Lecture Notes

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Lecture 4

5 The Dirac Equation

For a plane-wave wavefunction we have

$$\Psi \propto e^{i(\bar{p}\cdot\bar{x}-Et)},$$

which means we can define operators that gives the energy and momentum

$$-i\bar{\nabla}\Psi = \bar{p}\Psi \quad \text{and} \quad i\frac{\partial}{\partial t}\Psi = E\Psi,$$

or equivalently

$$i\partial_\mu \Psi = p_\mu \Psi.$$

We can then interpret the standard Schrödinger equation as a manifestation of the non-relativistic formula for kinetic energy

$$E = \frac{\bar{p}^2}{2m} \quad \rightarrow \quad i\frac{\partial}{\partial t}\Psi = -\frac{1}{2m}\bar{\nabla}^2\Psi.$$

Now we instead want to do this for the relativistic formula, which immediately gives us the Klein–Gordon equation

$$E^2 = \bar{p}^2 + m^2 \quad \rightarrow \quad -\frac{\partial^2}{\partial t^2}\Psi = \left(-\bar{\nabla}^2\Psi + m^2\right).$$

We note immediately that there are problems with negative energy solutions, but there is also a problem with normalisation, which you can read about in the book. The solution is QFT, where the K–G equation is valid for a *scalar* field. Paul Dirac nevertheless tried to overcome the problems, and managed in 1928 to derive an equation describing fermionic fields, for which he got the Nobel Prize in 1933.

Dirac tried to get a linear equation, and came up with

$$i\frac{\partial}{\partial t}\Psi = -i\left(\alpha_x\partial_x + \alpha_y\partial_y + \alpha_z\partial_z\right)\Psi - \beta m\Psi.$$

To figure out what α_i and β are, we note that if applied twice we must get something that looks like the K–G equation:

$$\frac{\partial^2}{\partial t^2}\Psi = \left(\partial_i^2 - m^2\right)\Psi.$$

This gives us

$$\left[\alpha_i\partial_i^2 + \sum_{i>j}\left(\alpha_i\alpha_j + \alpha_j\alpha_i\right)\partial_i\partial_j + im\left(\alpha_i\beta + \beta\alpha_i\right)\partial_i - m^2\beta^2\right]\Psi = \left(\partial_i^2 - m^2\right)\Psi,$$

and the following requirements

$$\begin{aligned} 1 &= \alpha_i^2 = \beta^2 = 1 \\ 0 &= \alpha_i\alpha_j + \alpha_j\alpha_i \equiv \{\alpha_i, \alpha_j\} \\ 0 &= \{\alpha_i, \beta\}. \end{aligned}$$

Hence, the α_i and β are anti-commuting numbers, which means we can represent them with matrices.

If we start with the massless case we can ignore β for a while and we already know about the anti-commuting Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for which $\sigma_i^2 = 1$. So we can simply put $\alpha_i = -\sigma_i$ and get

$$i\frac{\partial}{\partial t}\Psi = \bar{\sigma} \cdot \bar{p}\Psi,$$

where Ψ now needs to be a spin-doublet

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

For the massive case, it is not enough with 2×2 matrices, since we cannot find four different anti-commuting matices. We therefore go to 4×4 and now get four components of Ψ, and find that

$$\alpha_i = \begin{pmatrix} \sigma_i & \mathbf{0} \\ \mathbf{0} & -\sigma_i \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

will do the trick. We can, in principle, choose other matrices, but that basically only results in a reordering of the four components of Ψ (in fact I will use a different choice as compared to the book).

Rather than using α_i and β we will use the so-called Gamma-matrices

$$\gamma^\mu = \left(\gamma^0; \gamma^i\right) = \left(\beta; \beta\alpha_i\right),$$

or

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix},$$

for which we fairly easily get that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

We also get that

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

which we will use later.

So, we had before

$$\partial_0\Psi = \left(\beta m - i\alpha_i\partial_i\right)\Psi,$$

and multiplying with β and rearranging we a nice formula for a T-shirt:

$$\left(i\gamma^\mu\partial_\mu - m\right)\Psi = 0.$$

We have given the gamma matrices Lorentz indices, but they are not normal Lorentz vectors - they are constant matrices which are invariant under a Lorentz tranformation.

Furthermore we note that $\Psi^\dagger\gamma^\mu\gamma^\nu\gamma^\phi\cdots\Psi$ is a scalar. And introducing $\bar{\Psi} = \Psi^\dagger\gamma^0$ we find that

- $\bar{\Psi}\gamma^\mu\Psi$ transforms as a 4-vector and is on equal footing with p^μ and x^μ.
- $\bar{\Psi}\Psi$ is a scalar which is independent of Lorentz frame.
- $\bar{\Psi}\gamma^5\Psi$ is a Lorentz scalar, but is odd under parity ($x_i \to -x_i$). We call that a pseudo-scalar.
- $\bar{\Psi}\gamma^5\gamma^\mu\Psi$ is, similarly, odd under parity, and we call that a pseudo-vector.

We can now go on and look at currents and densities. We start by taking the Hermitian conjugate of the standard formula and get

$$i\gamma^\mu\partial_\mu\Psi - m\Psi = 0 \quad \to \quad -i\partial_\mu\Psi^\dagger\gamma^{\mu\dagger} - m\Psi^\dagger = 0,$$

and note that γ^0 is Hermitian, while $\gamma^{i\dagger} = -\gamma^i$ is anti-Hermitian. We also recall that $\{\gamma^0, \gamma^i\} = 0$, so if we multiply with γ^0 from the right we get

$$-i\partial_\mu\Psi^\dagger\gamma^{\mu\dagger}\gamma^0 - m\bar{\Psi} = 0 \quad \to \quad i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0.$$

Mutiplying this from the right by Ψ and adding the standard formula by $\bar{\Psi}$ mulitplied from the left gives us

$$\left(\partial_\mu\bar{\Psi}\right)\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\partial_\mu\Psi = \partial_\mu\left(\bar{\Psi}\gamma^\mu\Psi\right) = 0$$

which we identify as the equation for the conservation of the current

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi, \quad \partial_\mu j^\mu = 0.$$

This can now be rewritten in terms of the four momentum, remembering that $i\partial_\mu\Psi = p_\mu\Psi$:

$$0 = i\gamma^\mu\partial_\mu\Psi - m\Psi = \gamma^\mu p_\mu\Psi - m\Psi \quad \to \quad \gamma^\nu\gamma^\mu p_\mu\Psi - m\gamma^\nu\Psi,$$

and with $\gamma^\mu\gamma^\mu = g^{\mu\nu}$ we arrive at

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi = \frac{p^\mu}{m}\bar{\Psi}\Psi.$$

and we identiy $\bar{\Psi}\Psi$ with a probability density.

Now we look at free particle solutions to the Dirac equations. First we define the spinors with the anzatz

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}.$$

Then, going back to the Pauli matix basis of the gamma matrices, we can write $(\gamma^\mu\partial_\mu - m)\Psi = (\gamma^\mu p_\mu - m)\Psi = 0$ as

$$\begin{aligned} \begin{pmatrix} -m & p_0 + \bar{\sigma}\cdot\bar{p} \\ p_0 - \bar{\sigma}\cdot\bar{p} & -m \end{pmatrix}\begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} &= 0 \\ \Rightarrow -m\Psi_R + \left(p_0 + \bar{\sigma}\cdot\bar{p}\right)\Psi_L &= 0 \\ \left(p_0 - \bar{\sigma}\cdot\bar{p}\right)\Psi_R - m\Psi_L &= 0. \end{aligned}$$

Now the Pauli matrices measures spin ($\bar{S} = \bar{\sigma}/2$), and in particular the combination $\bar{\sigma} \cdot \bar{p}$ measures spin along the direction of motion, or *helicity*, where $\bar{\sigma} \cdot \hat{p}$ typically is ± 1. For the massless case the equations separate and

$$\begin{aligned} ((p_0 - \bar{\sigma} \cdot \bar{p})\, \Psi_R = 0 \quad &\Rightarrow \quad \frac{\bar{\sigma} \cdot \bar{p}}{p_0} = 1 \\ ((p_0 + \bar{\sigma} \cdot \bar{p})\, \Psi_R = 0 \quad &\Rightarrow \quad \frac{\bar{\sigma} \cdot \bar{p}}{p_0} = -1 \end{aligned}$$

And we say that R is right-handed and L is left-handed. Now for the massive case, a particle's handedness is not well defined (we can always go to a frame where we go faster than the particle), and indeed the equations do not separate. Instead we get

$$\begin{aligned} \Psi_R &= \frac{p_0 + \bar{\sigma} \cdot \bar{p}}{m} \Psi_L \\ \Psi_L &= \frac{p_0 - \bar{\sigma} \cdot \bar{p}}{m} \Psi_R \end{aligned}$$

but still, when we go to relativistic energies $p_0 \gg m$, we get that for left-handed particles $\Psi_L \gg \Psi_R$, and vice versa. In some sense we can say that the mass *measures* the interaction between the left- and right-handed parts of the field.

Now, we are considering free, plane-wave particles, which means we can factorise the wave function in one part which contains the x-dependent part and one that does not,

$$\Psi = u e^{i p^\mu x_\mu},$$

where u now will obey the dirac equation. We have the normalisation, $\bar{u}u = 2m$, for massive particles, or more generally $\bar{u}\gamma^\mu u = 2p^\mu$, which is also OK for massless particles.

We note that the Dirac equation also allows for negative energy solutions, and we identify these with positive energies of anti-particles. Since Ψ has four independent quantities, we identify these with one particle and one anti-particle, each with left- and right-handed components.

For fermions, we will have vertices of the type $\bar{\Psi}\gamma^\mu\Psi \approx \bar{u}\gamma^\mu u$, and we identify u as an incoming fermion or outgoing anti-fermion, and vice versa for $\bar{u}$.

Now, recall

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix},$$

and consider

$$\begin{aligned} P_L &= \frac{1 - \gamma^5}{2} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \text{and} \\ P_R &= \frac{1 + \gamma^5}{2} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \end{aligned}$$

These are now projection operators with the properties

$$\begin{aligned} P_L^2 &= P_L \\ P_R^2 &= P_R \\ P_L + P_R &= 1 \\ P_L P_R &= 0. \end{aligned}$$

We define

$$u = \begin{pmatrix} u_R \\ u_L \end{pmatrix}$$

and get

$$\begin{aligned} u_L &= P_L u = \begin{pmatrix} 0 \\ u_L \end{pmatrix} \quad \text{and} \\ u_R &= P_R u = \begin{pmatrix} u_R \\ 0 \end{pmatrix}. \end{aligned}$$

Now since helicity is frame-dependent for a massive particle, which means that the separation between u_L and u_R is also frame-dependent.

Consider also the parity operation: $\bar{x} \to -\bar{x}; \bar{p} \to -\bar{p}; t \to t; p_0 \to p_0$, which gives $\bar{\sigma} \sim \bar{r} \times \bar{p} \to \bar{\sigma}$. From

$$\begin{aligned} \begin{pmatrix} -m & p_0 + \bar{\sigma} \cdot \bar{p} \\ p_0 - \bar{\sigma} \cdot \bar{p} & -m \end{pmatrix} \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} &= 0 \\ \Rightarrow -m\Psi_R + (p_0 + \bar{\sigma} \cdot \bar{p})\, \Psi_L &= 0 \\ (p_0 - \bar{\sigma} \cdot \bar{p})\, \Psi_R - m\Psi_L &= 0. \end{aligned}$$

it is then clear that parity will flip $\Psi_L \leftrightarrow \Psi_R$. But it turns out that nature is not invariant under parity transformations, and the weak interaction only interact with left-handed fields.

Some useful relations:

- $\bar{\Psi}\gamma^\mu\Psi = \bar{\Psi}_L\gamma^\mu\Psi_L + \bar{\Psi}_R\gamma^\mu\Psi_R$, which means that helicity is preserved in such vertices.
- $\bar{\Psi}\Psi = \bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L$ means that helicity is flipped in interactions of the form $m\bar{\Psi}\Psi$.
- Vertices on the form

$$\begin{aligned} \bar{\Psi}_L\gamma^\mu\Psi_L &= \frac{1}{2}\bar{\Psi}\gamma^\mu\left(1 - \gamma^2\right)\Psi \qquad (V - A) \\ \bar{\Psi}_R\gamma^\mu\Psi_R &= \frac{1}{2}\bar{\Psi}\gamma^\mu\left(1 + \gamma^2\right)\Psi \qquad (V + A) \end{aligned}$$

  corresponds to vertices where only left- or right-handed particles interact.

Finally we write down the Lagrangian corresponding to the Dirac equation:

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi$$

To check we use the Euler–Lagrange equation,

$$\left(\frac{\partial}{\partial\Psi} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial(\partial_\mu\Psi)}\right)\mathcal{L} = 0.$$

Here Ψ and $\bar{\Psi}$ are treated independently and we might as well write it.

$$\left(\frac{\partial}{\partial\bar{\Psi}} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial(\partial_\mu\bar{\Psi})}\right)\mathcal{L} = 0.$$

Since the Largrangian equation does not depend on $(\partial_\mu\bar{\Psi})$, this reduces trivially to

$$\frac{\partial}{\partial\bar{\Psi}}\mathcal{L} = 0 \quad \text{and} \quad \left(i\gamma^\mu\partial_\mu - m\right)\Psi = 0.$$

Lecture 5

6 The Standard Model Lagrangian

We now basically have the ingredients needed to construct the full standard model Lagrangian.

We can divide things up in a number of terms:

- $\mathcal{L}_G$ – pure gauge terms (energy of photons, etc.)
- $\mathcal{L}_F$ – fermions and their interactions with the gauge fields.
- $\mathcal{L}_H$ – the higgs field and its interactions with the (weak) gauge bosons.
- $\mathcal{L}_Y$ – The Yukawa terms giving the masses of the fermions from the interaction with the Higgs field.

Let's start with the fermions. We know that they obey three separate symmetries:

- U(1): Phase-invariance giving rise to electro-magnetic interactions, (but which is mixed with the weak interaction) with the gauge boson γ (spin 1).
- SU(2): non-Abelian part of the weak interaction with gauge boson fields W^i_μ (again spin 1).
- SU(3): non-Abelian strong interaction with eight gauge bosons G^μ_a (spin 1).

The weak boson fields are typically rearranged as

$$\begin{aligned} W^+ &= \frac{-W_1 + iW_2}{\sqrt{2}} \\ W^- &= \frac{-W_1 - iW_2}{\sqrt{2}} \\ W^0 &= W_3 \end{aligned}$$

or

$$\sigma_i W_i = \begin{pmatrix} W^0 & -\sqrt{2}W^+ \\ -\sqrt{2}W^- & -W^0 \end{pmatrix}.$$

The fermion interactions depend on the spin, and we have

- $e^-_R = P_R \Psi_{e^-}$ which is an SU(2) singlet, and
- $e^-_L = P_L \Psi_{e^-}$ which makes up an SU(2) doublet together with ν_e:

$$L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$$

There may be a right-handed electron neutrino also, but let's leave that aside for a moment.

For quarks we also have weak interactions and we define

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad \text{(doublet) and} \quad u_R, d_R \quad \text{(singlets)}$$

Quarks also have colour and are therefor triplets under SU(3) (leptons are singlets):

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}$$

(with indice red, green, and blue). The gluons also carry colour: $r\bar{g}$, $r\bar{b}$, $g\bar{r}$, $g\bar{b}$, $b\bar{g}$, $b\bar{r}$, $(r\bar{r}-g\bar{g})/\sqrt{2}$, and $(r\bar{r}+g\bar{g}-2b\bar{b})/\sqrt{6}$. Note that the singlet combination $(r\bar{r}+g\bar{g}+b\bar{b})/\sqrt{3}$ is not a generator of the group and does not exist.

Now we take the Dirac kinetic energy term in the Lagrangian

$$\bar{\Psi} i\gamma^\mu \partial_\mu \Psi$$

and replace the derivatives and make them covariant with respect to gauge transformations

$$\partial_\mu \to D_\mu - ig_1 \frac{Y}{2} B_\mu - ig_2 \frac{\sigma^i}{2} W^i_\mu - ig_3 \frac{\lambda^a}{2} G^a_\mu$$

where g_i are the coupling strengths, and Y is the weal hyper-charge, which is not à priori known (unlike the charges of of SU(2) and SU(3).

Now we can write the fermion part of the Lagrangian:

$$\mathcal{L}_F = \sum_f \bar{f} i\gamma^\mu D_\mu f,$$

where $f = L, e_R, Q_L, u_r, d_r$ plus the cousins in the other two families. In general, there will be a lot of indices on f: Space×Spin×U(1)×SU(2)×SU(3), but since much of this decouples we rarely write out all of them. We note that G^a_μ acting on leptons vanishes, and that W^i_μ acting on e_R, u_R, d_R also vanishes.

Looking more specifically on the electroweak part of the Lagrangian we have for the off-diagonal part of $\sigma^i W^i_\mu$

$$\begin{aligned} & \begin{pmatrix} \bar{\nu}_L & \bar{e}L \end{pmatrix} i\gamma^\mu(-ig_2) \begin{pmatrix} \mathbf{0} & \sqrt{2}W^+ \\ \sqrt{2}W^- & \mathbf{0} \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ = \; & \frac{g_2}{\sqrt{2}} \left(\bar{\nu}_L \gamma^\mu W^+ e_L + \bar{e}_L \gamma^\mu W^- \nu_L \right) \end{aligned}$$

and for the quarks

$$\frac{g_2}{\sqrt{2}} \left(\bar{u}_L \gamma^\mu W^+ d_L + \bar{d}_L \gamma^\mu W^- u_L \right)$$

From which we can immediately draw the standard neutron decay.

For the neutral W^0_μ current, we note that it will have the same quantum numbers as the B_μ, so they will mix. Comparing the U(1) and SU(2) terms we get

$$\frac{g_1}{2}\left[Y_L\left(\bar{\nu}_L\gamma^\mu\nu_L+\bar{e}_L\gamma^\mu e_L\right)+Y_R\bar{e}_R\gamma^\mu e_R\right]B_\mu$$

and

$$\frac{g_2}{2}\left[\bar{\nu}_L\gamma^\mu\nu_L-\bar{e}_L\gamma^\mu e_L\right]W^0_\mu.$$

Adding them together, we have

$$\left[\frac{g_1}{2}Y_LB_\mu+\frac{g_2}{2}W^0_\mu\right]\bar{\nu}_L\gamma^\mu\nu_L+\left[\frac{g_1}{2}Y_LB_\mu-\frac{g_2}{2}W^0_\mu\right]\bar{e}_L\gamma^\mu e_L+\frac{g_1}{2}Y_RB_\mu\bar{e}_R\gamma^\mu e_R.$$

Now we assume that (B_μ, W^0_μ) are mixed together as

$$\begin{cases} A_\mu = & B_\mu\cos\theta_W+W^0_\mu\sin\theta_W \\ Z^0_\mu = & -B_\mu\sin\theta_W+W^0_\mu\cos\theta_W \end{cases}$$

where we will assume that A_μ is the EM photon field. By requiring that the coupling of A_μ to the neutrinos is zero and that the coupling to the electron is independent of handedness we get that

$$Z^0_\mu\propto\frac{g_1}{2}Y_LB_\mu+\frac{g_2}{2}W^0_\mu$$

so that the full neutrino coupling is to Z^0_μ and none is left for A_μ, and then we require that A_μ should be orthogonal to Z^0_μ which gives

$$A_\mu\propto\frac{g_2}{2}B_\mu-\frac{g_1}{2}Y_LW^0_\mu.$$

or with proper normalization:

$$\begin{aligned} A_\mu &= \frac{g_2B\mu-g_1Y_LW^0_\mu}{\sqrt{g_2^2+g_1^2Y_L^2}} \\ Z^0_\mu &= \frac{g_1Y_LB\mu+g_2W^0_\mu}{\sqrt{g_2^2+g_1^2Y_L^2}}. \end{aligned}$$

Transforming back to (B_μ, W^0_μ) we get

$$\begin{aligned} B_\mu &= \frac{g_1Y_LZ^0_\mu+g_2A\mu}{\sqrt{g_2^2+g_1^2Y_L^2}} \\ W^0_\mu &= \frac{g_2Z^0_\mu-g_1Y_LA\mu}{\sqrt{g_2^2+g_1^2Y_L^2}}. \end{aligned}$$

Inserting back we get

$$\begin{aligned} & \frac{1}{\sqrt{g_2^2+g_1^2Y_L^2}}\left[\frac{g_1}{2}Y_L\left(g_1Y_LZ^0_\mu+g_2A_\mu\right)+\frac{g_2}{2}\left(g_2Z^0_\mu-g_1Y_LA_\mu\right)\right]\bar{\nu}_L\gamma^\mu\nu_L \\ + & \frac{1}{\sqrt{g_2^2+g_1^2Y_L^2}}\left[\frac{g_1}{2}Y_L\left(g_1Y_LZ^0_\mu+g_2A_\mu\right)-\frac{g_2}{2}\left(g_2Z^0_\mu-g_1Y_LA_\mu\right)\right]\bar{e}_L\gamma^\mu e_L \\ + & \frac{1}{\sqrt{g_2^2+g_1^2Y_L^2}}\frac{g_1}{2}Y_R\left(g_1Y_LZ^0_\mu+g_2A_\mu\right)\bar{e}_R\gamma^\mu e_R. \end{aligned}$$

and for the A_μ terms

$$\frac{g_1 g_2}{\sqrt{g_2^2 + g_1^2 Y_L^2}} \left(Y_L \bar{e}_L \gamma^\mu e_L + \frac{Y_R}{2} \bar{e}_R \gamma^\mu e_R \right) A_\mu$$

and we choose to identify this with the EM vector current by setting $Y_L = Y_R/2$. We are then allowed to put $Y_L = -1$ as Y_L only appears in the combination $g_1 Y_L$ and whatever value we choose just means a redefinition of g_1.

