a group element brings you outside the set. E.g., the rotation group around the z-axis, and a vase(now decorated) together with all the possible angles it can stand. In order to be a group representation the action of the group elements on the set should be compatible with the group product. A representation is thus a set R and a series of functions f_g acting on this set, one for each group element $g \in G$:

$$\forall g \in G : \exists f_g : \forall x \in R : f_g(x) \in R \tag{7}$$

$$\forall x \in R \; \forall g_1, g_2 \in G : f_{(g_1 * g_2)}(x) = f_{g_1}(f_{g_2}(x)) \tag{8}$$

The last line is what is meant with being compatible with the group product.

3.1 Linear Representation

We will mostly be concerned with linear representations where R is a vector space and the functions can be represented as matrices.

For Lie groups and most simple discrete groups the complete structure of linear representations is known, and they can always be written using matrices. The set of objects can be written as a linear space, as a space of column vectors (elements real or complex $\Rightarrow$ real or complex representation) and the action of the group can be written as matrices acting on these column vectors. So the set of column-vectors is b and the set of matrices of group elements g is $M(g)$ with

$$b \longrightarrow M(g)b \tag{9}$$

under the group operation. In order to be a linear representation the requirement is:

$$M(g_1 * g_2) = M(g_1)M(g_2) \tag{10}$$

3.2 An example S_3

S_3 is a group of permutations of 3 objects a, b, c and consists of the six elements $abc, acb, bca, bac, cab, cba$ where I have denoted the group element by the way it shuffles a, b, c. abc is here the unit element. A representation is now:

$$\begin{aligned}
a &\to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad abc \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad acb \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
b &\to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad bca \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad bac \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
c &\to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad cab \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 13 & 0 \end{pmatrix} \quad cba \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{11}$$

You can easily check that this a representation of S_3, the first column gives the representation and the other columns the matrices the group elements become.

3.3 Fundamental representation of $SU(n)$ and $SO(n)$

The fundamental representation is, as the name says, the one in which the matrices representing the group elements are simply themselves, $M(g) = g$. For $SU(n)$ and $SO(n)$ these are the $n \times n$ matrices defined in Section. 2.1. The states operated on are the n-component column vectors, with complex components for $SU(n)$ and with real components for $SO(n)$.

For $SU(2)$ the fundamental representation is the column vector with two elements and the group elements are 2×2 unitary matrices: $M(g) = g$. E.g.

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad b \longrightarrow M(g)b = gb = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} \tag{12}$$

3.4 Adjoint representation of $SU(n)$ and $SO(n)$

The adjoint representation is the Lie algebra of the Lie group and the action of a group element U or O on a Lie algebra element T is given by

$$T \longrightarrow UTU^\dagger \text{ or } T \longrightarrow OTO^T \tag{13}$$

So in the sense that the Lie algebra is the group (as in eq. (6)), the adjoint representation is also the group. Here both ways of writing are used, the one from Eq. (13) and the standard notation with columnvectors. The latter corresponds to

$$T = \sum_{i=1,n} x_i T_i \longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} . \tag{14}$$

For $SU(2)$ again we take as example

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad U = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \quad T \longrightarrow UTU^\dagger = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{15}$$

This can be written in the standard notation with column matrices with 3 elements with

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad T_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \tag{16}$$

and $M(U)$ is then

$$M(U) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{17}$$