Solutions set 3, FYTN04, autumn 2014

1. Denote the common length of the three-vector \( p_C = |p_C| = |p_D| \), so that \( E_C = \sqrt{m_C^2 + p_C^2} \) and \( E_D = \sqrt{m_D^2 + p_C^2} \). The condition \( E_C + E_D = E_{cm} \) then gives

\[
E_C = E_{cm} - E_D \quad \Rightarrow \quad E_C^2 = m_C^2 + p_C^2 = E_{cm}^2 - 2E_{cm}E_D + m_D^2, p_C^2
\]

\[
E_D = \frac{E_{cm}^2 + m_D^2 - m_C^2}{2E_{cm}}
\]

\[
E_C = E_{cm} - E_D = \frac{E_{cm}^2 + m_C^2 - m_D^2}{2E_{cm}}
\]

\[
p_C^2 = E_C^2 - m_C^2 = \left(\frac{E_{cm}^2 + m_C^2 - m_D^2}{2E_{cm}}\right)^2 - 4E_{cm}^2 m_C^2 = \frac{\lambda(E_{cm}^2, m_C^2, m_D^2)}{4E_{cm}^2}
\]

introducing the so-called Källén function

\[
\lambda(a^2, b^2, c^2) = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2
\]

\[
= (a^2 - b^2 - c^2)^2 - 4b^2c^2
\]

\[
= (a + b + c)(a - b - c)(a - b + c)(a + b - c)
\]

2. (9.1 in book, with clarification/addition)
For the \( Z^0 \) use the branching ratios in eq. (9.39). When only \( \nu_4 \) light then

\[
BR(\nu_4) = \frac{\Gamma_Z^\nu}{3\Gamma_Z^e + 4\Gamma_Z^{\nu\mu} + 2\Gamma_Z^{\mu\mu} + 3\Gamma_Z^{\nu\nu}} = \frac{0.16}{3 \cdot 0.08 + 4 \cdot 0.16 + 2 \cdot 0.28 + 3 \cdot 0.36} = 0.063
\]

while with everything light

\[
BR(\nu_4) = \frac{\Gamma_Z^\nu}{4\Gamma_Z^e + 4\Gamma_Z^{\nu\mu} + 4\Gamma_Z^{\mu\mu} + 4\Gamma_Z^{\nu\nu}} = \frac{0.16}{4 \cdot 0.08 + 4 \cdot 0.16 + 4 \cdot 0.28 + 4 \cdot 0.36} = 0.045
\]

For the \( W \), if only \( \nu_4 \) light but \( \nu_4 \) heavy then \( BR(L_4\nu_4) = 0 \), while with everything light \( BR(L_4\nu_4) = \frac{1}{4(3+1)} = 0.062 \), where 3 is for three colours of quarks and 1 for the lepton pair of each generation.

3. (9.2 in book)
\( e_R \) does not couple to \( W \), \( \nu_\mu \) does not couple to \( \gamma \), neither couples to gluon, so must be \( Z^0 \).

\[
\nu_\mu
\]

\[
Z^0
\]

\[
e_R
\]

\[
e_R
\]

\[
\frac{1}{(Q^2 - m_Z^2)} \approx -\frac{1}{m_Z^2} \text{ (propagator)}
\]

\[
(g_2/\cos\theta_W)\bar{\nu}_L\gamma^\mu\nu_L T^\nu_3 \text{ (upper vertex)}
\]

\[
(g_2/\cos\theta_W)e_R\gamma^\mu e_R(-Q_e\sin^2\theta_W) \text{ (lower)}
\]
where $\bar{\nu}\nu, e\bar{e} \sim \sqrt{s}$, $T^3_{\nu} = 1/2$, $Q_e = 1$

$$M \approx \frac{g_2}{\cos \theta_W} \sqrt{s} \frac{1}{2} \frac{g_2}{m_Z^2} \cos \theta_W \sqrt{s} \sin^2 \theta_W = \frac{g_2^2}{2} \tan^2 \theta_W \frac{s}{m_Z^2} = 2\pi \alpha_2 \tan^2 \theta_W \frac{s}{m_Z^2}$$

so

$$\sigma \approx \frac{|M|^2}{32\pi^2 s^{3/2}} \frac{\sqrt{s} \theta_W}{4\pi \cos^4 \theta_W \frac{s}{m_Z^2}}$$

If $e$ fix then use $\alpha_2 = \alpha_{em} / \sin^2 \theta_W$, so that

$$\sigma \approx \frac{\pi \alpha_{em}^2 s}{4 \cos^4 \theta_W m_Z^2}$$

4. (10.1 in book)

$$s = (p_1 + p_2)^2 = (43 + 48)^2 - (43 - 48)^2 = 4 \cdot 43 \cdot 48 = 8256 = 90.9^2 \approx m_z^2$$

Two oppositely charged particles consistent with production and decay of a $Z^0$.