Fot the electron we now require that we obtain the electric charge $-e$ resulting in

$$e = \frac{g_1 g_2}{\sqrt{g_2^2 + g_1^2}}$$

We can now also get an expression of the so-called Wineberg angle θ_W by identifying

$$\begin{cases} \sin\theta_W = \frac{g_1}{\sqrt{g_2^2+g_1^2}} \\ \cos\theta_W = \frac{g_2}{\sqrt{g_2^2+g_1^2}} \end{cases} \Rightarrow \begin{cases} g_1 = \frac{e}{\cos\theta_W} \\ g_2 = \frac{e}{\sin\theta_W} \end{cases}$$

The coupling to Z_μ^0 is a bit more messy, but can be sorted out to

$$\frac{1}{\sqrt{g_2^2 + g_1^2}} \left[\left(\frac{g_1^2}{2} + \frac{g_2^2}{2} \right) \bar{\nu}_L \gamma^\mu \nu_L + \left(\frac{g_1^2}{2} - \frac{g_2^2}{2} \right) \bar{e}_L \gamma^\mu e_L + \frac{g_1^2}{2} \bar{e}_R \gamma^\mu e_R \right] Z_\mu^0$$

$$\frac{e}{\sin\theta_W \cos\theta_W} \left[\left(-\frac{1}{2} + \sin^2\theta_W \right) \bar{e}_L \gamma^\mu e_L + \sin^2\theta_W \bar{e}_R \gamma^\mu e_R + \frac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L \right] Z_\mu^0.$$

Here we can identify the *charges* wrt. the Z^0 for different fermions. Dividing these up wrt the charges wrt. W (third component of the weak iso-spin T_3^f) and B (the hyper charge Y_f) we get.

$$\begin{aligned} & \frac{e}{\sin\theta_W \cos\theta_W} \left(T_3^f \cos^2\theta_W - \frac{1}{2} Y_f \sin^2\theta_W \right) \\ = & \frac{e}{\sin\theta_W \cos\theta_W} \left(T_3^f - \left(T_3^f + \frac{1}{2} Y_f \right) \sin^2\theta_W \right) \\ = & \frac{e}{\sin\theta_W \cos\theta_W} \left(T_3^f - Q_f \sin^2\theta_W \right) \end{aligned}$$

where we now can generalize $Q^f = (T_3^f + Y^f/2)$ also to quarks, and we identify

	T_3^f	Y_f	Q_f
ν_L	$+\frac{1}{2}$	-1	0
e_L	$-\frac{1}{2}$	-1	-1
e_R	0	-2	-1
u_L	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
d_L	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
u_R	0	$+\frac{4}{3}$	$+\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$

Since EM and weak interactions mix, we expect $g_1 \approx g_2$ and they should have the same strength and indeed we have from the experimental value of θ_W

$$g_2 = \frac{e}{\sin\theta_W} \approx 2e.$$

So in fact the weak force is stronger than the EM one. But nevertheless we find in nature that the weak force is much weaker, and this is because of the Higgs mechanism which gives W a large mass (later).

Lecture 7

9 Cross sections, Decay widths and Lifetimes

Handout: The standard model in a nutshell

From now on we will be less formal. We will make many approximations, especially when it comes to the treatment of spin. Keeping track of the spin factors is in principle straight forward, but it is quite cumbersome and is better left to specialized computer programs, such as MadGraph. Throughout we will use *natural* units where $\hbar = c = 1$.

We describe the probability of a certain process in terms of its cross section. It is defined in analogy with the situation where a beam of point-like particles with a given number per unit area, f, is hitting an extended target of transverse size σ, where the probability of something a particle hitting the target is σf.

To obtain cross sections we need to compute the *scattering matrix*, S, describing the transition from an initial state, $|i\rangle$ to a final state $|f\rangle$:

$$|f\rangle = S|i\rangle,$$

giving the amplitude

$$S_{fi} = \langle f|S|i\rangle.$$

We then define the normalization of the *matrix element* according to

$$S_{fi} \equiv \delta_{fi} + (2\pi)^4\delta^{(4)}(P_f - P_i)(-i\mathcal{M}_{fi}) \prod_{j=f,i} \frac{1}{\sqrt{2E_j}}.$$

- δ_{fi} is the *Kronecker delta function* which is unity if $i = f$ and zero otherwise. In this case it describes what happens if nothing happens.
- P_i and P_f are the sum of the momenta of all particles in the initial and final state respectively
- $\delta^{(4)}(P)$ is the *Dirac delta function*, in this case in each of the four components of the momentum argument. The delta function is defined according to

  $$\int_{-\infty}^{\infty} h(x)\delta(x)dx \equiv h(0)$$

  for any reasonable function h. This ensures that four-momentum is conserved.
- The product runs over all initial and final state particles and is a normalization convention based on the separation of the space- and time-dependence of the wave-function[1]

  $$\psi = ue^{ip\cdot x} \quad \Rightarrow \quad \bar{u}u = 2E.$$

[1] Kane eq. 5.32.

The probability of a given transition is now propotional to the squared matrix element,

$$d\mathcal{P} \propto \overline{|\mathcal{M}_{fi}|^2}$$

Where we have averaged over all non-observed degrees of freedom (DoF) in the initial state and summed over all DoF in the final state. Eg. if we have an unpolarized fermion in the initial state, each spin state (*up* and *down*) contributes with a factor a half each. Similarly if we have a fermion in the final state where we do not measure the final spin, the two spin states are simply added. We have also summed the amplitudes for all possible diagrams with the same initial and final states.

Introducing an arbitrary volume, $V = L^3$, we get the density of states of a particle by assuming periodic boundary conditions:

$$e^{ip_x L} = 1 \quad \Rightarrow \quad p_x = 2\pi n/L.$$

Looking in three dimensions we get

$$dn = \frac{V d^3p}{(2\pi)^3}.$$

We now get the transition probability assuming fixed momenta in the initial state and a continuum in the final state by squaring the matrix element and multiplying with the appropriate phase space volumes and dividing by the normalisation of the wavefunctions in the volume,

$$\mathrm{d}\Gamma = V(2\pi)^4\delta^{(4)}(P_f - P_i)\overline{|\mathcal{M}_{fi}|^2}\prod_{j=f,i}\frac{1}{2E_jV}\prod_{j=f}\frac{Vd^3p_j}{(2\pi)^3}$$

From which we can get the cross section by dividing by the flux: $d\sigma = d\Gamma/\text{flux}$ (where the flux will contain the additional volume factors to cancel the dependence).

If we consider the scattering of two particles, A and B, to a final state f we get the relative flux factor by looking eg. in the rest frame of B: v_A/V which gives

$$v_A = \frac{\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}{E_A m_B}.$$

In general we get the relative flux factor depending on the relative velocity

$$v_r = \frac{\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}{E_A E_B}.$$

We now get the expression for the cross section

$$d\sigma = \frac{(2\pi)^4\delta^{(4)}(P_f - p_A - p_B)E_A E_B}{\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}\overline{|\mathcal{M}_{fi}|^2}\frac{1}{2E_A}\frac{1}{2E_B}\prod_{j=f}\frac{d^3p_j}{2E_j(2\pi)^3},$$

where the dependence on V is gone, and where the the factor $E_A E_B$ will cancel.

The phase space element is, in fact, Lorentz invariant. It can be derived from the four-momentum phase space element for a free particle

$$\delta(p^2 - m^2)\Theta(p_0)d^4p = \delta(E^2 - \bar{p}^2 - m^2)\Theta(E)dEd^3\bar{p}$$

where the δ-function ensures that the particle is *on-shell* and the Θ-function ensures that we have positive energy. The Dirac delta-function is a bit tricky when it comes to variable substitution. Formally

$$\delta(f(x)) = \sum_{x_0; f(x_0)=0} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

Take eg. a konstant factor k and a test function h, and make a substitution $y = kx$:

$$\int \delta(k(x - x_0))h(x) = \int \delta(y - kx_0)h(\frac{y}{k})\frac{dy}{|k|} = h(\frac{kx_0}{k})\frac{1}{|k|} = \frac{h(x_0)}{|k|}$$

Taylor expanding any function f around $f(x_0) = 0$ we need only take into account the first non-zero term since everything else is negligible, so $f(x) \approx f'(x_0)(x - x_0)$ and we get

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}$$

In our example we then get

$$\delta(E^2 - \bar{p}^2 - m^2)\Theta(E)dEd^3\bar{p} = \delta((E + \sqrt{\bar{p}^2 + m^2})(E - \sqrt{\bar{p}^2 + m^2})\Theta(E)dEd^3\bar{p} =$$

$$\frac{\delta(E - \sqrt{\bar{p}^2 + m^2})}{E + \sqrt{\bar{p}^2 + m^2}}dEd^3\bar{p} = \frac{d^3\bar{p}}{2E}.$$

In a scattering we define $s = (p_A + p_B)^2 = E_{cm}^2$. If $m_A, m_B \ll E_{cm}$ we have

$$s = (p_A + p_B)^2 = m_A^2 + 2p_A \cdot p_B + m_B^2 \approx 2p_A \cdot p_B$$

and

$$4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} \approx 4p_A \cdot p_B \approx 2s$$

And we get the standard form of the scattering cross section

$$d\sigma = (2\pi)^4\delta^{(4)}(P_f - p_A - p_B)\frac{\overline{|\mathcal{M}_{fi}|^2}}{2s}\prod_{j=f}\frac{d^3p_j}{2E_j(2\pi)^3},$$

In a $2 \to 2$ scattering, $A + B \to C + D$ we have in the restframe $p_A + p_B = p_C + p_D = (E_{cm}, 0, 0, 0)$ and

$$\delta^{(4)}(p_C + p_D - p_A - p_B)\frac{d^3p_C}{E_C}\frac{d^3p_D}{E_D} = \delta^{(3)}(\bar{p}_C + \bar{p}_D)\delta(E_C + E_D - E_{cm})\frac{d^3p_C}{E_C}\frac{d^3p_D}{E_D} =$$

$$\delta(E_C + E_D - E_{cm})\frac{d^3p_C}{E_C E_D}$$

Shifting to polar coordinates and introducing $d\Omega = d(\cos\theta)d\phi$ we get

$$\delta(\sqrt{\bar{p}_C^2 + m_C^2} + \sqrt{\bar{p}_C^2 + m_D^2} - E_{cm})\frac{\bar{p}_C^2 d|\bar{p}_C|d\Omega_C}{E_C E_D}$$

Again, we can transform the δ-function,

$$\frac{\delta(|\bar{p}_C| - p(E_{cm}, m_C, m_D))}{\left|\frac{|\bar{p}_C|}{\sqrt{\bar{p}_C^2+m_C^2}} + \frac{|\bar{p}_C|}{\sqrt{\bar{p}_C^2+m_D^2}}\right|}\frac{\bar{p}_C^2 d|\bar{p}_C|d\Omega_C}{E_C E_D}$$

and we can set $p = |\bar{p}_C| = p(E_{cm}, m_C, m_D)$ and $\Omega_C = \Omega$

$$\frac{p^2 d\Omega_C}{\left|\frac{p}{E_C} + \frac{p}{E_D}\right| E_B E_C} = \frac{p d\Omega}{E_C + E_D} = \frac{p d\Omega}{\sqrt{s}}$$

And the cross section becomes

$$d\sigma = (2\pi)^4 \delta^{(4)}(P_f - p_A - p_B) \frac{\overline{|\mathcal{M}_{fi}|^2}}{2s} \frac{d^3 p_C}{2E_C (2\pi)^3} \frac{d^3 p_D}{2E_D (2\pi)^3} =$$

$$\frac{\overline{|\mathcal{M}_{fi}|^2}}{(2\pi)^2 2s} \frac{1}{4} \frac{p}{\sqrt{s}} d\Omega = \overline{|\mathcal{M}_{fi}|^2} \frac{p}{32\pi^2 s^{3/2}} d\Omega$$

Lifetimes, Resonance widths and Decay probability

Now, look at the decay of an unstable particle $R \to C + D$. We get the transition probability

$$d\Gamma = \frac{(2\pi)^4 \delta^{(4)}(P_f - p_R)}{2E_R} \overline{|\mathcal{M}_{fi}|^2} \prod_{j=f} \frac{d^3 p_j}{2E_j (2\pi)^3}.$$

Again, looking at the rest frame of the decaying particle we have $E_R = m_R = \sqrt{s}$ and the same transformation as before of the final state particles gives

$$d\Gamma = \overline{|\mathcal{M}_{fi}|^2} \frac{p}{32\pi^2 m_R^2} d\Omega$$

and integrating over the solid angle (assuming unpolarized R)

$$\Gamma = \overline{|\mathcal{M}_{fi}|^2} \frac{p}{8\pi m_R^2}$$

Assume an exponential decay for a particle formed at $t = 0$:

$$|\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau} \Theta(t)$$

This will give the transition probability

$$\Gamma = -\frac{1}{|\psi(t)|^2} \frac{d|\psi(t)|^2}{dt} = \frac{1}{\tau}$$

Now, the time dependence of the wave function of a free particle in its rest frame is

$$\psi(t) = \psi(0) e^{-iE_\psi t}$$

so we now get

$$\psi(0) e^{-iE_0 t} e^{-t\Gamma/2} \equiv \psi(0) e^{-iE_\psi t}$$

and an unstable particle can be seen to have complex energy $E_\psi = E_0 - i\Gamma/2$. Let's do a Fourier transformation

$$\tilde{\psi}(E) = \frac{1}{\sqrt{2\pi}} \int_\infty dt\, e^{iEt} \psi(t) = \frac{\psi(0)}{\sqrt{2\pi}} \int_0^\infty dt\, e^{i(E - (E_0 - i\Gamma/2)} = \frac{i\psi(0)}{\sqrt{2\pi}} \frac{1}{E - E_0 + i\Gamma/2}$$

$$\Rightarrow |\tilde{\psi}(E)|^2 = \frac{|\psi(0)|^2}{2\pi} \frac{1}{(E - E_0)^2 + \Gamma^2/4}$$

Heissenberg says: undetermined lifetime means undetermined energy and the probability of finding an unstable particle with a given energy is a distribution around E_0.

Scattering through a resonance

We have the decay of a resonance $R \to C + D$ prepared by the fusion of $A + B \to R$ at time $t = 0$. We first remember the scattering in a spherically symmetric potential using partial wave expansion

$$f(\theta) = \sum_l (2l+1) \frac{e^{2i\delta_l} - 1}{2i\kappa} P_l(\cos\theta).$$

where l is the angular momentum, δ_l is the corresponding phase shift, $\kappa = |\bar{p}_C| = |\bar{p}_D|$ and P_l are the Legendre polynomials. The latter have the property

$$\int d\Omega P_l P_{l'} = \frac{4\pi}{2l+1} \delta_{ll'}$$

so squaring and integrating we get the cross section

$$\sigma = \frac{4\pi}{\kappa^2} \sum_l (2l+1) \sin^2 \delta_l$$

For $\delta_l(E) = \pi/2$ we have a large cross section, corresponding to a resonance in the l-channel. We can write

$$\frac{e^{2i\delta_l} - 1}{2i} = e^{i\delta_l} \sin\delta_l = \frac{\sin\delta_l}{\cos\delta_l - i\sin\delta_l} = \frac{1}{\cot\delta_l - i}$$

and Taylor expanding $\cot\delta_l$ around $\delta_l(E_R) = \pi/2$ we have

$$\cot\delta_l(E) = \cot\delta_l(E_R) + (E - E_R) \left.\frac{d\cot\delta_l}{dE}\right|_{E=E_R} + \cdots$$

We will identify $\left.\frac{d\cot\delta_l}{dE}\right|_{E=E_R}$ with $-2/\Gamma$ and get for the dominating l-term

$$f(\theta) \approx f_l(\theta) \approx \frac{2l+1}{\kappa} \frac{P_l(\cos\theta)}{(E - E_R)(-2/\Gamma) - i} = \frac{(2l+1)P_l(\cos\theta)}{\kappa} \frac{\Gamma/2}{(E - E_R) - i\Gamma/2}$$

Squaring again and we get the cross section

$$\sigma_R(E) = \frac{4\pi}{\kappa^2}(2l+1) \frac{\Gamma^2/4}{(E - E_R)^2 + \Gamma^2/4}$$

where the last factor is called the non-relativistic Breit-Wigner distribution and can be compared to the exponential decay above. This distribution is peaked around E_R and falls to half the peak value at $E = E_R \pm \Gamma/2$. Note that we can produce a resonance also away from its nominal rest energy. Also note that the expression is not relativistically invariant. We can making it invariant by by noting that $E + E_R \approx 2m_R \approx 2m$ in the rest frame:

$$\frac{\Gamma/2}{(E - E_R) - i\Gamma/2} = \frac{E + E_R}{E + E_R} \frac{\Gamma/2}{(E - E_R) - i\Gamma/2} \approx \frac{m\Gamma}{(m^2 - m_R^2) - i m_R \Gamma}$$

and with $m^2 = s$ we get the relativistic Breit-Wigner shape of the cross section

$$\sigma(s) = \frac{4\pi}{\kappa^2}(2l+1) \frac{s\Gamma^2}{(s - m_R^2)^2 + m_R^2\Gamma^2} = 16\pi(2l+1) \frac{\Gamma^2}{(s - m_R^2)^2 + m_R^2\Gamma^2}$$

where the last step is true for massless particles where $\kappa = \sqrt{s}/2$.

