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Constructing the p^8 purely mesonic chiral Lagrangian

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Abstract

In this thesis we construct the purely mesonic chiral Lagrangian at order p^8. We first build an operator basis considering all possible operator structures and permutations of Lorentz indices, then minimize the number of operators by using relations between them. These relations are found from the equation of motion, total derivatives, antisymmetry equations, the Bianchi identity, the Cayley-Hamilton theorem and the Schouten identity. The minimized number of operators at order p^8 we calculate suggests the number of low energy constants of order eight is still manageable.

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Contents

1 Introduction

Quantum chromodynamics (QCD) is the field theory of the strong interaction, which describes the interaction between quarks and gluons. In QCD, it is possible to apply perturbation theory at high energies, this as the running coupling constant α_s decreases with energy and one can make an expansion in it. However, in the low energy limit this coupling becomes large, and the usual perturbative techniques cannot be used. Chiral perturbation theory is an effective field theory of QCD, where the degrees of freedom are hadrons rather than quarks and gluons and a perturbative expansion is done in powers of momentum. This allows for a perturbative study of the strong interaction at low energies.

The Lagrangian of an effective field theory can be constructed by writing down all possible operators invariant under an imposed general symmetry of the system and using the relevant degrees of freedom [1], where an example is the approximate chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ of QCD. Several of the operators will be related through specific relations, and by using these one can minimize the total number of operators. It is in general non-trivial to find the minimal set, but by employing as many relations as possible the set of operators can be reduced significantly.

In this thesis, we use chiral symmetry to construct the effective Lagrangian of chiral perturbation theory at orders p^2, p^4 [2, 3], p^6 [4, 5] and p^8, of which the last never before has been calculated. The momentum order of an operator is related to counting the number of derivatives, since each ∂_x is of order p. For simplicity, we neglect quark masses and external fields, but consider both a general number of flavours N_f and the specific cases of $N_f = 2$ and $N_f = 3$. We apply the equation of motion (EOM) as well as other relations to minimize the number of operators in the Lagrangian. In section 2, we introduce chiral perturbation theory and represent the simplest example of $\mathcal{L}_2$. In section 3, we show those relations that we used in detail. In section 4, we practically represent the process of constructing high order Lagrangian in different bases.

2 Chiral perturbation theory

2.1 Chiral symmetry breaking

In the chiral limit, where the masses of the u, d and/or s quarks are small enough to be considered zero, QCD has a chiral symmetry $SU(N_f)_L \times SU(N_f)_R$, N_f is the number of flavours. It can be shown mathematically that the group $G = SU(N_f)_L \times SU(N_f)_R$ can be spontaneously broken to the group $H = SU(N_f)_V$ (equivalently written as $SU(N_f)_{L=R}$) by a non-vanishing quark vacuum expectation value, $\langle \bar{q}q \rangle \neq 0$. This symmetry breaking is therefore an assumption in QCD.

From the chiral symmetry breaking, we can define the building block U. First the Lie group takes the form $e^{i\pi_i T_i}$, where π_i is a number and T_i a group generator. The generators of the group G, the unbroken group H and the coset $G/H = SU(N_f) \times SU(N_f)/SU(N_f) =$

$SU(N_f)$ take the form

$$G : \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \;\; ; \; H : \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \;\; ; \; G/H : \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} \tag{2.1}$$

Here A,B,X and H are the group generators. Accordingly, the group elements of those three groups/cosets are

$$g = \begin{bmatrix} g_R & 0 \\ 0 & g_L \end{bmatrix} \;\; ; h = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \;\; ; \xi = \begin{bmatrix} u & 0 \\ 0 & u^\dagger \end{bmatrix} \tag{2.2}$$