5. (10.3 in book, with simplification/clarification)

$$\hat{\sigma}_{ud} = \frac{4\pi^2}{3} \frac{\Gamma_u^W \Gamma_f^W}{\Gamma_{ud}^W} \frac{1}{s} \delta \left( x_1 x_2 - \frac{m_W^2}{s} \right)$$

$$\hat{\sigma}_{u\bar{u}} = \frac{4\pi^2}{3} \frac{\Gamma_{u\bar{u}}^Z \Gamma_f^Z}{\Gamma_{u\bar{u}}^Z} \frac{1}{s} \delta \left( x_1 x_2 - \frac{m_Z^2}{s} \right)$$

$$\hat{\sigma}_{d\bar{d}} = \frac{4\pi^2}{3} \frac{\Gamma_{d\bar{d}}^Z \Gamma_f^Z}{\Gamma_{d\bar{d}}^Z} \frac{1}{s} \delta \left( x_1 x_2 - \frac{m_Z^2}{s} \right)$$

Neglect mass difference in phase space integral, but remember that two $u$'s in a proton but only one $d$. Therefore $I_{Z}^{u\bar{u}} = 2I_{u}^{ud}$ while $I_{Z}^{d\bar{d}} = (1/2)I_{ud}^{ud}$, but factor 2 for $W^\pm$.

$$\frac{\sigma_Z}{\sigma_W} = \frac{\Gamma_{ud}^W \Gamma_{f}^W}{\Gamma_{u\bar{u}}^Z \Gamma_{f}^Z} \frac{\Gamma_{ud}^W}{\Gamma_{ud}^W} \frac{\Gamma_{d\bar{d}}^Z}{\Gamma_{d\bar{d}}^Z} \frac{2}{2} = 2.1 \cdot 80 (0.28 + 0.36/4) \frac{2}{2} \cdot 0.08 = 2.5 \cdot 91 = 0.68 \cdot 0.23 = 0.14$$

6. (not in book)

(a) Argument of $\delta$ function for energy has to be zero:

$$q + k + \sqrt{q^2 + k^2 + 2kq \cos \theta} = m_{\mu}$$

Isolate square root on one side and square

$$q^2 + k^2 + 2kq \cos \theta = (m_{\mu} - q - k)^2 = m_{\mu}^2 + q^2 + k^2 - 2m_{\mu}q - 2m_{\mu}k + 2kq$$
which gives
\[ \cos \theta = \frac{m^2 - 2m_\mu (k + q) + 2kq}{2kq} = 1 + \frac{m_\mu (m_\mu - 2k - 2q)}{2kq} \]

(b) Energy conservation gives \( k + q + k' = m_\mu \). Triangle inequality \(|a + b| < |a| + |b|\) and three-momentum conservation gives that
\[ q = |q| = | - k - k'| < |k| + |k'| = k + k' + q - q = m_\mu - q \]
or \( q < m_\mu /2 \), and similarly for \( k \) and \( k' \). But
\[ k' = m_\mu - k - q < m_\mu /2 \Rightarrow k + q > m_\mu /2 \]
which restricts to upper right triangle in the \((q, k)\) plane.
Alternatively you can use expression in (a) with \( \cos \theta < 1 \) to derive same limit.

7. (12.1 in book)
(a) 43 GeV
(b) 450 GeV
(c), (d) 2 TeV

Some relevant points:
• \( p \) beam can only use part of energy, so lose factor 3 – 10
• (a) and (c) need sea antiquark for annihilation, so further suppressed
• in \( ep \) you cannot create new particle by annihilation (except leptoquark?)
• antiproton gives lower luminosity, but also cheaper machine
• fixed target always loser for energy, but useful for extreme rates, e.g. \( \nu \) beams

8. (13.2 in book)
\( Z^0 \rightarrow \nu\bar{\nu} \) not visible, \( BR = 3 \cdot 0.07 = 0.21 \).
They are indirectly observed by the \( g \) jet and the opposite missing \( p_\perp \).

Comment: \( Z^0 \rightarrow e^+e^-/\mu^+\mu^- \) best bet, \( BR = 2 \cdot 0.03 = 0.06 \). \( \tau \) decay gives neutrino(s) so no mass peak. Decays to quark masked by QCD processes.