If we have more than one decay channel the widths are additive, $\Gamma_{tot} = \sum_f \Gamma_f$, and since $\Gamma \propto 1/\tau$ the more decay channels the faster decay. We get the cross section for a particular final state

$$\sigma(A + B \to R \to f) = \sigma(A + B \to R)\frac{\Gamma_f}{\Gamma_{tot}}.$$

Now we reverse the time and get a factor $\frac{\Gamma_i}{\Gamma_{tot}}$ there as well, but here we average rather than sum. Eg. assuming we have spin multiplicities $2s_A + 1$, $2s_B + 1$ and colour multiplicities c_A and c_B for the initial state and the corresponding $2s_R + 1$ and c_R for the resonance decaying to a particular final state, f, we get the cross section

$$\sigma(A + B \to R \to f) \approx 16\pi \left[\frac{(2s_R + 1)c_R}{(2s_A + 1)(2s_B + 1)c_A c_B}\right] \frac{\Gamma_{AB}\Gamma_f}{(s - m_R^2)^2 + m_R^2\Gamma_{tot}^2}$$

A rough estimate of the cross section can be obtained by

$$\sigma \leq \sigma(s = m_R^2) \approx 16\pi\,[\sim 1]\,\frac{\Gamma_{AB}\Gamma_{tot}}{m_R^2\Gamma_{tot}^2} \leq \frac{16\pi}{m_R^2}$$

We can get also an estimate of an upper limit of the cross sectionn for scattering through a resonance by considering an impact parameter picture. Here we assume that two particles are approaching each other at a (transverse) distance b, with longitudinal momentum $\pm p = m_R/2$. The total angular momentum must be conserved:

$$L = \left|\sum \bar{r} \times \bar{p}\right| \sim 2 \cdot \frac{b}{2} \cdot \frac{m_R}{2} = \frac{bm_R}{2}.$$

So for a resonance with spin of the order one we have $b = \frac{2L}{m_R}$ and

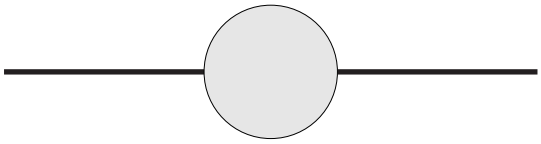
$$\sigma \sim \pi b^2 = \frac{4\pi}{m_R^2}$$

Important cross section formulae

Work in the rest frame of the process.
Assume incoming masses $m_A, m_B \approx 0$.
Define $s = E_{\text{cm}}^2 = (p_A + p_B)^2 = (p_C + p_D)^2$.

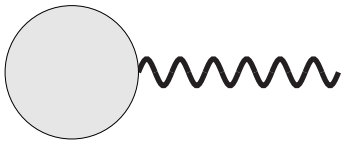
$$\begin{aligned}
A + B \to C + D: &\quad \mathrm{d}\sigma = \overline{|\mathcal{M}_{fi}|^2}\,\frac{p_C}{32\pi^2 s^{3/2}}\,\mathrm{d}\Omega_C \\
R \to C + D: &\quad \mathrm{d}\Gamma = \overline{|\mathcal{M}_{fi}|^2}\,\frac{p_C}{32\pi^2 m_R^2}\,\mathrm{d}\Omega_C \\
\Longrightarrow &\quad \Gamma = \overline{|\mathcal{M}_{fi}|^2}\,\frac{p_C}{8\pi m_R^2} \quad \text{if isotropic} \\
A + B \to R \to C + D: &\quad \sigma = 16\pi\left[\frac{2s_R + 1}{(2s_A + 1)(2s_B + 1)}\,\frac{c_R}{c_A c_B}\right]\frac{\Gamma_R^{AB}\Gamma_R^{CD}}{(s - m_R^2)^2 + m_R^2\Gamma_R^2}
\end{aligned}$$

Approximate Feynman rules



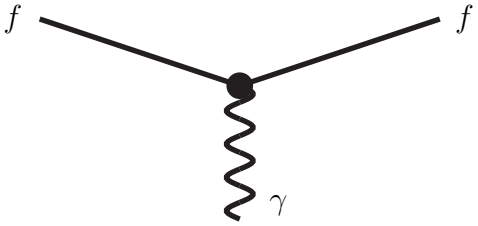
external fermion (pair)

$\overline{u}u = 2E \quad \Rightarrow \quad u \sim \sqrt{2E}$



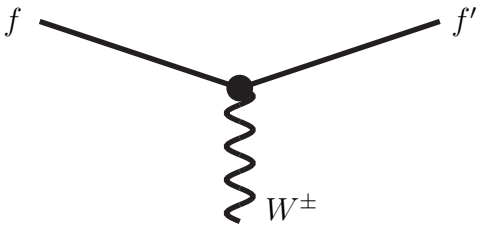
external gauge boson

$\epsilon_\mu \approx 1$



photon vertex

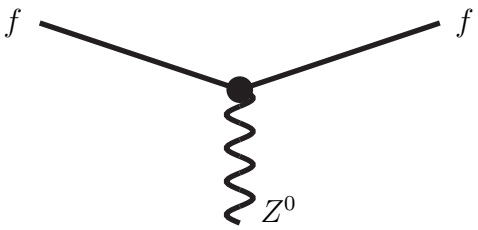
$eQ_f\gamma^\mu \approx eQ_f$



$W^\pm$ vertex

$\frac{g_2}{\sqrt{2}} V_{ff'}^{\mathrm{CKM}}$ for lefthanded fermions

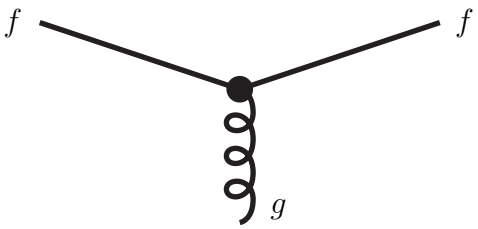
0 for righthanded fermions



Z^0 vertex

$\frac{g_2}{\cos\theta_W}(T_f^3 - Q_f \sin^2\theta_W)$ for lefthanded fermions

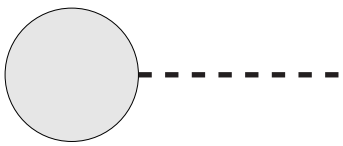
$\frac{g_2}{\cos\theta_W}(-Q_f \sin^2\theta_W)$ for righthanded fermions



gluon vertex

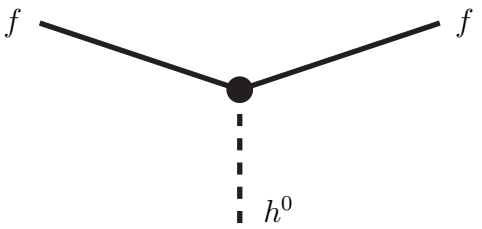
$g_3\gamma^\mu \approx g_3$ for quarks

0 for leptons



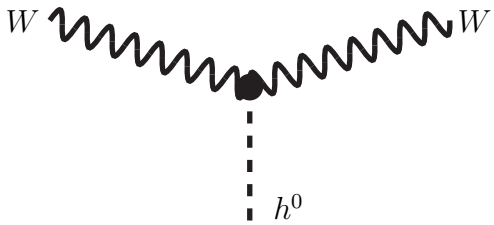
external scalar

1



Higgs–fermion vertex

$g_f = \frac{g_2 m_f}{2m_W}$



Higgs–W vertex

$g_W\, \epsilon\epsilon' \approx g_W = g_2 m_W$

(except if $m_H \gg m_W$)

Boson resonance propagator

$(R = \gamma, W^\pm, Z^0, g, \ldots)$

$$\frac{1}{m^2 - m_R^2 + i m_R \Gamma_R}$$

Feynman rules and Lagrange densities

Looking at the Lagrange density we can identify different kinds of propagators and vertices.

Terms quadratic in a field corresponds to a propagator, i.e. a particle created at some point (source) propagating and destroyed at another point (sink). E.g. for a real scalar we have

$$\mathcal{L} = \frac{1}{2}\left(\delta_\mu \phi \delta^\mu \phi - m^2 \phi^2\right)$$

which gives the requirement on the field

$$\delta^\mu \delta_\mu \phi + m^2 \phi = 0 \quad \Rightarrow \quad -p^2 \phi + m^2 \phi = 0$$

and the propagator

$$\frac{1}{p^2 - m^2}$$

which is the Fourier transform of the so-called Yukawa potential

$$-\frac{1}{4\pi}\frac{e^{-mr}}{r}$$

Ternary terms (with three fields) corresponds to three-vertices. E.g. the interaction term in the electro-magnetic Lagrange density with fermion fields:

$$\mathcal{L}_{int} = -J_\mu A^\mu = Q_f \bar{\psi}_f \gamma_\mu \psi_f A^\mu$$

where $\psi_f = u_f(p) e^{-ipx}$ destroys a fermion, f (or creates the corresponding anti-fermion) with momentum p and $\bar{\psi}_f = \bar{u}_f(p) e^{ip'x}$ creates f (or destroys $\bar{f}$) with momentum p'.

Similarly $A^\mu = \varepsilon^\mu e^{ikx}$ creates (or destroys) a photon with momentum k. Momentum conservation means $e^{-ipx}e^{ip'x}e^{ikx} = 1$, and we are left with the external wave functions

$$\bar{u}_f(p'), \quad u_f(p), \quad \varepsilon^\mu$$

and the vertex factor $Q_f\gamma_\mu$.

Quartic terms will give rise to four-particle vertices, which we will not discuss here.

The W width

From Kane eq. (7.24) we get the part of the Lagrange density for the fermion–W$^\pm$ vertex

$$\mathcal{L} = \ldots + \frac{g_2}{\sqrt{2}}\left[\bar{u}_L\gamma^\mu d_L + \bar{\nu}_e\gamma^\mu e_L\right] W^+_\mu + h.c. + \ldots$$

repeated for the other two families. This gives us the decay modes of the W$^+$:

$$W^+ \to e^+\nu_e, \quad \mu^+\nu_\mu, \quad \tau^+\nu_\tau, \quad u\bar{d}, \quad c\bar{s}, \quad (t\bar{b})$$

The last one is forbidden for real W$^+$ by energy conservation. Note that the quarks come in three colours. We can write down the matrix element for the first decay

$$\mathcal{M}(W^+ \to e^+\nu_e) = \frac{g_2}{\sqrt{2}}\bar{\nu}_L\gamma^\mu e_L\varepsilon_\mu \approx \frac{g_2}{\sqrt{2}}m_W$$

where we have used the standard normalization $\bar{\nu}e \sim 2E \sim m_W$ and the polarization of the photon, $\varepsilon_\mu = \mathcal{O}(1)$. The squared matrix element then becomes $\overline{|\mathcal{M}|^2} = g_2^2 m_W^2/2$, while a full calculation would give $\overline{|\mathcal{M}|^2} = g_2^2 m_W^2/3$. This will give us the width

$$d\Gamma_W^{e\nu} = \overline{|\mathcal{M}|^2}\frac{p}{32\pi^2 m_W^2}d\Omega$$

$$\Rightarrow \Gamma_W^{e\nu} = \frac{g_2^2 m_W}{48\pi} = \frac{\alpha_2 m_W}{12}$$

Summing up to the total width we get a factor three from the three generations of leptons and another factor three for each of the two first generation of quarks:

$$\Gamma_W^{tot} = (1+1+1+3+3)\frac{\alpha_2 m_W}{12} = \frac{3\alpha_2 m_W}{4}.$$

Inserting numbers ($m_W \approx 80$ GeV, $\alpha_2 = 1/30$) gives $\Gamma_W \approx 2$ GeV. Recent experimental results: $m_W = 80.385 \pm 0.015$ GeV, $\Gamma_W = 2.085 \pm 0.042$ GeV.

The Z^0 width

As for the W we have the relevant term in the Lagrange density

$$\mathcal{L} = \ldots + \frac{g_2}{\cos\theta_W}\sum_f \left[\bar{f}_L\gamma^\mu f_L(T_f^3 - Q_f\sin^2\theta_W) + \bar{f}_R\gamma^\mu f_R(-Q_f\sin^2\theta_W)\right] Z_\mu + \ldots$$

$$\Rightarrow \overline{|\mathcal{M}|^2} = \frac{g_2^2}{\cos^2\theta_W} m_Z^2 \left[(T_f^3 - Q_f \sin^2\theta_W)^2 + (-Q_f \sin^2\theta_W)^2\right]$$

and with $\alpha_2 = g_2^2/4\pi$ and a factor 2/3 from spin factors as in the W case we get

$$\Gamma_Z^f = \frac{\alpha_2 m_Z}{6\cos^2\theta_W} N_C^f \left[(T_f^3)^2 - 2T_f^3 Q_f \sin^2\theta_W + 2Q_f^2 \sin^4\theta_W\right]$$

We have very precise measurements from LEP1: $m_Z = 91.1876 \pm 0.0021$ GeV and $\Gamma_Z = 2.4925 \pm 0.0023$ GeV.

We can also measure the partial widths and branching fractions of Z^0 into quarks (BR$(u\bar{u}) \approx 0.12$, BR$(d\bar{d}) \approx 0.15$) and charged leptons (BR$(l^+l^-) \approx 0.033$). From this we can infer the partial width or branching fraction into neutrinos (BR$(\nu\bar{\nu}) \approx 0.07$) — the invisible width — which means we can measure the number of (massless) neutrino species and hence also the number of families:

$$n_\nu = \frac{\Gamma_Z^{\text{invisible}}}{\Gamma_Z^\nu} = 2.983 \pm 0.009$$

10 Production of $W^\pm$ and Z^0

The cleanest way to produce real and *on-shell* W and Z is by lepton fusion. At LEP1 (1989–94) (and at the SLC) they used $e^+e^- \to Z^0 \to$ fermions. The corresponding production of W, eg. $e^-\bar{\nu}_e \to W^- \to$ fermions is not technically feasible since we cannot create a beam of neutrinos which is well enough focused. Instead at LEP2 (1996–2000) they used the processes $e^+e^- \to W^+W^-$ (exchanging a neutrino) and $e^+e^- \to Z^\star/\gamma^\star \to W^+W^-$ (via an off-shell Z^0 or γ). The latter uses the fact that $SU(2)$ is a non-abelian theory so that we have terms in the Lagrange density which couples the W field to itself.

Alternatively we can use the fusion of quarks into W/Z. For this we can collide hadrons which consist of quarks and use processes such as $u\bar{u} \to Z^0 \to$ fermions and $u\bar{d} \to W^- \to$ fermions. At LEP the processes involving W/Z are dominating if we tune our collision energy to hit the resonance peak. Here, we have controll of the hadronic collision energy but not of the energy of the colliding quarks. In addition, quarks interact much more strongly with gluons (QCD), so any electro–weak process will drown in a lot of QCD processes. Therefore the dominant decay of Z^0 into quarks cannot be used as a singnal. Neither can the decay into neutrinos. However the decay into chagred leptons can be used (but remember the branching ratio is small). Similarly for the W the decay into quarks will drown in the background of QCD processes. The only way is to look for a charged lepton and then look for the fact that the non-observed neutrino will induce a imbalance of the transverse momentum of all the detected particles. The W and Z was first discovered at a proton–anti-proton collider at CERN in 1983.

To calculate the cross section of W/Z at hadron colliders we have to try to describe the quarks (and gluons) inside the hadron. We know that a proton consists of three *valence* quarks (uud) together with a number of gluons and virtual quark–anti-quark pairs (*sea* quarks). We call the quarks and gluons inside the proton *partons*.

To calculate the cross section we use the concept of a *parton density*: $f_{i/p}(x)$ is the probability to find a parton of type i inside a proton carrying a fraction x of the protons

momentum. Unfortunately we cannot calculate the $f_{i/p}$ (however some of its behavior can be calculated, as we will see later),[2] but we can infer some *sum-rules* such as $\sum_i \int_0^1 x f_{i/p}(x)dx = 1$ and we can measure them. It turns out that the (logarithmic) density of valence quarks peaks around 0.15 and the total fraction of the protons momentum carried by the valence quarks is about 50%. Also it turns out that the gluon density increases dramatically at small x as does the sea-quark density (although it is typically much smaller than the gluon density).

Let's look at the reaction $p\bar{p} \to$ proton remnants $+ [u\bar{d} \to W^+]$. We define

$$s = (p_p + p_{\bar{p}})^2$$

and we look in the rest frame of the hadronic collision, so

$$p_p \approx \frac{\sqrt{s}}{2}(1;0,0,1), \qquad p_{\bar{p}} \approx \frac{\sqrt{s}}{2}(1;0,0,-1),$$

and the momentua of the colliding quarks are

$$p_u = x_1 p_p, \qquad p_{\bar{d}} = x_2 p_{\bar{p}}.$$

Then we define

$$\hat{s} = (p_u + p_{\bar{d}})^2 \approx 2p_u p_{\bar{d}} = x_1 x_2 p_p p_{\bar{p}} \approx x_1 x_2 (p_p + p_{\bar{p}})^2 = x_1 x_2 s$$

We can write the cross section

$$\sigma(W^+) = \int dx_1 dx_2 f_{u/p}(x_1) f_{\bar{d}/\bar{p}}(x_2) \hat{\sigma}_{u\bar{d}\to W^+}(\hat{s})$$

We have basically already calculated the *partonic* cross section

$$\hat{\sigma}(\hat{s}) = 16\pi \left[\frac{3}{2\cdot 2}\cdot\frac{1}{3\cdot 3}\right] \frac{\Gamma_W^{u\bar{d}}\Gamma_W^f}{(\hat{s}-m_W^2)^2 + m_W^2\Gamma_W^2}.$$

To make life simple we use the narrow-width approximation where we note that

$$\int_{-\infty}^{\infty} \frac{d\hat{s}}{(\hat{s}-m_W^2)^2 + m_W^2\Gamma_W^2} = \int_{-\infty}^{\infty} \frac{d\tilde{s}}{\tilde{s}^2 + m_W^2\Gamma_W^2} = \frac{1}{m_W\Gamma_W}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{m_W\Gamma_W},$$

with the successive substitutions $\hat{s} - m_W^2 \to \tilde{s}$, $\tilde{s}/m_W\Gamma_W \to x$. Hence, if we want to replace the distribution in $\hat{s}$ with a delta-function we use

$$\frac{d\hat{s}}{(\hat{s}-m_W^2)^2 + m_W^2\Gamma_W^2} \to \frac{\pi\delta(\hat{s}-m_W^2)}{m_W\Gamma_W}$$

and we get

$$\hat{\sigma}(\hat{s}) \approx \frac{4\pi}{3}\Gamma_W^{u\bar{d}}\Gamma_W^f \frac{\pi}{m_W\Gamma_W}\delta(\hat{s}-m_W^2).$$

[2] In Kane one could get the impression that there are formulae available at `pdg.lbl.gov`, but that it not the case. There are, however, numerical parameterisations available as computer code at `lhapdf.hepforge.org`.

With $\hat{s} = x_1 x_2 s$ we get the total cross section

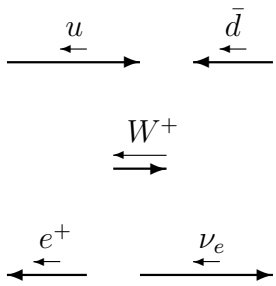
$$\begin{aligned}
\sigma(W^+) &= \int dx_1 dx_2 f_{u/p}(x_1) f_{\bar{d}/\bar{p}}(x_2) \frac{4\pi^2}{3} \frac{\Gamma_W^{u\bar{d}} \Gamma_W^f}{m_W \Gamma_W} \frac{1}{s} \delta(x_1 x_2 - m_W^2/s) \\
&= \frac{4\pi^2}{3} \frac{\Gamma_W}{s m_W} \mathrm{BR}(W \to u\bar{d}) \mathrm{BR}(W \to f) \int dx_1 dx_2 f_{u/p}(x_1) f_{\bar{d}/\bar{p}}(x_2) \frac{1}{x_1} \delta(x_2 - \frac{m_W^2}{s x_1}) \\
&= \frac{4\pi^2}{3} \frac{\Gamma_W}{s m_W} \mathrm{BR}(W \to u\bar{d}) \mathrm{BR}(W \to f) \int_{m_W^2/s} \frac{dx}{x} f_{u/p}(x) f_{\bar{d}/\bar{p}}(\frac{m_W^2}{s x_1}).
\end{aligned}$$

In the same way we can get the cross section for $u\bar{u} \to Z^0$, etc.

