A non-Abelian field strength and kinetic term

In terms of the underlying gaugefields W_μ^i we defined the matrix quantity $W_\mu = (1/2)\tau_i W_\mu^i$ (the τ_i are the Pauli matrices for $SU(2)$, or for other groups the corresponding matrices, e.g. for $SU(3)$ we use instead $G_\mu = (1/2)\lambda_a G_\mu^a$).

As we saw before we had the transformation

$$W_\mu \longrightarrow W'_\mu = UW_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \tag{1}$$

so what happens now if we try to construct the equivalent of the electromagnetic field strength and kinetic term,

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ? \tag{2}$$

Let's try:

$$\begin{aligned}
\partial_\mu W_\nu \quad &\longrightarrow \quad \underbrace{\partial_\mu U W_\nu U^\dagger}_{(4)} + U\partial_\mu W_\nu U^\dagger + \underbrace{UW_\nu \partial_\mu U^\dagger}_{(5)} - \frac{i}{g}\underbrace{\partial_\mu \partial_\nu U U^\dagger}_{(1)} - \frac{i}{g}\underbrace{\partial_\nu U \partial_\mu U^\dagger}_{(2)} \\
-\partial_\nu W_\mu \quad &\longrightarrow \quad -\underbrace{\partial_\nu U W_\mu U^\dagger}_{(6)} - U\partial_\nu W_\mu U^\dagger - \underbrace{UW_\mu \partial_\nu U^\dagger}_{(7)} + \frac{i}{g}\underbrace{\partial_\nu \partial_\mu U U^\dagger}_{(1)} + \frac{i}{g}\underbrace{\partial_\mu U \partial_\nu U^\dagger}_{(3)}
\end{aligned}$$

Well, we would like the sum of those two terms to transform simply[1], i.e. $\partial_\mu W_\nu - \partial_\nu W_\mu \longrightarrow U(\partial_\mu W_\nu - \partial_\nu W_\mu)U^\dagger$, and it obviously does not do that. The terms without extra numbers are the ones we would like to keep and the only extra term that disappears in the sum is the one labelled (1).

We now try to improve matters by adding things. After all, when we constructed the covariant derivative we solved a problem with extra terms by adding the gauge field term to it via $D_\mu = \partial_\mu - igW_\mu$.

The quantity we want has two Lorentz-indices, $\mu\nu$, the simplest other combination we can construct with W_μ and derivatives is $W_\mu W_\nu$. We also add the one with the indices interchanged.

$$\begin{aligned}
W_\mu W_\nu \quad &\longrightarrow \quad UW_\mu W_\nu U^\dagger - \frac{i}{g}\underbrace{\partial_\mu U W_\nu U^\dagger}_{(4)} - \frac{i}{g}\underbrace{UW_\mu U^\dagger \partial_\nu U U^\dagger}_{(7)} - \frac{1}{g^2}\underbrace{\partial_\mu U U^\dagger \partial_\nu U U^\dagger}_{(3)} \\
-W_\nu W_\mu \quad &\longrightarrow \quad -UW_\nu W_\mu U^\dagger + \frac{i}{g}\underbrace{\partial_\nu U W_\mu U^\dagger}_{(6)} + \frac{i}{g}\underbrace{UW_\nu U^\dagger \partial_\mu U U^\dagger}_{(5)} + \frac{1}{g^2}\underbrace{\partial_\nu U U^\dagger \partial_\mu U U^\dagger}_{(2)}
\end{aligned}$$

where we already used $UU^\dagger = 1$. If we multiply the last two lines with $-ig$ we see that the terms labelled (4) and (6) nicely cancel somewhat improving the unwanted terms.

[1] Taking the trace and squaring would then be invariant, see Eq. (4).

But $UU^\dagger = 1$ leads to $\partial_\mu U U^\dagger + U \partial_\mu U^\dagger = 0$ and by multiplying with $U^\dagger$ in front to $U^\dagger \partial_\mu U U^\dagger = -\partial_\mu U^\dagger$. Using this we see that all the other unwanted terms also cancel. The generalization of the electromagnetic (Abelian) field strength $F_{\mu\nu}$ is thus

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig\left(W_\mu W_\nu - W_\nu W_\mu\right) \quad \longrightarrow \quad W'_{\mu\nu} = U W_{\mu\nu} U^\dagger \,. \tag{3}$$

The kinetic terms of the photon field was $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. The generalization[2]

$$-\frac{1}{2} \mathrm{tr}\left(W_{\mu\nu} W^{\mu\nu}\right) \tag{4}$$

is thus the kinetic term for the non-Abelian gauge bosons.

But here there is another difference. $W_{\mu\nu}$ contains terms with two W^i_μ fields. Eq. (4) thus also contains terms with 3 or 4 W^i_μ fields and not only terms with two fields as in the case of the photon. As we saw earlier, terms with 3 or more fields correspond to interactions. Or *the nonabelian gauge symmetry requires interactions among the gauge bosons themselves*. Loosely speaking, the force carriers carry charge themselves. This is why the strong interaction, non-Abelian, behaves so different from the Abelian electromagnetic interaction.

[2]The 1/2 in front is such that the terms with $\left(\partial_\mu W^i_\nu - \partial_\nu W^i_\mu\right)^2$ have the same normalization as in the photon case.