Here g, g_R, g_L, h and ξ are group elements of $SU(N_f)$. We then transform the element of the group G/H as

$$g\xi h^\dagger = \begin{bmatrix} g_R & 0 \\ 0 & g_L \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & u^\dagger \end{bmatrix} \begin{bmatrix} h^\dagger & 0 \\ 0 & h^\dagger \end{bmatrix} = \begin{bmatrix} g_R u h^\dagger & 0 \\ 0 & g_L u^\dagger h^\dagger \end{bmatrix} \tag{2.3}$$

Since the unbroken symmetry group is $SU(N_f)_{L=R}$, we get

$$g_R u h^\dagger = h u {g_L}^\dagger \tag{2.4}$$

We define $U = uu$, so

$$U \mapsto g_R u h^\dagger h u {g_L}^\dagger = g_R U {g_L}^\dagger \tag{2.5}$$

Therefore, under chiral symmetry, U transforms as U $\mapsto g_R U {g_L}^\dagger$. The properties of U also satisfy $UU^\dagger = U^\dagger U = 1$ and $\det U = 1$. Also, according to Goldstone's theorem, every broken generator will generate one Goldstone boson. Therefore, for $SU(N_f)$ symmetry, there are $N^2 - 1$ Goldstone bosons. We can see these as the lightest pseudoscalar mesons, which are composed of a quark and an antiquark.

Using meson fields, the building block U takes the form $U = \exp(\frac{i}{2F_0}\Phi)$, where F_0 is the pion decay constant and Φ is the meson matrix. For $N_f = 2$

$$\Phi(N_f = 2) = \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix} \tag{2.6}$$

and when $N_f = 3$

$$\Phi(N_f = 3) = \begin{bmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\overline{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{bmatrix} \tag{2.7}$$

The mesons (π, K, η) exist in the lightest pseudoscalar octet.

2.2 The u basis

Instead of using U as the fundamental building block of the effective field theory (EFT), we consider the matrix u defined through $U = u^2$. We first define the important building structure u_μ as (excluding external fields)

$$u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \tag{2.1}$$

which has good transformation properties, i.e. it transforms as $u_\mu \mapsto h u_\mu h^\dagger$. Also, ${u_\mu}^\dagger = u_\mu$ and $\langle u_\mu \rangle = 0$. Since we construct the Lagrangian in four dimensions, μ=0,1,2,3. Below we show the proof of the transformation property of u_μ:

Using the transformation property of U and the definition of u, the transformation properties of u and $u^\dagger$ are

$$u \mapsto g_R u h^\dagger = h u {g_L}^\dagger; \; u^\dagger \mapsto h u^\dagger {g_R}^\dagger = g_L u^\dagger h^\dagger$$

h is an element of the broken group H and depends on u, g_L, g_R, so $\partial_\mu h \neq 0$. In this way, $u^\dagger \partial_\mu u$ transforms as

$$u^\dagger \partial_\mu u \mapsto h u^\dagger {g_R}^\dagger \partial_\mu (g_R u h^\dagger) = \; h u^\dagger {g_R}^\dagger g_R (\partial_\mu u h^\dagger + u \partial_\mu h^\dagger) \; = h u^\dagger \partial_\mu u h^\dagger - \partial_\mu h h^\dagger$$

Similarly, $u \partial_\mu u^\dagger \mapsto h u \partial_\mu u^\dagger h^\dagger - \partial_\mu h h^\dagger$. Therefore,

$$\begin{aligned}(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \mapsto (h u^\dagger \partial_\mu u h^\dagger - \partial_\mu h h^\dagger) - (h u \partial_\mu u^\dagger h^\dagger - \partial_\mu h h^\dagger) \\ = h(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) h^\dagger\end{aligned}$$

We have thus proved the transformation property $u_\mu \mapsto h u_\mu h^\dagger$.