Experimentally we find the Z^0 by looking for two oppositely charged leptons, e^+e^- or $\mu^+\mu^-$ (the tau decay hadronically and is more difficult to detect). These can, of course, come from other sources, mainly from the weak decay of heavy quarks, but if we in each event where we find eg. an e^+e^--pair, calculate the quantity $m_{+-}^2 = (p_{e^+} + p_{e^-})^2 \approx 2E_{e^+}E_{e^-}(1 - \cos\theta_{+-})$ and look at the distribution of these, we should find a (Breit–Wiegner) peak around $m_{+-} \approx m_Z$.

For the W, things are a bit more complicated since we cannot calculate the corresponding mass $m_{l\nu}^2 \approx 2E_l E_\nu (1 - \cos\theta_{l\nu})$ as we do not know E_ν. We can, however, measure the transverse momentum components of the neutrino, assuming this was the only particle not detected: $p_{\nu\perp} \approx |\sum_i \bar{p}_{i\perp}|$, where the sum runs over all detected particles (energy) in an event. If we look at the distribution of $m_\perp^2 = 2p_{l\perp}p_{\nu\perp}(1 - \cos\Delta\phi_{l\nu})$ and we assume that all Ws decay in a plane transverse to the beam axis, we would expect to find a Breit–Wigner peak around $m_\perp \approx m_W$. The fact that the decay plane is has a distribution polar angle means the peak will become smeared and slightly shifted towards lower $m_\perp$ (a Jacobian peak).

Then we have the spin properties of the W which can be checked. We know that W^+ only couples to left-handed (spin oposite to the direction of motion) fermions and right-handed (spin parallel to the direction of motion) anti-fermions. Hence if a proton comes along the positive z-axis and the anti-proton along the negative, a W^+ will normally have been produced from a u in the proton and a $\bar{d}$ in the anti-proton and its spin will then be along the negative z-axis. This means that when it decays into a $e^+\nu_e$ the e^+ cannot go along the positive z-axis since its spin must be paralell to its direction of motion. Hence if we look at the distribution in azimuth angle of e^+ in events believed to be W^+ events we expect it to be peaked in the direction opposite to the incoming proton. This is exactly what was found.



Lecture 8

11 Measuring Standard Model parameters

The Standard Model of particle physics contains of the order of 20 parameters: the masses of the fundamental particles, a number of *mixing angles* (later), and the couplings constants

$$\left.\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right\} \Rightarrow \left\{\begin{array}{c}\alpha_{\rm EM} = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\\ \sin^2\theta_W = \frac{\alpha_1}{\alpha_1+\alpha_2}\end{array}\right.$$
$$\alpha_3 \quad \Leftrightarrow \quad \alpha_S$$

$\alpha_{\rm EM}$ is best measured with the quantum hall effect (solid state) giving an impressive accuracy: $\alpha_{\rm EM}(0) = 1/137.035999139(31)$.

The precision of $\sin^2\theta_W$ is not as good, but it can be measured in many ways to ensure consistency. One of the best measurements is from the width of the Z at LEP1:

$$\Gamma_Z = \frac{\alpha_{\rm EM} m_Z}{\sin^2\theta_W \cos^2\theta_W} f(\sin^2\theta_W).$$

We can also use the simpler $\sin^2\theta_W = 1 - m_W^2/m_Z^2$. The current best value[3] is $\sin^2\theta_W(M_Z) = 0.23122(15)$.

α_S is even trickier to measure. The best measurement is from the ratio between the cross sections $\sigma(e^+e^- \to Z^0 \to q\bar{q}g)$ and $\sigma(e^+e^- \to Z^0 \to q\bar{q})$. But we cannot detect neither quarks or gluons, all we see is jets of hadrons, and jets are nototiously difficult to define in a way so that we can calculate this ratio to a reasonable precision. Current best value is $\alpha_S(m_Z) = 0.1180(12)$.

The place to find the best measurements of coupling constants, masses, widths, mixing angles and other stuff is in the Particle Data book `http://pdg.lbl.gov/`.

The muon lifetime

Let's try to calculate the lifetime of a muon. This is a useful exercise as most weak deacys of particles are calculated in the same way. We can draw the decay with a Feynman diagram where the muon first decays into a muon-neutrino and a W^- immediately followed by the decay of the W^- into an electron and an anti-electron-neutrino. Denoting the momenta of the different particles, $\mu(p) \to \nu_\mu(k) + [W^-(Q) \to e^-(q) + \bar{\nu}_e(k')]$, we use the Fynman rules to write down the matrix element

$$\mathcal{M} = \frac{g_2}{\sqrt{2}}\left(\bar{\nu}_\mu\gamma^\lambda P_L\mu\right)\frac{1}{Q^2 - m_W^2}\frac{g_2}{\sqrt{2}}\left(\bar{e}\gamma_\lambda P_L\nu_e\right)$$

with $P_L = 1 - \gamma_5$. The first parenthesis is the vertex factor for the $\mu\nu_\mu W$ vertex and the second the vertex factor for the $e\nu_e W$ vertex. Sandwiched inbetween is the *propagator*

[3] The fact that the coupling *constants* are given as a function of an energy will be addressed later on.

factor corresponding to the intermediate W. Since the maximum value of $Q^2 \sim m_\mu^2$ is much smaller than m_W^2, the propagator is to a very good approximation $1/m_W^2$. If we ignore the spinor complication and remember that the wavefunctions are basically just given by the square root of the energy we get m_μ for each of the vertices and the full matrix element is

$$\mathcal{M} = \frac{g_2^2}{2}\frac{m_\mu^2}{m_W^2} = 2\sqrt{2}G_F m_\mu^2.$$

Now we can write down the decay rate differential in the momenta of the incoming and outgoing particles:

$$\begin{aligned} d\Gamma_\mu &= \frac{(2\pi)^4\delta^{(4)}(k+q+k'-p)}{2m_\mu}\overline{|\mathcal{M}|^2}\frac{d^3k}{2E_k(2\pi)^3}\frac{d^3q}{2E_q(2\pi)^3}\frac{d^3k'}{2E_{k'}(2\pi)^3} \\ &= \frac{1}{(2\pi)^5}\cdot\frac{8G_F^2m_\mu^4}{2m_\mu}\cdot\frac{1}{2}\delta^{(4)}(k+q+k'-p)\frac{d^3k}{2E_k}\frac{d^3q}{2E_q}\frac{d^3k'}{2E_{k'}}, \end{aligned}$$

where the extra factor one half comes from the fact that only left-handed muons are involved. We want to move to the rest frame of the muon, so $p = (m_\mu; 0, 0, 0)$ so the differential can be written

$$\delta^{(3)}(\bar{k}+\bar{q}+\bar{k}')\delta(E_k+E_p+E_{k'}-m_\mu)\frac{d^3k}{2E_k}\frac{d^3q}{2E_q}\frac{d^3k'}{2E_{k'}},$$

and if we ignore the masses of the decay products we get

$$\delta(|\bar{k}|+|\bar{q}|+|\bar{k}+\bar{q}|-m_\mu)\frac{d^3kd^3q}{8\,|\bar{k}|\cdot|\bar{q}|\cdot|\bar{k}+\bar{q}|}.$$

We now go to polar coordinates (with k relative to a given axis and with q relative to k [**draw diagram**])

$$\delta(|\bar{k}|+|\bar{q}|+\sqrt{\bar{k}^2+\bar{q}^2+2|\bar{k}||\bar{q}|\cos\theta_q}-m_\mu)\frac{\bar{k}^2d|\bar{k}|d\Omega_k\cdot\bar{q}^2d|\bar{q}|d\Omega_q}{8|\bar{k}||\bar{q}|\sqrt{\bar{k}^2+\bar{q}^2+2|\bar{k}||\bar{q}|\cos\theta_q}}$$

Here we can always do the integral over Ω_k ($\Rightarrow 4\pi$) and over the azimuth angle of q around the k direction ($\Rightarrow 2\pi$), leaving us with

$$\delta(|\bar{k}|+|\bar{q}|+\sqrt{\bar{k}^2+\bar{q}^2+2|\bar{k}||\bar{q}|\cos\theta_q}-m_\mu)\frac{4\pi|\bar{k}|d|\bar{k}|\cdot 2\pi|\bar{q}|d|\bar{q}|d\cos\theta_q}{8\sqrt{\bar{k}^2+\bar{q}^2+2|\bar{k}||\bar{q}|\cos\theta_q}}$$

Integrating also over $\cos\theta_q$, the delta function gives

$$\sqrt{\bar{k}^2+\bar{q}^2+2|\bar{k}||\bar{q}|\cos\theta_q} = m_\mu - |\bar{k}| - |\bar{q}|$$

and an inverse jacobian

$$\frac{\sqrt{\bar{k}^2+\bar{q}^2+2|\bar{k}||\bar{q}|\cos\theta_q}}{|\bar{k}||\bar{q}|}$$

Simplifying miraculously to

$$\pi^2 d|\bar{k}|d|\bar{q}|$$

and we get the differential deacy width

$$d\Gamma_\mu = \frac{2G_F^2 m_\mu^3}{(2\pi)^5}\pi^2 d|\bar{k}|d|\bar{q}|$$

which can be integrated ($|\bar{k}| < m_\mu/2$, $|\bar{q}| < m_\mu/2$, $m_\mu - |\bar{k}| - |\bar{q}| < m_\mu/2$) giving the total width

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{128\pi^3}$$

or, if you do the spin and kinematics correctly,

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

So, by measuring Γ_μ and m_μ we can get a good value for g_2 (and, indirectly, $\sin^2\theta_W$).

[Draw the Dalitz plot for k and q.]

18.10 The top quark

The top quark was discovered in 1995 at Fermilab outside of Chicago in USA. The mass has been measured to be $m_t = 174.2 \pm 2.0 \pm 2.6$ GeV. The way they found it was to use the fact that the top will decay (before it hadronizes) to a b and a W^+. Top quarks are pair-produced on $p\bar{p}$ collisions through $u\bar{u} \to g^\star \to t\bar{t} \to bW^+\bar{b}W^-$. The W can be detected in the usual way with high-transverse-momentum leptons and missing transverse momentum (a bit trickier since the Ws have a substantial transverse momentum and the missing momentum is the sum of the two neutrinos). The b-quarks will hadronize into B mesons which have a very long lifetime, giving rise to secondary vertices around a millimeter from the collision vertex.

Even before the top quark was directly observed there was plenty of evidence that it must exist (if the Standard Model is correct). One way is to consider the decay of the Z^0 into $b\bar{b}$-pairs:

$$\Gamma_Z^{b\bar{b}} \propto (T_b^3 - Q_b \sin^2\theta_W)^2 + Q_b^2 \sin^2\theta_W$$

The first term is for left-handed b-quarks and here the T_b^3 is $-1/2$ if the b is part of a doublet, but 0 if it is a singlet. Alternatively one may consider the *forward–backward asymmetry* as in Kane.

0.-1 The tau-neutrino

In the same way there was indirect evidence for ν_τ (also from $\Gamma_Z^{\text{invisible}}$) before it was directly observed at the DONUT experiment at Fermilab in 2000. At DONUT they constructed a beam believed to consist of tau-neutrinos. If so, a ν_τ may exchange a W with a nuclear target, producing a τ which is detected as it decays $\tau^- \to \nu_\tau e^- \bar{\nu}_e$ a couple of tenths of a millimeter away from the primary collision vertex. They only observed four such events, but that was enough.

Lecture 9

12 Accelerators

We study the fundamental particles by colliding them at high energies. To get them to high energies we use accelerators. These use electrical fields to accelerate beams of charged particles. They also use magnetic fields to bend and focus the beams.

In a linear accelerator the energies achievable depends on the length, giving approximately 5 MeV/m (but new techniques are being developed to increase this).

Alternatively the beams are bent around in storage rings which enables repeated acceleration. The only set-back here is the synchrotron radiation which means that the beam loses energy $dE/dt \propto \gamma^4/R^2 \propto E^4/(m^4R^2)$ (larger ring or heavier particles give less energy loss)[4]. Hence for $e^\pm$ beams this is a real problem. For (anti-) proton beams there is, on the other hand, the problem that we need very strong magnets to bend the beam, and also that the proton is composite, which means that the quarks (gluons) which interacts only carries around 10% of the energy.

Major accelerators

name	beams	location	period	energy (GeV)	main physics
S$p\bar{p}$S	$p\bar{p}$	CERN	81–89	630	$Z^0, W^\pm$
Tevatron	$p\bar{p}$	FNAL	85–11	2000	top
LEP1	e^+e^-	CERN	89–95	91	Z^0
SLC	e^+e^-	SLAC	89–98	91	Z^0
HERA	ep	DESY	91–07	30×900	QCD
LEP2	e^+e^-	CERN	95–00	–209	$W^\pm$
PEP-II	e^+e^-	SLAC	99–06	10	B
KEKB	e^+e^-	KEK	99–	10	B
RHIC	AuAu	BNL	00–	200×A	QGP
LHC	pp(PbPb, pPb)	CERN	10–13	$7-8\,000$	H, BSM(?)
LHC	pp(PbPb)	CERN	15–	$13-14\,000$	H, BSM(?)
HL-LHC	pp(PbPb)	CERN	24–	same ($10\times\mathcal{L}$)	H, BSM(?)

Major LHC experiments

name	physics orientation	public info
ATLAS	big general-purpose, incl. Lund	`http://atlas.ch/`
CMS	big general-purpose	`http://cms.cern.ch/`
ALICE	heavy-ion physics, incl. Lund	`http://aliceinfo.cern.ch/Public/`
LHCb	B physics	`http://lhcb-public.cern.ch/`

[4]Different quantity in Kane: energy loss per revolution.

Proposed future accelerators

None of these are approved projects.
Only ILC has a complete Technical Design Report, and an ongoing political process.
FCC = FEC + FHC (Future Circular Collider, ditto Electron, ditto Hadron)
feasibility studies now starting up at CERN.
Chinese CPEC + SppC plans getting serious; site selected.
Dates are extremely speculative, and machine parameters only approximate.
BSM = Beyond the Standard Model

name	beams	location	period	energy (GeV)	main physics
LEP3	e^+e^-	CERN	2025–	240	H
CPEC	e^+e^-	China	2028–	240	H
ILC	e^+e^-	Japan	2028–	250–1000	H, t, BSM
CLIC	e^+e^-	CERN	2030–	3000	H, BSM
FEC	e^+e^-	CERN	2035–	90–350	H, t, BSM
VLHC	pp	FNAL	2035–	100 000	BSM
FHC	pp	CERN	2040–	–100 000	BSM
SppC	pp	China	2042–	50 000–	BSM
$\mu\mu$C	$\mu^+\mu^-$	FNAL(?)	2040–	–5000	BSM
LHeC	ep	CERN	2025-	60×7000	QCD

The performance of an accelerator is, besides the collision energy, determined by the *luminosity*, $\mathcal{L}$. The number of events we can expect for a given process with cross section, σ, is determined by the integrated luminosity:

$$N = \sigma \int dt \mathcal{L}$$

We can use the accelerated beam to hit a stationary target. The luminosity is then given by the number of particles hitting the target per unit time, n, and the density, ρ, and length, l, of the target. A beam of transverse size, σ drills a *hole* in the target of volume $V = l\sigma$ where we can find $n' = V\rho$ particles in the target, giving us a luminosity

$$\mathcal{L} = n \cdot n'/\sigma = n\rho l$$

For a fixed-target machine the (invariant) collision energy is approximately given by $\sqrt{(p_{\text{beam}} + p_{\text{target}})^2} \approx \sqrt{2E_{\text{beam}} m_{\text{target}}}$, so the energy increases only like the square root of the beam energy.

For colliding beams, on the other hand we get a collision energy of $\sqrt{4E_+E_-}$, which is much more economic, if you are able to focus the beams against eachother properly. If we have a storage ring with n bunches of k particles per bunch going round at a frequency f and the cross section of the interaction area is A we get a luminosity[5]

$$\mathcal{L} = fnk^2/A$$

(note error in the book).

[5]The formula in Kane is wrong. we can only have one bunch crossing at the time in a detector, hence $\mathcal{L} \propto fn$. in each bunch crossing we have $\mathcal{L} \propto k^2/A$

13 Experiments and Detectors

At LHC we expect of the order of a hundred particles being produced per collision. We wish to detect as many of them as possible so we want a good solid-angle coverage of detectors $\Omega \to 4\pi$.

For charged particles it is typically possible to track them as they interact electrically with detector elements. If we also have a magnetic field we can determine their momentum by the way they bend. For neutral particles we can use calorimeters to detect the energy deposition in the detector parts. Typical examples of detectors are:

- Micro-vertex detectors: (CCD) silicon pixels $(20-50\mu m)^2$ layered very close to the interaction point. Detects charged particles.
- Wire chambers: $\sim 1m$ in radius, filled with gas and wires under high voltage. The passing of a charged particle induces a discharge between the wires which can be measured.
- Cherenkov counters: Particles which traverses a medium at a speed larger than the speed of light in that medium will emit Cherenkov radiation (similar to a supersonic bang). The photons are detected and gives information about the mass of the particle.
- Electro-magnetic calorimeter: Eg. lead glass and photon detectors. Also a neutral particle will interact with nuclei in the detector giving flashes of photons.
- Hadronic calorimeter: Hadron interacts strongly with matter (typically iron) and we get ionization (kicking out charged hardons from the nucleus) of the detector which can be detected.
- Muon Chambers: Any charged particle which can penetrate the calorimeters, typically meters thick, is most likely to be a muon.

Nice pictures

No detector is perfect. For identified particles there is always uncertainties due to non-perfect *efficiencies* and *purities*. For measurements of energy and momenta there is always an uncertainty, typically increasing with energy $\Delta E \propto \sqrt{E}$.

14 Low Energy and Non-Accelerator Experiments

The Standard Model works extremely well. Especially now that the final piece of the puzzle, the Higs particle, has been found. There is really no hard evidence of any physics *Beyond* the Standard Model (BSM). Of course every particle physicists dream is to find something which does not fit into the standard model (others, like myself, tries to improve our understanding of the SM so that we have a chance to find deviations). To do this one can either try to crank up the energy in the collision experiments, hoping to find new particles, which necessarily will be associated with BSM physics. Building larger and larger accelerators is, however, very expensive and difficult. Alternatively one can use

"natural" accelerators which seem to exist in the universe, which constantly bombard us with high energy particles (these are, however, rather difficult to controll).

There is one thing one can do without high collision energies, and that is to search for low-energy processes which are very rare or even forbidden in th SM.

One example thing is to look for effects of massive neutrinos in nuclear β-decay (later).

Examples of decays which are forbidden in the standard model are lepton-family-number-violating processes such as $\mu^- \to e^-\gamma$ (current limit $BR < 5 \cdot 10^{-11}$) or $\mu^- \to e^-e^+e^-$ ($BR < 10^{-12}$). Also some hadron decays are forbidden: $K^0 \to \mu e$ ($BR < 3 \cdot 10^{-11}$), $K^+ \to \pi^+ e\mu$ ($BR < 2 \cdot 10^{-10}$). Especially forbidden is proton decay ($\tau_p > 10^{32}$ y).

15 Finding the Higgs

To complete the Standard Model we need to find the Higgs particle, and this has therefore been the main objective of many accelerators in the past. Before it was eventually found at the LHC, there were indirect measurements that favoured a light Higgs. LEP2, as it was about to close down, had a couple of events which could be Higgs, but the significance turned out to be too low. They did, however set the best limit at the time on the mass: $m_h \geq 113$ GeV (95% confidence). Even the Tevatron had something which looked like a Higgs event, but the significance was even lower. Finally in 2012, the Higgs was discovered at LHC.

To understand how we can produce and detect the Higgs, we start by looking at the relevant terms in the Lagrange density and find that there are fermion–fermion–Higgs, W–W–Higgs and Z–Z–Higgs vertices, but no vertices containing gluons or photons.

For the fermion vertex we have the coupling

$$g_f = \frac{g_2 m_f}{2m_W}$$

so we can produce a Higgs by annihilating a ferminon–anti-fermion pair, and conversely the Higgs can decay into a ferminon–anti-fermion pair. We can approximate the matrix element by

$$\overline{|\mathcal{M}|^2} \approx (g_f m_h)^2 c_f 2$$

where c_f is three for quarks and 1 for leptons and 2 comes from the sum over spin states times a fudge factor. From this we get the partial width

$$\Gamma^h_{f\bar{f}} \approx \frac{|\bar{p}_f|}{m_h^2} \frac{\int d\Omega}{32\pi^2} \overline{|\mathcal{M}|^2} = \alpha_2 \frac{c_f}{8} \frac{m_f^2}{m_W^2} m_h.$$

Hence, we should use heavy fermions if we want to produce Higgs, and when the Higgs decays it will predominately do so to heavy fermions. In particular, considering the hierarchy of fermion masses, the Higgs will mainly decay to the heaviest possible fermion (with $m_f < m_h/2$). Note that close to threshold there will be a kinematical damping of the decay width

$$\beta^2 = (1 - \frac{4m_f^2}{m_h^2})^{3/2}$$

For the W–W–Higgs vertex we have to calculate a bit more since we cannot make the approximation $u \sim \sqrt{2E}$ for the W wavefunctions, mainly because the W can be longitudinally polarized. The calculation in is performed in Kane (21.2) and the result is

$$\overline{|\mathcal{M}|^2} \approx \frac{g_2 m_h^4}{2m_W^2}$$

giving the patial width

$$\Gamma_{WW}^h \approx \frac{\alpha_2 m_h^3}{16m_W^2}.$$

Similarly we get

$$\Gamma_{ZZ}^h \approx \frac{\alpha_2 m_h^3}{32m_W^2}.$$

So we get a width of the Higgs which is proportional to the mass^3. This means that a light Higgs is very narrow, while a heavy Higgs is extremely broad (in TeV units we get $\Gamma \sim 0.5 m_h^3$).

In $e^+e^- \to Z^0$ annihilation we can look for higgs using $Z^0 \to Z^0 h$, where either the incoming (LEP2) or the outgoing (LEP1) Z^0 is off-shell (reduced cross section). Detecting the final Z^0 decaying into leptons we can calculate the missing mass, $m_{\text{miss}}^2 = (p_{e^+} + p_{e^-} - p_{l^+} - p_{l^-})^2$, where we should find a peak corresponding to the Higgs mass. The higgs will then mainly decay into $b\bar{b}$ (jets) and we can study the angular distribution to confirm the scalar nature of the produced resonance. At LEP2 they found some events, but not significan enough.

At the LHC we can produce the Higgs in the same way ($u\bar{u} \to Z^0 \to Z^0 h$), but there we do not have access to the missing mass method, so we have to see the Higgs directly. However, there is a HUGE background of bottom jets at LHC to completely drown any such Higgs signal.

If the Higgs is lighter than $2m_W$ the dominant decay mode is to bottom quarks which are difficult to separate from the background. So we need to look at less likely decay channels wich are easier to find.

One such channel is $h \to \gamma\gamma$. Since the photon doesn't couple directly to the Higgs this decay goes through a fermion loop (typically top) or a W loop. The branching ratio is tiny, $\sim 10^{-3}$, but the background of (isolated) high transverse momentum photons is even smaller.

If the Higgs heavier than $2m_Z$ we can look for the decay $h \to Z^0 Z^0 \to l^+l^-l'^+l'^-$, the gold-plated signal. Again the combined branching ratio into Z and subsequently into leptons is small, $\sim 10^{-3}$, but the background is virtually non-existent. Note that even if the Higgs mass is below $2m_Z$ (or $2m_W$) we can still find look for $h \to Z^0 Z^{0\star} \to l^+l^-l'^+l'^-$

At the LHC we can also produce Higgs via gluon–gluon fusion. As for the $\gamma\gamma$ decay, this proces includes a top-quark loop and the partial width is small. However, there is a huge amount of gluons in the colliding protons, making this one of the main production channels for light to medium-mass Higgs.

We can also produce Higgs with W^+W^- fusion, where the Ws are radiated off quarks in the protons.

Nice pictures

What if we didn't find the Higgs at the LHC? There are some cross sections in the Standard Model, in particular the longitudinal $W^+W^- \to W^+W^-$ cross section, which would grow with energy in the absence of the Higgs and will eventually violate unitarity ($\sim$ probability of interaction larger than unity). So, if the Higgs does not exist, or is too heavy ($m_h \gg 1$ TeV) some other mechanism beyond the Standard Model must enter the scene. People have made calculations "proving" that if we do not find the Higgs at the LHC, we will find something else.

Lecture 10

17 Strong interactions

Strong interactions are quite different from electro–magnetic ones. In QED we have the potential (of eg. an electron around a proton in an atom or around a positron in *positronium*):

$$V(r) = -\frac{\alpha_{\text{EM}}}{r} + k_1 \bar{L} \cdot \bar{S} + k_2 \mu_e \mu_p$$

Where the last two terms are the fine-structure (spin–orbit interaction) and hyper-fine (spin–spin) splitting terms respectively. Note that we cannot really calculate the properties of bound states with perturbation theory, as the binding corresponds to a constant exchange of infinitely many force carriers.

For strong interactions (QCD) we expect to have approximately the same thing,

$$V(r) = -\frac{4}{3}\frac{\alpha_S}{r} + k_1' \bar{L} \cdot \bar{S} + k_2' \mu_q \mu_{\bar{q}}$$

where the Casimir factor 4/3 comes from group theory, the fact that we have three charges (r, g, b) and an (in principle arbitrary) normalization. So we expect to have bound states of quarks (hadrons) and, if enough energy is added, free quarks.

But because gluons self-interact (contrary to photons) we get *confinement*. While the electro-magnetic field between two electrically charged particles extend out to infinity, the colour-field between two colour-charged particles is compressed into a flux-tube with approximately constant transverse size $d_\perp \approx 0.7$ fm – a string. This will give us

$$V(r) = -\frac{4}{3}\frac{\alpha_S}{r} + \kappa r,$$

where κ is a constant energy per unit length ~ 1 GeV/fm ~ 10 metric tons per meter.

Hence we can never give a quark enough energy to escape from a bound stata and become free. What happens instead if a quark is given enough energy is that the field (string) is stretched to the point where enough energy is stored in the field to make a virtual $q\bar{q}$-pair real. The colours of this pair may then screen the endpoint charges causing the string to break. The original quark is still confined but not to the same hadron as before. If the energy is high enough, the string will break in several places giving many hadrons, where the ones created close to the original quark in the string carries most of the quarks momentum, while the other hadrons have a smaller fraction. In this way, hadrons are typically emitted in a narrow cone around the original quark — a jet.

According to Heissenberg if a $q\bar{q}$ is produced inside a transverse size $d_\perp$ will have an uncertainty in transverse momenta of $p_\perp \sim 1/d_\perp \approx 0.3$ GeV.

Carful analysis indicates that different break-ups along the string are causually disconnected, this means that the average "distance" between the break-ups must remain the same independent of the "length" of the string. It turns out that a reasonable definition of "distance" is *rapidity*,

$$y = \frac{1}{2}\ln\frac{E + p_z}{E - p_z} = \ln\frac{e + p_z}{m_\perp}$$

since the difference between two rapidities is invariants under Lorentz transformations along the string direction (z). Hence we expect dn/dy to be constant, where n is the number of break-ups, or the number of hadrons. Now, if we have a string between a q and a $\bar{q}$ in an e^+e^--annihilation with a collision energy $\sqrt{s}$, the maximum rapidity of a hadron is $y_{\max} \sim \ln \sqrt{s}/m_\perp$ and the average number of hadrons are

$$N_{\text{tot}} \propto \ln \frac{\sqrt{s}}{\langle m_\perp \rangle} + \text{const}$$

Light hadrons

All hadrons are colour-singlets. Note that it is not enough that a $q\bar{q}$-pair in a meson has opposite colour charge. Compare to the spin of a two-electron system, where the individual spins can combine in four different ways, either as a spin-one system

$$S = 1, \quad \begin{cases} S_z &= 1: & |\uparrow\uparrow\rangle \\ &= 0: & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ &= -1: & |\downarrow\downarrow\rangle \end{cases}$$

or a spin 0 system

$$S = 0, \quad S_z = 0 : \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

So there are two way of getting a "spin 0" system. In the same way a $|r\bar{r}\rangle$ is colourless, but not a singlet as colour will appear under a phase rotation. Instead a meson is given by

$$q\bar{q} = \frac{1}{\sqrt{3}}(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle)$$

(where the + is a phase convention)

Similarly we have baryons with

$$qqq = \frac{1}{\sqrt{6}}(|rgb\rangle - |rbg\rangle + |brg\rangle - |bgr\rangle + |gbr\rangle - |grb\rangle) = \frac{1}{\sqrt{6}}\varepsilon_{ijk}|q_i q_j q_k\rangle$$

Note that for any $SU(N)$ group, we will get "baryons" with N "quarks". In QCD we can, in principle, have more complicated hadrons with quark content $q\bar{q}q\bar{q}$ (*tetraquarks*), $q\bar{q}qqq$ (*pentaquarks*) etc. The first of these to be discovered was ($uudd\bar{s} \to nK^+$) corresponding to a nK^+ binding energy of $\mathcal{O}(100$ MeV). This should be compared to the typical binding energy in nuclear physics where the binding energies are $\mathcal{O}(10$ MeV) which is then what one would expect if it was an "atomic state" of n and K^0.

Another option for colour-singlet states is the glue-ball gg, ggg, ... This is allowed by the gluon self interaction (we have no $\gamma\gamma$ bound states). However these states will mix with ordinary meson states and are difficult to observe. So far there are only indications.

In principal we can expect states with a valence gluon (eg. $q\bar{q}g$). Again there is mixing with normal mesons and no clear evidence for these *hermaphodites* has been found.

We classify hadrons by:

- flavour contents (gives charge, decay patterns etc.)
- mass
- internal spin S. Mesons $S = 0, 1$. Baryons $S = \frac{1}{2}, \frac{3}{2}$.
- internal angular momentum $L = 0, 1, 2, \ldots$
- total spin J, $\bar{J} = \bar{S} + \bar{L}$. (Often also simply called spin.)
- Radial excitation n.
- Relativistic corrections and mixing (makes everything quite messy).

It turns out to be very messy to calculate the masses of the hadrons, but we can try to use the atom as an analogy to get

$$m = \sum_i m_i + k \sum_{i<j} \langle \bar{\mu}_i \bar{\mu}_j \rangle = \sum_i m_i + k \sum_{i<j} \frac{\langle \bar{S}_i \bar{S}_j \rangle}{m_i m_j}$$

where $k \propto |\Psi(0)|^2$ and is $\approx m_u^2 \cdot 640$ MeV for mesons and $\approx m_u^2 \cdot 200$ MeV for baryons. This formula does not include the binding energy, instead this is included by using *constituent* quark masses:

quark	*current* mass (MeV)	*constituent* mass (MeV)
u	2	300
d	5	300
s	100	500
c	1 250	1 600
b	4 200	5 000
t	173 000	–

$m_u \approx m_d \approx 330$ MeV, $m_s \approx 500$ MeV, $m_c \approx 1600$ MeV and $m_b \approx 5000$ MeV. For the light quarks these masses are very different from the mass of a freely propagating quark.

Lets look at a $\pi^+ = u\bar{d}$. It is a spin-0 state so

$$\bar{S}_1 \cdot \bar{S}_2 = \frac{1}{2}(\bar{S}^2 - \bar{S}_1^2 - \bar{S}_2^2) = \frac{1}{2}(S(S+1) - S_1(S_1+1) - S_2(S_2+1)) = -\frac{3}{4}$$

so

$$m_\pi = 2 \cdot 330 - \frac{3}{4} \cdot 640 = 180 \text{ MeV}$$

(OK, it's not perfect: $m_{\pi^0} = 135$ MeV, $m_{\pi^\pm} = 140$ MeV).

In the book there are lists of meson and baryon states. For the mesons the main ones are the vector mesons and the pseudo-scalars (scalars since spin-0, but pseudo, since odd parity). Note that the flavour-diagonal states are special since they can mix:

$$u\bar{u}, d\bar{d}, s\bar{s} \Rightarrow \pi^0, \eta, \eta'$$

If you want to see more states: `pdg.lbl.gov`

We always have the standard charge (C) and parity (P) symmetries. These can always be applied to hadron state, although all states are not eigenstates of these symmerties.

The absolute parity of a state is not completely defined, rather it is relative. If we have a state of two particles a and b with a parities P_a and P_b with an angular momentu L (assuming it is an eigenstate to L)

$$P|ab;L\rangle = |P(a)P(b);L\rangle(-1)^L = P_aP_b(-1)^L|ab;L\rangle$$

where the $(-1)^L$ comes from the spherical harmonics, and $P_aP_b = -1$ for a fermion–anti-fermion pair and $+1$ for a boson–anti-boson pair.

For charge conjugation the mesons are not in general eigenstates: $C|a\bar{b};L,S\rangle \propto |b\bar{a};L,S\rangle \neq \pm|\bar{a}b;L,S\rangle$, but for the flavour-diagonal states we have

$$C|a\bar{a};L,S\rangle = |\bar{a}a;L,S\rangle = (-1)(-1)^L(-1)^{S+1}|a\bar{a};L,S\rangle = (-1)^{L+S}|a\bar{a};L,S\rangle$$

There are three different ways a hadron can decay:

Strong Decay will always dominate if it is possible, ie. if the quark numbers and energy can be conserved. eg. $\Delta^{++} \to p\pi^+$ ($uuu \to uud + \bar{d}u$). The Δ^{++} has a lifetime $\tau \approx 10^{-23}$ s $\Leftrightarrow \Gamma \approx 100$ MeV (difficult to calculate).

Strong decays respect *strong isospin* conservation, which is not due to a gauge symmetry, rather it is due to the fact that the strong force (gluon) cannot tell the difference between a u and a d quark. It is only an approximate symmetry due to the (accidental?) fact that $m_u \approx m_d$. Strong isospin symmetry means that u and d can be thought of as one particle which is in an SU(2) doublet, with iso-spin one half and different projections $u = |\frac{1}{2},\frac{1}{2}\rangle$ and $d = |\frac{1}{2},-\frac{1}{2}\rangle$.

The stong iso-spin was invented before we knew about quarks, and looking at different ways hadrons interacted one found that strong interactions seemed to conserve iso-spin, such that protons ($|\frac{1}{2},\frac{1}{2}\rangle$) and neutrons ($|\frac{1}{2},-\frac{1}{2}\rangle$) behaved in the same way. Also $\pi^\pm$ ($|1,\pm1\rangle$)and π^0 ($|1,0\rangle$) behaves the same in strong interactions.

This would mandate that $\Gamma(\Delta^{++} \to p\pi^+) = \Gamma(\Delta^+ \to n\pi^+)$ up to spin factors. Also we can predict the branching ratios of the $K^{\star+}$ which can decay into $K^+\pi^0$ or $K^0\pi^+$ ($u\bar{s} \to u\bar{s} + u\bar{u}$ and $u\bar{s} \to d\bar{s} + u\bar{d}$). In the iso-spin notation we then have

$$|\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|\frac{1}{2},\frac{1}{2}\rangle|1,0\rangle \pm \frac{2}{\sqrt{3}}|\frac{1}{2},-\frac{1}{2}\rangle|1,1\rangle,$$

so $BR(K^0\pi^+) = 2BR(K^+\pi^0) = \frac{2}{3}$.

Electromagnetic decay Will dominate if possible and no strong decays are allowed. Conserves quark number but not necessarily strong isospin eg. $\pi^0 \to \gamma\gamma$ ($|1,0\rangle \to |0,0\rangle|0,0\rangle$). Also we have eg. $\Sigma^0 \to \Lambda\gamma$ which includes a spin flip in the light di-quarks ($(ud)_1 s \to (ud)_0 s$). The lifetimes are $10^{-20} - 10^{-10}$ s and are typically approximately calculable.

Weak decay does not conserve quark numbers as in $\pi^+ \to \mu^+\bar{\nu}_\mu$ ($u\bar{d} \to W^+ \to \mu^+\bar{\nu}_\mu$). These are only important if no strong or electromagnetic decays are possible. The lifetimes are $10^{-10} - 10^3$ s and are typically approximately calculable.

Heavy Quarks

In November 1974 the *October revolution* occurred in particle physics. One found an extremely narrow resonance in $e^+e^- \to \mu^+\mu^-$ and also in pp ($u\bar{u} \to \mu^+\mu^-$). The resonance was called J by one experiment and ψ by another. Today we call it J/ψ and it has a mass of 3.1 GeV and a width of only 70 keV. The only possible explanation was the bound state of a new heavier quark–anti-quark flavour - the charm quark.

It turns out that $m(J/\psi) < 2m_D$ (where $D = c\bar{d}, c\bar{u}$) so strong decay is impossible and the particle is long-lived. Also the mass of the charm mass is $\approx m(J/\psi)/2 \approx 1.6$ GeV, so we have non-relativistic motion of the quark inside the meson and it is like a $c\bar{c}$ atom which means we can do "spectroscopy".

Sure enough they found excited states ${}^3S_1(1s) = J/\psi$, ${}^3S_1(2s) = \psi'$, ${}^3S_1(3s) = \psi''$, ${}^3P_J = \chi_c$. They eventually also found a pseudo-scalar ${}^1S_0 = \eta_c$. Note that as soon as the mass of these states bcome larger that $2m_D$ (the effective ionization energy), strong decays are possible and the resonances become very wide.

The D mesons come in different flavours depending on the light quark: $D^+ = c\bar{d}$, $D^0 = c\bar{u}$ and $D_s^+ = c\bar{s}$, and they can only decay weakly, ie. they are rather long-lived. We can calculate the decays in the same way as for the muon $c \to s[W^+ \to \mu^+\bar{\nu}_\mu]$ and similarly for $W^+ \to e^+\bar{\nu}_e$ and $W^+ \to u\bar{d}$, and we get

$$\Gamma_c \sim \frac{G_F^2 m_c^5}{192\pi^3}(1+1+3)$$

In 1977 the b quark was discovered (more or less the same story, but it was not predicted in the same way as the c quark). However, here the corresponding decay $b \to t[W^- \to \nu_\mu\mu^-]$ is kinematically forbidden due to the large mass of the top. Exchanging the top with a charm quark works, however, but it is heavily suppressed by the *Cabbibo-Kobayashi-Mascawa* matrix element[6] and b is therefore much more long-lived as compared to the charm quark. This is why we can observe it with a secondary vertex, $c\tau \sim 0.4$ mm.

[6] more about this in the lecture about quark mixing

Lecture 11

-1 Deeply inelastic scattering

Rutherford found the atomic nuclei by scattering α particles off gold atoms. In 1968 at SLAC the did a similar thing, firing electrons with a fixed energy at protons and measure how they scattered: $e^-(k)p(P) \to e^-(k')X(P')$. We interpret this as an exchange of a virtual photon with momentum $q = k - k'$. The final momentum of the scattered proton system is $P' = P + q = P + k - k'$. The virtuality of the exchanged photon propagator is denoted

$$Q^2 = -q^2 = -(k-k')^2 = 2kk' - 2m_e^2 \approx 2EE'(1-\cos\theta) = 4EE'\sin^2\theta/2.$$

We also denote the invariant mass squared of the hadronic final state

$$W^2 = (P+q)^2 = m_p^2 + 2Pq - Q^2.$$

If $W^2 = m_p^2$ we have elastic scattering. If $W^2 > m_p^2$ we have inelastic scattering, the proton "breaks up". If $W^2 \gg m_p^2$ we have deeply inelastic scattering.

We can sketch the matrix element of this scattering

$$\mathcal{M} \propto e(\bar{k}'\gamma^\mu k)\frac{1}{Q^2}e(\bar{P}'\dots P)$$

we don't measure the hadronic final state so we write the last factor as $e(\bar{P}'\gamma_\mu P)F(Q^2)$, ie. as an elestic scattering, but with a form factor $F(Q^2)$ which can be identified with the Fourier transform of the spatial charge distribution in the proton. We use the normalization $F(0) = 1$ and for a smeared charge distribution we expect $F(Q^2) \to 0$ as $Q^2 \to \infty$. Note that if the proton had been-like, we could have calculated the $ep \to ep$ exactly with a unit form factor. The fact that $F(Q) < 1$ means that the proton is not point-like (see discussion in Kane 16).

Now, if the proton consists of point-like constituents, we can interpret the results as if we are elastically scattering on these *partons*. If the partons has a momentum fraction x of the protons momentum, $\hat{P} = xP$, we are looking at the process $e^-(k)q(\hat{P}) \to e^-(k')q(\hat{P}')$, where $\hat{P}^2 = \hat{P}'^2 \approx 0$ for nearly massless partons. Hence we have

$$\hat{P}'^2 = (xP+q)^2 = x^2P^2 + 2xPq + q^2 \approx 2xPq - Q^2 = 0 \quad \Rightarrow \quad x = \frac{Q^2}{2Pq} \equiv x_{\mathrm{Bj}}$$

the *Bjorken* $x \approx Q^2/(Q^2+W^2)$.

Here we can write down the matrix element of the partonic scattering

$$\mathcal{M} \approx e(\bar{k}'\gamma^\mu k)\frac{1}{-q^2}e_i(\bar{\hat{P}}'\gamma_\mu\hat{P}) \approx ee'\frac{\hat{s}}{-Q^2}$$

where $\hat{s} = (k-xP)^2 \approx x(k+P)^2 = xs$. In the rest frame of the partonic scattering we have

$$Q^2 = -(k-k')^2 \approx 2\frac{\sqrt{\hat{s}}}{2}\frac{\sqrt{\hat{s}}}{2}(1-\cos\hat{\theta})$$

so

$$dQ^2 = \frac{\hat{s}}{2} d(\cos\hat{\theta}) = \frac{\hat{s}}{2 \cdot 2\pi} d\hat{\Omega}$$

and with $Q_i = e_i/e$ and $\alpha = e^2/4\pi$ we get the cross section

$$\frac{d\hat{\sigma}_i}{dQ^2} = \frac{\pi\alpha^2}{Q^4} Q_i^2.$$

Compare this with standard Rutherford scattering using $Q^2 = \sin^2\theta/2$:

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\theta/2}.$$

Now we can do the same thing as when we calculated W production in $p\bar{p}$ collisions, ie. we use the parton density, $f_i(x)$, the probability of finding a parton, i, with momentum fraction, x, and get the full differential cross section

$$\frac{d\sigma_i}{dQ^2 dx} \approx \frac{\pi\alpha^2}{Q^4} Q_i^2 f_i(x).$$

If we now define the *structure function*

$$F_2(x) = \sum_i Q_i^2 x f_i(x)$$

we can write the cross section

$$\frac{d\sigma}{dQ^2 dx} = \frac{\pi\alpha^2}{Q^4} \frac{F_2(x)}{x} 2(1 + (1-y)^2)$$

where the last factor is obtained from proper spin counting with $y = Pq/Pk$, the fraction of the electron momentum taken by the photon.

From this we can actually measure the density of quarks species. But not the individual ones.

$$F_2(x) = \frac{4}{9}\left(xf_u(x) + xf_{\bar{u}}(x) + xf_c(x) + xf_{\bar{c}}(x)\right) + \frac{1}{9}\left(xf_d(x) + xf_{\bar{d}}(x) + xf_s(x) + xf_{\bar{s}}(x)\right)$$

If we instead do DIS on a deuteron target we can measure another combination of denisties since we know from strong iso-spin invariance that

$$f^p_{u_v} = f^p_u - f^p_{\bar{u}} \approx f^n_{d_v} \quad \text{and} \quad f^p_{\bar{u}} \approx f^n_{\bar{u}} \approx f^n_{\bar{u}} \approx f^n_{\bar{d}}$$

We can also do "charged current" DIS, where the electron exchanges a $W^\pm$ with the target: $e^- + p \to \nu_e + X$. This is of course a bit tricky, since we cannot detect the neutrino. But we caninfr its momentum by measuring all particles in the X system.

Here we can do the same calculation as before with some important modifications. First the propagator particle is now massive, so we will get

$$\frac{1}{Q^4} \Rightarrow \frac{1}{(m_W^2 - q^2)^2} \approx \frac{1}{m_W^4}$$

Also, not all quark species can be involved, only $u \to d$, $\bar{d} \to \bar{u}$, $c \to s$ and $\bar{s} \to \bar{c}$. So

$$\sum_i Q_i^2 x f_i(x) \Rightarrow x f_u(x) + x f_{\bar{d}}(x) + x f_c(x) + x f_{\bar{s}}(x).$$

At HERA they also used positrons on protons, which in charged current the gives scattering proportional to

$$x f_d(x) + x f_{\bar{u}}(x) + x f_s(x) + x f_{\bar{c}}(x).$$

The picture of the proton becomes a bit more difficult if we consider virtual fluctuations. A quark can emit a gluon and re-absorb it again. Also a gluon can emit a gluon, and a gluon can split into a virtual $q\bar{q}$ pair. The valence quarks are surrounded by a cloud of virtual partons. On the other hand we have the virtual photon, it is short-lived with a small wave function centered along the direction of the electron. We can approximate the transverse size of the photon $\Delta r \sim 1/\Delta p_\perp \sim 1/Q$. The higher the virtuality, the finer details in the proton we can see and also the more details of the cloud of virtual partons around the valence quarks. This is called scaling violations. If we increase Q^2 we can see that a parton at high x actually consist of two (or more) partons at lower x, so the $xf(x)$ is depleted at high x and increased at low x.

The end result is that the parton densities are Q^2-dependent, $f_i(x) \Rightarrow f_i(x, Q^2)$. The Q^2-dependence is only logarithmic, and it is calculable

$$\frac{\delta f_(x, Q^2)}{\delta \ln Q^2} \propto \frac{\alpha_s}{2\pi} \left(-P_i f_i(x, Q^2) + \sum_j \int_x^1 \frac{dz}{z} f_j(z, Q^2) P_{j \to i}(\frac{x}{z}) \right)$$

Here the first term corresponds to the loss of a parton at x being resolved into two with lower momentum fractions, while the second is the gain of a parton at a higher momentum fraction being resolved into one at x. The P functions (where $P_i = \sum_j \int dz P_{i \to j}(z)$) are called (Altarelli–Parisi) splitting functions.

This formalism was developed in the seventies by Altarelli and Parisi, but later on it was realised that the same formalism was simultaneously developed in the Soviet Union by Gribov, Lipatov and Dokshitser. The formalism therefore known as DGLAP.

e^+e^- physics

We have the process $e^+e^- \to \gamma^\star \to f\bar{f}$ (where the $\gamma^\star \to Z^0$ or a mixture of the two at higher energies. The matrix element is easy to write down (for $s << m_Z^2$)

$$\mathcal{M} = e_e(\bar{e}\gamma_\mu e)\frac{1}{s} e_f(\bar{f}\gamma^\mu f)$$

giving the cross section

$$\sigma = \frac{4\pi}{3} \frac{Q_f^2 \alpha^2}{s}$$

One of the main successes of the quark model is the experimental confirmation of the ratio

$$R_{\text{had}} = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \sum_{q;2m_q<\sqrt{s}} Q_q^2$$

demonstrating the charges of the quarks and the fact that they come in three different colours. It is also possible to confirm the spin-1/2 nature of the quarks by looking at the angular distribution of jets

$$\frac{d\sigma(e^+e^- \to q\bar{q})}{d\Omega} \propto 1 + \cos^2\theta$$

At DESY they also managed to confirm the existence and spin-1 nature of gluons by studyng three-jet events $e^+e^- \to q\bar{q}g$. By stydying four-jet events $e^+e^- \to q\bar{q}gg$ and $e^+e^- \to q\bar{q}q'\bar{q}'$ it has been possible to confirm the self-coupling among gluons.

19 Running couplings

The couplings in the standard model Lagrangian are in principle impossible to measure, since they are *bare* couplings, while in any measurement we never have eg. a completely "bare" electron as it is always surrounded by an electromagnetic field which will contain virtual fluctuations.

Hence, if we want to calculate an observable R (eg. a cross section or a decay width) we get a perturbative series

$$R = \alpha^n(R_0 + R_1\alpha + R_2\alpha^2 + \ldots)$$

R_0 is typically a finite number calculable with using a Feynman *tree* diagram. R_1, however, typically involves a divergent integral over a momentum in a *loop* diagram. But we know we can make a measurement of R (at some scale q^2) so we should be able to calculate $R(q^2) = R_{\text{obs}}(q^2)$, even though we cannon directly measure α. The trick is to redefine α so that the divergencies in R_1 are cancelled, $\alpha \to \alpha(q^2)$.

The way we redefine α is easiest to see if we consider a $e\gamma$ vertex $e(k) \to \gamma(q)e(k')$. Here the second order diagram contains a loop where the photon splits into an $e^+(p)e^-(q-p)$ pair which then annihilates into a photon again. In the matrix element we will get the two terms

$$e\bar{u}(k')\gamma^\mu u(k)\varepsilon_\mu - \int \frac{d^4p}{(2\pi)^4}[e\bar{u}(k')\gamma^\mu u(k)]\frac{1}{q^2}\frac{[e\bar{u}(p)\gamma_\mu u(p-q)][e\bar{u}(p-q)\gamma^\lambda u(p)]}{(p^2-m_e^2)((p-q)^2-m_e^2)}\varepsilon_\lambda$$

We here recognize the three propagators and we get a minus sign due to the fact that this is a fermion loop (a boson loop would give a plus sign). We can simplify this to

$$e\bar{u}(k')\gamma^\mu u(k)\varepsilon_\mu(1 - I(q^2))$$

where the integral $I(q^2)$ is dimensionless an can be simplified to

$$I(q^2) = \frac{\alpha}{3\pi}\int_{m_e^2}^{\infty}\frac{dp^2}{p^2} - \frac{2\alpha}{\pi}\int_0^1 dx \cdot x(1-x)\ln\left(1 - \frac{q^2x(1-x)}{m_e^2}\right)$$

where the first term is clearly divergent. To continue we need to introduce an *ultra-violet* cutoff, Λ^2, for this integral (which we will send to infinity later). If $Q^2 = -q^2 \gg m_e^2$ we have

$$\ln\left(1 - \frac{q^2 x(1-x)}{m_e^2}\right) \approx \ln\frac{Q^2}{m_e^2}$$

and we have

$$I(Q^2) \approx \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m_e^2} - \frac{\alpha}{3\pi}\ln\frac{Q^2}{m_e^2} = \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{Q^2}.$$

Now, if we can have one loop, we can have another, but that will just give the same integral squared

$$\mathcal{M} = \alpha\mathcal{M}_0\left[1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{Q^2} + \left(\frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{Q^2}\right)^2 + \dots\right] = \mathcal{M}_0\frac{\alpha}{1+\frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{Q^2}} \to \mathcal{M}_0\alpha_{\text{obs}}(Q^2)$$

Assume we have α_0, some "original" α which we "normalize" by measuring $\alpha(\mu^2)$

$$\alpha(\mu^2) = \frac{1}{\frac{1}{\alpha_0} + \frac{1}{3\pi}\ln\frac{\Lambda^2}{\mu^2}} \Rightarrow \frac{1}{\alpha_0} = \frac{1}{\alpha(\mu^2)} - \frac{1}{3\pi}\ln\frac{\Lambda^2}{\mu^2}$$

then we can calculate

$$\alpha(Q^2) = \frac{1}{\frac{1}{\alpha(\mu^2)} - \frac{1}{3\pi}\ln\frac{\Lambda^2}{\mu^2} + \frac{1}{3\pi}\ln\frac{\Lambda^2}{Q^2}} = \frac{1}{\frac{1}{\alpha(\mu^2)} + \frac{1}{3\pi}\ln\frac{\mu^2}{Q^2}} = \frac{\alpha(\mu^2)}{1 + \frac{\alpha(\mu^2)}{3\pi}\ln\frac{\mu^2}{Q^2}}$$

which is quite finite. We can now send $\Lambda \to \infty$ (while the *bare* $\alpha_0 \to 0$) and we can express the coupling as measured at one scale in terms of the coupling at another scale, and the coupling is *running*.

The physical interpretation is that the charge around an electron is screened, but for larger Q^2, we probe the electron at smaller distances where it is less screened and the effective coupling increases.

So far we only had electrons in the loop, but we can also have muons and quarks as long as their squared mass is less than Q^2 (in principle we can also have heavier fermions in the loop, but their contribution is tiny. It is easy to see that we now get

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - N\frac{\alpha(\mu^2)}{3\pi}\ln\frac{Q^2}{\mu^2}}$$

with

$$N = n_l + 3\left(\frac{2}{3}\right)^2 n_u + 3\left(\frac{1}{3}\right)^2 n_d$$

Note that be can have arbitrarily complex bubbles, which can be resummed in the same geometric series:

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - N\frac{\alpha(\mu^2)}{3\pi}\ln\frac{Q^2}{\mu^2} + \beta_2\alpha^2(\mu^2) + \cdots}$$

The same procedure can be repeated for QCD, looking at a quark–gluon vertex and inserting quark-loops. We then get

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 - \frac{n_f}{2}\frac{\alpha_s(\mu^2)}{3\pi}\ln\frac{Q^2}{\mu^2}}$$

but we will also have gluon loops in the gluon propagator, and since this is a boson loop it comes with the oposite sign. Doing the math we get

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(33 - 2n_f)\ln\frac{Q^2}{\mu^2}}$$

Again we have a coupling which *runs*, but in the other direction. For large Q^2 the coupling decreases and we get *asymptotic freedom*. On the other hand for small Q^2 the coupling diverges and we expect perturbation theory to break down. The picture is that if a red quark emits a red–anti-blue gluon and turns blue it can be viewed as if the original red charge is smeared out. The closer we look the more smeared out it is.

Now let's define Λ_{QCD} as the scale where α_s diverges ie. where

$$\frac{12\pi}{\alpha_s(\mu^2)} = -(33 - 2n_f)\ln\frac{\Lambda^2_{\text{QCD}}}{\mu^2}$$

We then get

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f)\ln\frac{Q^2}{\Lambda^2_{\text{QCD}}}}$$

As Q^2 increases, more and more quarks will contribute to the running, and to have a continuous coupling one usually have $\Lambda_{\text{QCD}}(n_f)$.

Conventionally we present measurements of α_s at the scale m_Z^2.

Lecture 14

20 Mixing angles

If we for a moment limit ourselves to two families, we can write the charged current in the Lagrangian as

$$J^\mu_{\text{ch}} = \begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix} \gamma^\mu P_L \begin{pmatrix} d \\ s \end{pmatrix}.$$

Here we assume that the quark fields are both mass eigenstates **and** weak eigenstates. But his is not necessarily true. In fact it turns out to be false.

One analogy is to consider a light source which emits transversely polarized light, which then traverses a medium where positive helicities travels faster than negative helicities, and is finally detected after a tranverse polarizer. Hence the production and detection eigenstates are the transverse polarizations $(\varepsilon_x, \varepsilon_y)$ while the propagating eigenstates (the "mass" eigenstates) are the circular polarizations $(\varepsilon_+, \varepsilon_-)$.

Now, it turns out that the mass and weak eigenstates of quarks are not the same. What enters into the quark–W vertex is the weak eigenstates, but what we want to have in matrix element, when we calculate cross sections for "free" propagating quarks, is the mass eigenstates. Let's call the weak eigenstates of the down-type quarks $\begin{pmatrix} d' \\ s' \end{pmatrix}$. These can be tranformed by a unitary transformation to the mass eigenstates and vice versa.

$$\begin{pmatrix} d' \\ s' \end{pmatrix}_L = V \begin{pmatrix} d \\ s \end{pmatrix}_L$$

The unitary matrix can be written completely generally as follows

$$V = \begin{pmatrix} \cos\theta e^{i\alpha} & \sin\theta e^{i\gamma} \\ -\sin\theta e^{-i\gamma} & \cos\theta e^{-i\alpha} \end{pmatrix}$$

And we get the charged current

$$J^\mu_{\text{ch}} = \begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix} \gamma^\mu P_L \begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \bar{u} & \bar{c} \end{pmatrix} \gamma^\mu P_L V \begin{pmatrix} d \\ s \end{pmatrix}.$$

In principle we could do the same thing with the up-type quarks and get both a V^u and V^d, but these would always show up together in any electro-weak vertex as $V^{u\dagger}V^d$ which itself is a unitary matrix, so we only need to consider the down-type quarks.

The phases in V above can be absorbed in the quark states and we are left with

$$V = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Expressed in the mass eigenstates we now have the current

$$J^\mu_{\text{ch}} = \bar{u}\gamma^\mu P_L d\cos\theta + \bar{u}\gamma^\mu P_L s\sin\theta - \bar{c}\gamma^\mu P_L d\sin\theta + \bar{c}\gamma^\mu P_L s\cos\theta +$$

And we get the new couplings to the W by multiplying by $g_2/\sqrt{2}$. Note that the "Cabbibo" angle is a completely free parameter of the Standard Model. If nature has chosen the value 0, there would have been no mixing between families and the s-quark (and the lightest strange hadrons) would be stable. Note that this does not change the neutral current at all since, just looking at the left-handed s and d part of current

$$J^\mu_{\text{neu}} = \begin{pmatrix} \bar{d} & \bar{s} \end{pmatrix} V^\dagger \left[T_3^L - Q_f \sin^2\theta_W\right] V \begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{s} \end{pmatrix} \left[T_3^L - Q_f \sin^2\theta_W\right] \begin{pmatrix} d \\ s \end{pmatrix}$$

it will be unchanged since $V^\dagger V = 1$.

We can generalize this to three generations

$$J^\mu_{\text{ch}} = \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix} \gamma^\mu P_L V \begin{pmatrix} d \\ s \\ b \end{pmatrix}.$$

V is now a 3×3 unitary matrix which means that it can be described by nine different parameters. As for the two-flavour matrix we can get rid of some of these by redefining the phases of the quark wave functions. But we are left with four parameters which are all completely free and need to be measured. Three of these can be expressed as real angles similar to the Cabbibo angle, but there is stil one relative phase which we cannot get rid of in this "Cabbibo-Kobayashi-Mascawa" mixing matrix. This means that we get a camplex current, and also complex terms in the Hamiltonian, $W_\mu J^\mu_{\text{ch}}$, which furthermore means that the the theory is not completely invariant under time-reversal transformations.

It turns out that the absolute values of the matrix elements coupling the third generation to the others are fairly small ($|V_{cb}| \approx 0.03$ and $|V_{ub}| \approx 0.003$). Since the b quark cannot decay into a top, the dominant decay mode is $b \to cW^-$ where the width is suppressed by $|V_{cb}|^2$. This is different from the corresponding lepton, the τ which can simply decay into a neutrino. One way of measuring $|V_{cb}|$ is therefore to compare

$$\frac{\Gamma_b}{\Gamma_\tau} \approx \frac{9}{5}|V_{cb}|^2 \frac{m_b^5}{m_\tau^5}$$

where the 9/5 comes from the number of decay modes available from the W.

21 *CP* Violation

There are three important discrete symmetries in nature, the charge (C), parity (P) and time (T) reversal symmetries. Now if we look at the standard model we immediately find that P is explicitly broken in the coupling to the W, since P will change a right-handed particle to a left-handed one, and while the latter couples to the W, the former does not. In the same way it is clear that C is violated, since a left-handed particle couples to the W while a left-handed ant-particle does not. However, we could expect the theory to be invariant under the simultaneous CP transformation. There is a theorem saying that any quantum field theory is invariant under CPT transformation, but it turns out that the standard model is not quite invariant under CP and therefor not under T transformations.

The reason for this "CP violation" is the remaining the phase in CKM matrix which we couldn't get rid of by redefining the phases in the quark wavefunctions. (Note that this could never happen if there was just two generations).

The first place where CP violation was measured was in the decay of neutral kaons. If we look at $K^0 = d\bar{s}$ and $\bar{K}^0 = s\bar{d}$, neither of these are CP eigenstates. But if CP is conserved we can create two combinations which are eigenstates: $K_L = K_0 - \bar{K}^0$ and $K_S = K_0 + \bar{K}^0$, where the former is odd and the latter is even under CP transformation. Now, these combinations have different decay modes (eg. $K_L \to \pi\pi\pi$ and $K_S \to \pi\pi$), and it turns out that the K_L is much more ling-lived that K_S (hence the names *long* ($\approx 5 \cdot 10^{-8}$ s) and *short* ($\approx 9 \cdot 10^{-11}$ s)).

But if CP is slightly violated, the K_L and K_S are not necessarily mass eigenstates. If they were, we could have a "beam" of K^0, where we expect all K_S will decay and quickly leaving a pure K_L beam. But if K_L is not a mass eigenstate, but made up from two states with different masses, travelling at different velocity, after a while we will have different mixtures of these states, which will give us back a component corresponding to K_S.

If CP was exactly conserved, the decay $K_L \to \pi^- e^+ \nu_e$ would be exactly the same as $K_L \to \pi^+ e^- \bar{\nu}_e$. But it turns out that we can measure a small asymmetry:

$$\frac{\Gamma(K_L \to \pi^- e^+ \nu_e) - \Gamma(K_L \to \pi^+ e^- \bar{\nu}_e)}{\Gamma(K_L \to \pi^- e^+ \nu_e) - \Gamma(K_L \to \pi^+ e^- \bar{\nu}_e)} = 0.00333 \pm 0.00014$$

So CP, and therefore also T, is violated in the Standard Model.

We can expect other sources of CP violation besides the phase in the CKM matrix. One way of checking this is to measure also CP violation in the $B^0\bar{B}^0$, if that is inconsistent with the $K^0\bar{K}^0$ CP violation, there may be other sources.

If we study CP violation, we may eventually understand why there are so many more particles tha anti-particles in the Universe.

Lecture 12

24 Neutrino masses and oscillations

In the standard model the neutrinos are typically assumed to be massless. And upuntil a few years ago they seemed to be. The experimental limits on their masses were, however, not extremely strict. Eg. from beta-decay of tritium we had $m(\nu_e) < 3$ eV, the decay $\pi^+ \to \mu^+\nu_\mu$ gave the limit $m(\nu_\mu) < 0.19$ MeV, and $m(\nu_\tau) < 18$ MeV came from $\tau \to \nu_\tau + n\pi$.

[**draw Curie plot:** dN/dE_e **vs** E_e]

Also there are cosmological constraints from fluctuations in the CMB which gives $\sum m_\nu \lesssim 0.23$ eV.

There is, however, no à priori reason we know about for the neutrinos to be massless (or not). The masses could arise in (at least) two different ways.

If there exists a right-handed neutrino (and left-handed anti-neutrino) we would have a coupling to the higgs field: $g_f\bar{\nu}_L\phi\nu_R$ and a (Dirac) mass term $m_D\bar{\nu}_L\nu_R$. So how do we see a right-handed neutrino? Well, it has zero charge, it would have $T_3 = 0$ and no colour, hence the only way it would interact is with the Higgs and via gravity. So we will never see it directly.

The second way we could have neutrino masses is if the neutrino, just as the photon and Z^0, was its own anti-particle. In that case we could get terms in the Lagrangian looking like $m_M\bar{\nu}_L\nu_L^c$, where ν_L^c is right-handed. But this would give us $\nu_L \leftrightarrow \bar{\nu}_R$, which violates fermion number. This hasn't been seen. Yet. One place where it could show up is in neutrino-less double-beta decay:

$$(A, Z) \to (A, Z+1) + e^- + \bar{\nu}_e \to (A, Z+1) + e^- + \nu_e \to (A, Z+2) + e^-e^-$$

So far, the non-observation of this sets the limit $m_M < 0.2$ eV.

If the neutrinos are massive, there is a possibility that the mass-eigenstates are not the same as the weak eigenstates, just like what happened for their cousins, the down-type quarks.

Consider a light in an optically active medium where left- and right-handed circular polaised photons travel at different speeds. If we have a photon with spin initially along the x-axis, this can be rewritten as a combination of right- and left- handed components, which travels at different speeds, and hence after some distance the spin may have moved away from the x-axis.

Denoting the mass eigenstates ν_i we have

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

U is a unitary matrix, with four independent DoF, and is typically parameterised accord-

ing to PMNS (Pontecorvo–Maki–Nakagawa–Sakata)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.
(OK, a unitary 3×3 matrix has 9 DoF, but 5 of them can be absorbed into the arbitrary phases of the weak and massive eigenstates).

It turns out that θ_{13} is quite small, while θ_{12} is surprisingly large.

Now a neutrino would be produced (eg. in $\pi^+ \to \mu^+\nu_\mu$) in a weak eigenstate at time 0:

$$\nu_\mu(0) = U_{\mu,1}\nu_1(0) + U_{\mu,2}\nu_2(0) + U_{\mu,3}\nu_3(0)$$

but then we will have a propagation of the mass eigenstates, so that

$$\nu_\mu(t) = U_{\mu,1}\nu_1(0)e^{iE_1t} + U_{\mu,2}\nu_2(0)e^{iE_2t} + U_{\mu,3}\nu_3(0)e^{iE_3t}$$

Let's simplify and only use two generations, which means that we can describe the mixing matrix with only one angle, α

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

Now, if we start out with a pure ν_e we have

$$\nu_e(0) = \nu_1(0)\cos\alpha + \nu_2(0)\sin\alpha$$

and

$$\nu_\mu(0) = -\nu_1(0)\sin\alpha + \nu_2(0)\cos\alpha = 0$$

but it is the mass eigenstates which propagates, so after some time t we will have

$$\nu_\mu(t) = -\nu_1(0)e^{-iE_1t}\sin\alpha + \nu_2(0)e^{-iE_2t}\cos\alpha$$

and there is a probability that the ν_e has turned into a ν_μ

$$|\langle\nu_e(0)|\nu_\mu(t)\rangle|^2 = \sin^2\alpha\,\cos^2\alpha\left|e^{-iE_1t} - e^{-iE_2t}\right| = \sin^2 2\alpha\;\sin^2\left(t(E_2 - E_1)/2\right)$$

where

$$E_2 - E_1 = \frac{E_2^2 - E_1^2}{E_1 + E_2} \approx \frac{m_2^2 - m_1^2}{2E} = \frac{\Delta m^2}{2E} \approx \frac{2m\Delta m}{2E} = \frac{\Delta m}{\gamma}$$

where the latter can be seen as a boosted version of the fact that also a neutrino at rest has no definite weak eigenstates.

Including also the third family we can write the probability for oscillation from family α to β

$$\begin{aligned} P(\alpha \to \beta) = \delta_{\alpha\beta} \quad &- \quad 4\sum_{i>j} \Re(U^\star_{\alpha i}U_{\beta i}U_{\alpha j}U^\star_{\beta j})\sin^2\left[1.27\Delta m^2_{ij}\cdot E/L\right] \\ &+ \quad 2\sum_{i>j} \Im(U^\star_{\alpha i}U_{\beta i}U_{\alpha j}U^\star_{\beta j})\sin^2\left[2.54\Delta m^2_{ij}\cdot E/L\right] \end{aligned}$$

where we have scaled things so that L should be given in km, E in GeV and Δm_{ij}^2 in eV^2.

Note that there is also some oscillations induced when neutrinos traverses matter. This is because neutrinos interacts, however weakly, with matter with Z^0 exchange. While Z^0 exchange is exactly the same for electron- and muon-neutrinos, electron neutrinos can also react with the electrons in matter through W exchange, $\nu_e e^- \to e^- \nu_e$. This will give an effective mass of the neutrino which is different from the mass in vacuum. We can write

$$i\frac{d}{dt}\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} U^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} + \begin{pmatrix} \sqrt{2}G_F N_e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

which will give us

$$P(\nu_e \to \nu_\mu) = \frac{\sin^2 2\theta}{W^2} \sin^2\left(1.27 \frac{W \Delta m^2 L}{E}\right)$$

with

$$W^2 = \sin^2 2\theta + \left(\sqrt{2}G_F N_e \frac{2E}{\Delta m^2} - \cos 2\theta\right)^2$$

where N_e is the density of electrons in the medium.

How do we measure this? Well, first we need a source of neutrinos:

- The sun: Abundant source of ν_e, but low energies. The main process $p + p \to d + e^+ + \nu_e + \gamma$ gives too low energies to detect. But $^7Be + p \to^8 Be + e^+ + \nu_e + \gamma$ works.
- Cosmic rays: Gives high energies, approximately 1:2 ratio of ν_e and ν_μ and corresponding anti particles.
- Accelerators: Generate eg. a beam of π^+ which decays to $\mu^+\nu_\mu$, with controllable energy.
- Reactors: Radioactive materials, mainly ν_e with different energies.

Then we have to detect them at some distance away.

- Homestead Mine (USA) (Ray Davies NP2002): $\nu_e +^{37} Cl \to^{37} Ar + e^-$, ^{37}Ar is radioactive so that we can count them.
- Sudbury Neutrino Observatory (Canada) (Arthur McDonald NP2015): Heavy water using the inverse *pp* cycle. But also netral current interactions with d, ann elastic scattering on electrons. In all cases look for tiny flashes of light.
- Super Kamiokande (Japan) (Takaaki Kajita NP2015): Huge underwater cave, with walls covered by photo multipliers, filled with normal water.

The signal for this is eg. the appearance of a ν_μ where we only expect ν_e or the disappearance of ν_e. The latter has been suspected for a long time from the fact that only a third of the expected number of neutrinos from the sun were found – but nobody really trusted the models for neutrino production in the sun enough to make a definite statement.

Now there is ample evidence for neutrino oscillations and hence also for neutrino masses (or rather mass differences). The evidence comes both from solar and atmospheric neutrinos as well as from 'man made' neutrino sources (reactors and accelerators).

We can study atmospheric neutrinos. These are produced when charged particles from space hits the atmosphere and produces mainly pions which, in turn, produces mainly muons (together with ν_μ) which in turn decays (giving another ν_μ and a ν_e). These are typically much more energetic than solar neutrinos. Now if we detect these and look at the direction they come from, if they come from above, they have traveled a short distance in the atmosphere, while if they come from below they have been traveling $\sim 10^4$ km through the earth. Hence we can vary L by varying the polar angle and we should see oscillations. And we do. By looking at both ν_e and ν_μ it is found that there is a big effect of ν_μ disappearing, not to ν_e but to ν_τ. From this one gets a mass difference which is larger than the mass difference responsible for the $\nu_e \leftrightarrow \nu_\mu$ oscillations. The current numbers are approximately

$$\begin{aligned}
\Delta m^2_{\text{sol}} \equiv \Delta m_{12} &\approx 7.5 \cdot 10^{-5}\,\text{eV}^2 \\
\Delta m^2_{\text{atm}} \equiv \Delta m_{13} \approx \Delta m_{23} &\approx 2.4 \cdot 10^{-3}\,\text{eV}^2 \\
\theta_{\text{sol}} \equiv \theta_{12} &\approx 34^\circ \\
\theta_{\text{atm}} \equiv \theta_{23} &\approx 42^\circ \\
\theta_{13} &\approx 9^\circ
\end{aligned}$$

We still need to measure one additional phase, and the absolute masses.

Lecture 13

22 Open issues

So there you have it. The standard model. It works very well and we hate it! Why?

- There are too many free parameters. Twelve fermion masses, eight mixing parameters, three couplings and the higgs field parameters $(\mu, \lambda) \Leftrightarrow (m_h, m_Z)$. In total 25 parameters (26 assume there is CP-violation in QCD) Wouldn't it be much nicer if we had a theory where these could be predicted?
- Many things seems arbitrary. Why are there three generations. Why are the left-handed fermions in $SU(2)$ doublets and the right-handed in singlets?
- Why do we just have $SU(3) \times SU(2) \times U(1)$? Nature could have picked any symmetry!
- Why is there charge quantization? In principle Y in $U(1)$ could be anything.

Then there is the obvious problem that gravity does not fit in anywhere. And the fact that we have such huge ratios of similar parameters, eg. $m_W/m_{\rm Pl} \sim 10^{-17}$, $m_e/m_W \sim 10^{-5}$, $m_\nu/m_e \sim 10^{-7}$. There is the fact that baryonic matter only makes up a fraction of the matter of the universe and that matter only makes up a fraction of the energy of the universe (the cosmological constant).

As if this isn't enough, there is a problem with the higgs mass. Masses, just as couplings depend on the scale due to loop corrections and need to be renormailized (looking just at QED)

$$m \to m\left(1 + \frac{\alpha}{3\pi}\int_{m_e^2}^{\infty}\frac{dp^2}{p^2}\right) \to m\left(1 + \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m_e^2}\right)$$

For the higgs the renormalization turns out to be worse

$$\delta m_h^2 \propto \int_{m_h^2}^{\infty} dp^2 \propto \Lambda^2$$

If there is no additional structure for particles until we reach the Planck scale, the Higgs mass we observe at electroweak scales must be the tiny difference between two huge numbers (the hierarchy problem). We don't like that.

23 Grand Unification

Just as the electro-weak symmetry is spontaneously broken by the Higgs field, $U(1)_Y \times SU(2)_L \to U(1)_{\rm EM}$, we could try to imagine that the whole structure of the standard model is a spontaneously broken version of a higher symmetry.

The simplest such *Grand Unified Theory* (GUT) was invented by Georgi and Glashow and is now excluded by data. It is however instructive to look at it anyway. It assumes

that there is only one symmetry group which is spontaneously broken into the standard model,

$$SU(5) \to SU(3) \times SU(2) \times U(1)$$

We then have the fermions in a quintet

$$\begin{pmatrix} \begin{pmatrix} \bar{d}_r \\ \bar{d}_b \\ \bar{d}_g \end{pmatrix} \\ \begin{pmatrix} e^- \\ \nu_e \end{pmatrix} \end{pmatrix}_L$$

where now $\bar{d}_L = \bar{d}_R$ is an weak $SU(2)$ singlet. The quarks are still associated with the $SU(3)t$ colour-triplet and the leptons are still in the weak $SU(2)$ doublet. In the fundamental representation of an $SU(N)$ group the generators are traceless $N \times N$ matrices, where the diagonal generator will give the charge, which reuires $\sum Q_i = 0$, so that $3Q_d + Q_e = 0$ and $Q_d = -1/3$ (and also $Q_u = 2/3$) and we have charge quantization, $Q_p + Q_e = 0$.

So where are the other quarks and leptons? Well the quintet above is actually the anti-quintet of the group. We also have the anti-symmetric 5×5 decuplet (cf. $SU(3)$ have triplet quarks, but also octet gluons)

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \begin{pmatrix} 0 & \bar{u}_b & -\bar{u}_g \\ -\bar{u}_b & 0 & \bar{u}_r \\ \bar{u}_g & -\bar{u}_r & 0 \end{pmatrix} & \begin{matrix} -u_r & -d_r \\ -u_g & -d_g \\ -u_b & -d_b \end{matrix} \\ \begin{matrix} u_r & u_g & u_b \\ d_r & d_g & d_b \end{matrix} & \begin{pmatrix} 0 & -e^+ \\ e^+ & 0 \end{pmatrix} \end{pmatrix}_L$$

Note that there are ten independent fields here. Then we have the quintet of the right-handed fields

$$(d_r, d_g, d_b, e^+, \bar{\nu}_e)_R$$

and a corrsponding anti-decuplet for the right-handed up-type quarks and electrons. For the decuplets we have $Q_{ij} = Q_i + Q_j$ so eg. $Q_u = Q_{4,1} = Q_4 + Q_1 = Q_d + Q_e$.

The $SU(5)$ group has $5^2 - 1 = 24$ generators, which will give us 24 gauge bosons, lets denote them

$$\begin{pmatrix} g_{ij} - \frac{2B}{\sqrt{30}}\delta_{ij} & \begin{matrix} \bar{X}_r & \bar{Y}_r \\ \bar{X}_g & \bar{Y}_g \\ \bar{X}_b & \bar{Y}_b \end{matrix} \\ \begin{matrix} X_r & X_g & X_b \\ Y_r & Y_g & Y_b \end{matrix} & \begin{pmatrix} \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ W^- & -\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{pmatrix} \end{pmatrix}$$

We can write the running of the couplings in the following way

$$\frac{1}{\alpha_i(M^2)} = \frac{1}{\alpha_i(\mu^2)} + \frac{b_i}{4\pi} \ln \frac{M^2}{\mu^2}$$

where we have $b_3 = 11 - 4n_F/3$, $b_2 = 22/3 - 4n_F/3$ and $b_1' = -4n_F/3$ (b_1' is related with the hypercharge rather than the electric charge). We assume that there is actually only

one coupling at som large *unification* scale, $M^2_{\rm GUT}$, but the broken symmetry gives them different values at lower scales. Then we would have eg.

$$\frac{1}{\alpha_3(\mu^2)} + \frac{b_3}{4\pi}\ln\frac{M^2_{\rm GUT}}{\mu^2} = \frac{1}{\alpha_2(\mu^2)} + \frac{b_2}{4\pi}\ln\frac{M^2_{\rm GUT}}{\mu^2}$$

So we can in principle calculate this scale

$$\ln\frac{M_{\rm GUT}}{\mu} = \frac{6\pi}{11}\left(\frac{1}{\alpha_2(\mu^2)} - \frac{1}{\alpha_3(\mu^2)}\right)$$

Inserting experimental data ($1/\alpha_2(m_Z^2) \approx 30$ and $1/\alpha_3(m_Z^2) \approx 10$) we will find $M_{\rm GUT} \sim 10^{18}$ GeV, but things do not exactly match up.

If $SU(5)$ is the true symmetry group at high scales the coupling g_5 would be the only copuling in the theory. All other couplings, g_1, g_2, g_3, would be derived from the breaking of $SU(5)$ into the $SU(3)\times SU(2)\times U(1)$. In particular we should be able to derive the ratio between g_1 and g_2 which is governed by θ_W.

Let's look at the covariant derivative of $SU(5)$

$$D^\mu = \partial^\mu - ig_5 T_a U^\mu_a$$

And Pick out the parts relevant to the electro-weak sector using

$$\begin{aligned} B^\mu &= A^\mu\cos\theta_W + Z^\mu\sin\theta_W \\ W^\mu_3 &= -A^\mu\sin\theta_W + Z^\mu\cos\theta_W \end{aligned}$$

$$D^\mu = \partial^\mu - ig_5(T_3W^\mu_3 + T_1B^\mu + \ldots) = \partial^\mu - ig_5\sin\theta_W(T_3 + \cot\theta_W T_1)A^\mu + \ldots = \partial^\mu - ieQA^\mu + \ldots$$

From this we can identify $e = g_5$ and $Q = T_3 - \cot\theta_W T_1 \equiv T_3 + cT_1$. Now, for any representation, R, of a group we have orthogonality and equal normalization of the generators T_a, so that

$$\mathrm{Tr}_R T_a T_b = N_R\delta_{ab}$$

So

$$\mathrm{Tr}Q^2 = \mathrm{Tr}(T_3 + cT_1)^2 = (1-c^2)\mathrm{Tr}T_3^2$$

since $\mathrm{Tr}T_3^2 = \mathrm{Tr}T_1^2$ and

$$\sin^2\theta_W = \frac{1}{1+c^2} = \frac{\mathrm{Tr}T_3^2}{\mathrm{Tr}Q^2} = \frac{0+0+0+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+1+0} = \frac{3}{8}$$

Now, this is fairly far from 0.23, but remember $\sin^2\theta_W$ is a ratio between couplings, and couplings are running and, hence, so is $\sin^2\theta_W$. And evolving down to the weak scale we are not too far off.

The 24 gauge bosons included the eight gluons and the $W^\pm$, Z^0 and γ. The other twelve are X, $\bar{X}$, Y and $\bar{Y}$ which all come in three different colours. Sandwiching the gauge boson matrix between the fundamental representations, $\bar{5}A5$, $\bar{10}A5$, etc. we get terms in the Lagrangian corresponsing to new vertices, eg.

$$X \to uu, \quad X \to e^+\bar{d}, \quad Y \to ud, \quad Y \to \bar{d}\bar{\nu}_e, \quad Y \to e^+\bar{u}$$

and we conclude that X and Y has charges 4/3 and 1/3 respectively. The problem with this is that the proton can decay:

$$p = u\,ud \to u\,Y \to u\,\bar{u}e^+ \to \pi^0 e^+$$

The anology with weak muon decay

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^2} = \frac{\alpha_2^2}{384\pi}\frac{m_\mu^5}{m_W^4}$$

we get something like

$$\Gamma_p \propto \alpha_5^2 \frac{m_p^5}{m_Y^4}$$

A proper calculation using $m_Y \sim m_{\mathrm{GUT}} \sim 10^{15}$ GeV will give $\tau_p \sim 10^{31\pm 2}$ years. The current limit $p \to \pi^0 e^+$ is $\tau_p > 10^{33}$ years.

Careful analysis tells us that the simplest GUT, based on $SU(5)$ is excluded by data, but one could easily imagine higher symmetry groups. An example is $SO(10)$ which includes a right-handed neutrino an extra Z boson and much more. Most of these suffer from baryon-number non-conservation and other maladies, and currently GUTs are not very popular.

26 Supersymmetry

Postulate there being a symmetry between fermions and bosons, with an operator Q changing one into the other

$$|b_i\rangle = Q|f_i\rangle \quad \text{and} \quad |f_i\rangle = \bar{Q}|b_i\rangle$$

but leaving any other quantum number as it is. The transformation is actually defined in terms of an algebra where

$$\{Q, \bar{Q}\} = Q\bar{Q} + \bar{Q}Q = 2\sigma^\mu P_\mu$$

where $P_\mu = i\partial_\mu$ is a translation. In some sense the transformation of a boson into a fermion is the square root of a translation. (Remember that $\sqrt{-1} = i$ and $\sqrt{\partial^\mu \partial_\mu} = \gamma^\mu \partial_\mu$.) We also have

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0 \quad \text{and} \quad [Q, P] = [\bar{Q}, P] = 0.$$

There are many ways of defining supersymmetry, we will only look at the simplest *minimal* supersymmetric extension of the standard model, the MSSM. Basically supersymmetry means that all particles we know must have a supersymmetric partner with exactly the same quantum numbers but different spin.

normal	partner	spin	
q_L	$\tilde{q}_L$	0	squarks
q_R	$\tilde{q}_R$	0	can mix
l_L	$\tilde{l}_L$	0	sleptons
l_R	$\tilde{l}_R$	0	also mix
ν_L	$\tilde{\nu}_L$	0	sneutrinos
γ	$(\tilde{\gamma})$	1/2	(photino zino higgsino)
Z^0	$(\tilde{Z}($	1/2	all mix together into
h^0		1/2	neutralinos $\tilde{\chi}_i^0$
H^0	$(\tilde{H})$	1/2	$2+3+1+1+1=8$ spin states for the bosons
A^0		1/2	gives four neutralinos with two spin states each.
$W^\pm$	$\tilde{W}^\pm$	1/2	(wino higgsino) mix together
$H^\pm$	$\tilde{H}^\pm$	1/2	charginos $\tilde{\chi}_i^\pm$
g	$\tilde{g}$	1/2	gluino

It turns out we need we need two Higgs doublets, one to generate mass for u, ν and one for d, l. Two doublets give eight degrees of freedom, three are "eaten" by the W/Z masses, leaving us with five Higgs particles. Their partners have the same spin and mixes into neutralinos and charginos. It is possible to derive mass relations between the Higgses and the W/Z, in particular $m_h < m_Z$ which would be excluded by experiments if it wasn't for some radiative corrections giving $m_h < 103$ GeV.

If SUSY was unbroken the masses of any particle would be the same as its super-partner. But this is not the case, so SUSY must be broken. Exactly how it is broken, nobody knows, but there are some suggestions.

We can define a "parity" relating to SYSU, called R-parity

$$R = (-1)^{L+3B+2S}$$

where L is lepton number, B is baryon number and S is spin. So all ordinary particles have $R = +1$ and their super-partners (sparticles) have $R = -1$. If R-parity is conserved, sparticles can only be produced in pairs. This also means that the lightest super-symmetric particle (LSP) is stable.

Even if the masses are not the same, the couplings do not care if we have particles or sparticles. Hence we have that vertices such as eg.

$$W^- \to e^-\nu_e, \quad W^- \to \tilde{e}^-\tilde{\nu}_e, \quad \tilde{W}^- \to \tilde{e}^-\nu_e, \quad \tilde{W}^- \to e^-\tilde{\nu}_e$$

all have the same coupling (g_2). So as soon as we come above the mass-threshold for producing sparticles, they will be produced at the same rate as their ordinary particle equivalents.

Now it was said before that the loop-corrections to the higgs mass, eg. from a top-loop, gave $\delta m_h^2 \propto -\int_{m_t^2}^{\infty} dp^2$, leading to the hierarchy problem. Now we will also have stop-loops which come in with oposite sign since they are bosons (but the same coupling). The end result is that $\delta m_h^2 \propto m_{\tilde{t}}^2 - m_t^2$ which is finite. This solves the hierarchy problem and many other problems with the standard model.

Another thing that SUSY can explain is dark matter. If there is a stable LSP, which only interacts weakly (eg. $\tilde{\chi}^0$) it would be produced copiously at the big bang and would basically still be around. The interaction with matter would be weak since $\sigma(\tilde{\chi}^0 p) \propto m_{\tilde{q}}^{-2}$ (cf. $\sigma(\nu p) \propto m_W^{-2}$). However, if R-parity is not conserved we could have decays like $\tilde{\gamma} \to \nu\gamma$ – it could still contribute to dark matter if the decay is slow enough. Note that we can get R-parity violation by either lepton number or baryon number violation, but if we have both we will get proton decay.

With R-parity conserved we would get characteristic decay chains of sparticles according to their mass hierarchy., eg.

$$\tilde{u} \to d + [\tilde{\chi}_1^+ \to \nu_\tau + [\tilde{\tau}^+ \to \tau^+ \tilde{\chi}_1^0]]$$

This comes with a price though. There are double the amount of particles and hence particle masses which are free parameters. There are also a bunch of extra mixing parameters for which there are no predictions. In total we have over hundred free parameters which need to be measured. If SUSY is discovered it will give a lot of hard (and boring) work to check its consistency.

-2 String theory

How do we include gravity in the standard model. Naively we would try to make a quantum field theory of general relativity and we would get a spin-2 graviton. But there would also be non-renormalizable divergencies. The problem is that in normal QFT we consider particles to be point-like. The higher the scale the closer we get, and also the higher the gravitational potential. In addition, the higher the scale, the higher energy i.e. mass and, hence, the higher the force. The result is that no known renormalization works.

A possible solution started to emerge 1984 when string theory suggested as a viable explanation to what may be going on. The idea is that elementary particles, rather than being point-like are actually modes of vibrations on small pieces of string. This means that particles are intrinsically extended, which also means that the divergencies are muffled.

While a particle will describe a *world line*, $x^\mu(\tau)$, the four-position at any given proper time, a string will describe a *world sheet*, $x^\mu(\tau, \sigma)$, where σ is an additional coordinate along the string. The strings can either be open or closed ($x^\mu(\tau, \sigma + 2\pi) = x^\mu(\tau, \sigma)$).

Just as particles are not point-like, nor are the vertices. Rather the vertices are smeared out in a way such that several different particle vertices (eg. s-, t- and u-channel diagrams) correspond to one and the same string vertex.

[**draw picture**]

To construct a string theory which is consistent with QFT is a bit tricky. To avoid tachyons (particles with $m^2 < 0$ which travel and convay information faster than light, not to be confused with normal off-shell virtual particles), it turns out that we need 26 dimensions. Things improve if we allow for supersymmetry, in which case we only need nine space- and one time dimension, as long as the basic QFT Lagrangian is invariant

under $SO(32)$ or $E_8 \times E_8$. Now since

$$E_8 \supset E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$$

we can live with this restriction, even though we have to live with a whole new zoo of particles living in the other E_8 which only interacts gravitationally.

It turns out you can construct exactly five such (super-)string theories, called type-I, type-IIA, type-IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$ (heterotic has closed strings). So, which one has nature chosen? Well, recently it has been realized that they may all just be different aspect of the same underlying theory, M-theory, and that they are related to eachother through *duality* transforms (cf. solid state physics where we can describe the same phenomena either by atomic vibrations or by phonons). From M-theory one can then derive other string theories. In fact the current estimate is that one can derive 10^{500} different string theories consistent with (SuSY) SM.

It turns out that M theory requires eleven dimensions. But let's go back to ten dimensions where we can construct five theories which all lack unphysical divergencies and all include $SU(3) \times SU(2) \times U(1)$ together with supersymmtry, a spin-2 graviton and a spin-3/2 gravitino. How do you explain that there are ten dimensions when we only live in four? Well the extention of the universe in the other six dimensions is extremely small. The universe is curled up into tubes with circumference $L \sim m_{\mathrm{Pl}}^{-1}$ (compactification).

Recently it was realized that we haven't really tested gravity at small distances. This means that we can imagine that all standard model fields live on a four-dimensional *brane* in a higher dimensional universe. What if gravity was allowed to propagate throughout all dimensions, what would then experiments tell us about the maximum size of these dimensions? Well, not much. Gravity has only been studied down to millimeter scales. Lets look at the gravitational potential around a particle with mass m generalized to $4+n$ dimensions:

$$V(r) \sim \frac{m}{m_{\mathrm{Pl}}^{n+2}} \frac{1}{r^{n+1}},$$

Now, if the n extra dimensions are of size R, for distances much larger than that the potential would look like

$$V(r) \sim \frac{m}{m_{\mathrm{Pl}}^{n+2}} \frac{1}{R^n} \frac{1}{r}$$

But these are the distances we are used to, where we have measured the potential to be

$$V(r) \sim \frac{m}{m_{\mathrm{Pl}4}^2} \frac{1}{r}$$

where we have found $m_{\mathrm{Pl}4} \approx 10^{19}$ GeV. But if the extra dimensions are *large* this means that the *true* Planck scale is much smaller

$$m_{\mathrm{Pl}} \sim R^{-\frac{n}{n+2}} m_{\mathrm{Pl}4}^{\frac{2}{n+2}}$$

In fact, for suitable values of n and R the Planck scale may be almost as small as the weak scale. Not only would this cure the hierarchy problem, it also means that it could be accessible at the LHC. Contrary to standard super string theory, this means that we could enter into the realm of quantum gravity at the LHC (with the production of mini black holes and lots of other fun stuff) — truly an exciting prospect.

Lecture 15

-3 Particle Physics and the Universe

This is a rather active research area right now and we have to change numbers rather often. An example here is that the Hubble constant which was only know to within a factor of two is now quite well known. So keeping these notes up to date is always a bit of a problem, espescially since the new developments often involve a large technical background.

There are many areas of contact, there exists in fact a whole field called astroparticle physics for which by now many books and reviews have appeared.

The most exciting recent developments have been in the contact with cosmology but even this relation goes back quite some time.

Astrophysics constraints on Particle Physics

These consists of knowing the physics between a variety of astronomical objects and then using these to obtain limits on extra effects not included normally.

These typically appear in the radiation of particles in stellar (or their successors) which can have sizable effects. Here are some

- In stars the thermonuclear process is rather well understood. If there were more weakly interacting particles that would get radiated at a reasonable rate they would carry of heat from the thermonuclear fusion engine running in the core of the star. This provides limits on axions and all light weakly interacting particles. They have to be weakly interacting to get out the core. The best limits tend to come from red giants since these have the highest temperatures in the core.
- A celebrated example is the fact that neutrino oscillations were really first discovered from the solar neutrinos.
- When a star finally collapses and produces a neutron star, very many neutrinos are radiated. Detection of these and their arrival times have allowed many limits on neutrino properties.

Nuclear Formation in the Big Bang

In the early universe the neutrons were caught into (mainly) ^{4}He nuclei. The precise amount of neutrons that get caught in various light nuclei versus the total number of protons and neutrons available is very sensitive to many conditions around. This has allowed to put some limits on the numbers of light neutrinos before limits from LEP appeared. It also allows limits on the numbers of relativistically moving particles (usually called radiation) in general at the time of nuclear synthesis.

The dark matter problem

It is known that there exists a lot more matter in the universe than there is in stars and other normal objects. This was first discovered in the velocity curves of distant galaxies. Particle physics has many candidates for this in beyond the standard model scenarios.

- Massive neutrinos
- Axions
- Various supersymmetric particles, these come in a class known as WIMPs (Weakly Interacting Massive Particles)

There are basicallythree types, hot dark matter (neutrinos), cold dark matter (WIMPs) and the ones where various fields get expectation values (which may vary with time) and as such produce the dark matter content.

Baryon Number Generation

One of the mysteries is why is there any baryon number left at all. In the beginning, pick your favourite energy, there are very many quarks and antiquarks around, as well as leptons. When the temperature drops below that required to have creation reactions running the annihilation of quarks and leptons starts. But they do not annihilate fully, some of them are left and the universe now exists mainly of quarks and leptons, not of anti-quarks and anti-leptons.

This tiny left over is rather puzzling and in order to solve it many conditions need to be satisfied (CPT conservation assumed here):

1. There have to be baryon number violating processes
2. Some things need to go out of thermal equilibrium, otherwise the equal balance principle would have baryon and anti-baryon number equal.
3. CP violation needs to be present.

In fact the standard model has all these three properties but not in a way sufficient to create the observed asymmetry (but it makes it worse for other higher energy mechanisms to produce the asymmetry). Grand Unified Theories provide one option for producing this imbalance.

Inflation

There are many problems associated with the fact that the universe is very big and looks rather homogenuous and isostropic on very large scales, as well as being very flat. These problems are:

- Flatness problem: this arises because the universe now is rather flat but the evolution from general relativity is such that the universe moves away from flatness with time. So in order to be as flat as observed now it had to be extraordinarily flat at early times.
- Horizon problem: Many of the areas of the universe can not have exchanged signals with each other because of the expansion of the universe. But still when we see those they look remarkably alike. How could this be?
- Monopoles, Cosmic Strings and Domain Walls: a problem: similar to the previous one. We expect several cases of spontaneous symmetry breaking to have occurred in the history of the universe. If the first breaking happens in different directions in different areas of the universe they start relaxing together but then socalled topological defects (similar to the domains in a ferromagnet) can occur and we should have observed these if they existed.

All of these are solved (at least in principle) by inflation. Note that the absolute simplest version of inflation does not work and no simple nice working model is known but many general features are well supported.

A really short Big Bang primer from General Relativity

We only discuss here the simple Friedman-Lemaître universes. Assume the universe is homogeneous and isostropic. That means that the energy-momentum tensor can be written in the simple form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{1}$$

and $\rho(t)$ and $p(t)$ are only functions of time. The matter is supposed to be at rest and behave like a socalled perfect fluid. The metric can be written in the general form

$$ds^2 = dt^2 - R(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right). \tag{2}$$

The parameter k can take the values $\pm 1, 0$ leading to the following general geometry of the universe

$k = +1$ Finite in space universe, closed, spherical geometry

$k = 0$ Infinite in space, open, flat geometry

$k = -1$ Infinite in space, open, hyperbolic geometry.

Notice that I have not said anything about whether the universe has a finite lifetime. From Einstein's equations we then get

$$H(t)^2 \equiv \left(\frac{\dot{R}(t)}{R(t)} \right)^2 = \frac{8\pi G_N \rho(t)}{3} - \frac{k}{R(t)^2}. \tag{3}$$

The pressure p satisfies

$$d\left(\rho R^3\right) = -pd\left(R^3\right) \ . \tag{4}$$

This has to be supplemented with an equation coupling ρ and p, known as the equation of state. The latter is in general complicated but for most purposes some simplifications are used. Usually it is assumed that

$$p(t) = w\rho(t)\,. \tag{5}$$

The following cases are of interest

- $w = 0$ corresponding to particles at rest, known as a matter dominated universe. $\rho \sim R^{-3}$
- $w = 1$ corresponding to particles moving at the speed of light whose energies is dominated by the kinetic energy and thus gets redshifted, known as a radiation dominated universe. $\rho \sim R^{-4}$
- $w = -1$ Or a cosmological constant. ρ is constant.
- $-1 \leq w \leq 0$ known as quintessence.

Note that in general relativity energy does not need to be conserved.

The later cases are normally realized as being produced by the values of a field ϕ which then obeys the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V(\phi)}{\partial \phi},. \tag{6}$$

For $k = 0$ these have the behaviour

matter	$\rho(t) = \rho_0/R(t)^3$	$R(t) = \left(\sqrt{6\pi G_N \rho_0}\, t\right)^{2/3}$
radiation	$\rho(t) = \rho_1/R(t)^4$	$R(t) = \left(\sqrt{32\pi G_N \rho_1/3}\, t\right)^{1/2}$
cosmological constant	$\rho = \rho_2$	$R(t) = e^{\sqrt{8\pi G_N \rho_2/3}\, t}$

So a possibility of exponential expansion exists if at some point we have a constant density. This is known as inflation.

A short history of the universe can now be described as.

1. For very small times all particles are relativistic and they dominate by far the energy density. The universe expands as a radiation dominated one and the particles slowly redshift.
2. The the energies of the ϕ field becomes low enough that the effect of the $V(\phi)$ becomes important and we shift to an exponentially expanding universe. The large inflation here means that the entire observable universe was reall a very small part beforehand.

3. The ϕ has slowly reached the minimum of the potential such that inflation stops and the universe is again radiation dominated.

4. Finally all relativistic particles get redshifted to such low energies that their rest mass dominates and we have the present (almost) matter dominated universe.

5. A newer observation is that we actually again have something like a (now much smaller one) cosmological constant around. This is known as the dark energy.

How does this solve the various problems. Basically all in the same way. Since the entire universe came from a very small pre-inflation area it had ample time to get homogenuous and organize itself. During the inflation any curvature present would also have gotten stretched away so that the universe afterwards will be flat with an incredible precision.

Cosmic Microwave Background

The cosmic microwave background has many interesting properties. First its existence by itself was a major success for the Big Bang model and was in fact what really made it universally accepted. Now the fine inhomogeneities are of major interest see e.g. [2]. In 2003 there was a major step forward with the results from the WMAP satellite, the successor of the COBE satellite, the results of which led to the Nobel prize in Physics 2006.

Structure Formation

This is the question how the present structures seen in the universe at scales from galaxies and up are formed from the initial small perturbation caused by quantum effects and/or whatever in very early universe. This is done by galaxy surveys and complements the structures seen in the microwave background by covering different time scales and size scales. Here come names like "the great wall, the gigantic bubble,..."

Other

There are many more speculative connections involving strings, extra dimensions gravity and gauge interactions operating in a different number of dimensions, quantum gravity and gravitational radiation,....

References

[1] L. Bergström and A. Goobar, Cosmology and Particle Astrophysics

[2] M. Kamionkowski, http://arXiv.org/abs/hep-ph/0210370