Having defined u_μ we can also define the covariant derivative ∇_μ as

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X] \tag{2.2}$$

X is here an arbitrary operator, transforming as $X \mapsto h X h^\dagger$. Γ_μ is called a connection (named from differential geometry), and is here given by

$$\Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) \tag{2.3}$$

This is needed for the covariant derivative to transform properly, i.e. $\nabla_\mu X \mapsto h \nabla_\mu X h^\dagger$. The field strength of Γ_μ, here denoted as $\Gamma_{\mu\nu}$, is defined according to

$$\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = \frac{1}{4}[u_\mu, u_\nu] \tag{2.4}$$

2.3 $\mathcal{L}_2$ and total derivative

$\mathcal{L}_2$ is generally called the leading order Lagrangian as it is the first non-trivial Lagrangian possible to write down and is of order p^2. As mentioned above, there must therefore be two derivatives in each possible operator, whose Lorentz indices must be contracted. The general structure of $\mathcal{L}_2$ is thus

$$\mathcal{L}_2 = a_1\langle\partial^2 UU^\dagger\rangle + a_2\langle U\partial^2 U^\dagger\rangle + a_3\langle\partial_\mu U\partial^\mu U^\dagger\rangle \tag{2.5}$$

where"$\langle\ \ \rangle$" is the trace in flavour space and $a_{1,2,3}$ are numbers. We can, however, find relations between these three terms by considering the total derivatives

$$\partial_\mu(\langle U\partial^\mu U^\dagger\rangle) = \langle\partial_\mu U\partial^\mu U^\dagger\rangle + \langle U\partial^2 U^\dagger\rangle = 0 \tag{2.6}$$

$$\partial_\mu(\langle\partial^\mu UU^\dagger\rangle) = \langle\partial^2 UU^\dagger\rangle + \langle\partial^\mu U\partial_\mu U^\dagger\rangle = 0 \tag{2.7}$$

Eq. (2.5) can therefore be simplified to

$$\mathcal{L}_2 = (a_3 - a_1 - a_2)\langle\partial_\mu U\partial^\mu U^\dagger\rangle = \frac{F^2}{4}\langle\partial_\mu U\partial^\mu U^\dagger\rangle \tag{2.8}$$

Where F is a constant.

The total derivatives we used above do not contribute to the action, hence the $= 0$ in eqs. (2.6)–(2.7). This property can be explained by Gauss' law. The action S for a Lagrangian containing a total derivative can be written as

$$S = \int d^4x\mathcal{L} = \int d^4x\mathcal{L}_0 + \int d^4x\partial_\mu J^\mu \tag{2.9}$$

where $\mathcal{L}_0$ is the part of Lagrangian without total derivatives, and $\int d^4x\partial_\mu J^\mu$ represents the part with total derivatives. Thus the problem is to derive $\int d^4x\partial_\mu J^\mu = 0$. For four dimensions, Gauss' law which changes a volume integral into a surface integral is

$$\int d^4x\partial_\mu J^\mu = \int dS_3\, n_\mu J^\mu \tag{2.10}$$

Here dS_3 is the three dimensional surface and n_μ is the direction vector perpendicular to the surface. Since the fields vanish at infinity, the integral in eq. (2.10) is equal to zero.

2.4 Parity and Charge conjugation

Parity P is a discrete transformation of spacetime, such that it inverts the spatial part and leaves time invariant, i.e. $\mathbf{x} \to -\mathbf{x}$ and $t \to t$. Since the mesons considered here are pseudoscalars, they obtain an extra minus sign under parity, and the total transformation of u_μ under parity is $u_\mu \to -\varepsilon(\mu)u_\mu$, where $\varepsilon(0) = -\varepsilon(\mu \neq 0) = 1$.

Charge conjugation, denoted C, is defined by exchanging each particle for its antiparticle. For the meson matrix, this means

$$C(\Phi(N_f = 2)) = \begin{bmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{bmatrix} = (\Phi(N_f = 2))^T \tag{2.11}$$

which mathematically means taking the transpose of the matrix. Thus, $u_\mu \to u_\mu^T$.

3 Relations

Depending on the chiral order, there are different relations to relate the various operators allowed by the chiral symmetry. In this section, we discuss those relations in detail.

3.1 Equation of Motion

To derive the lowest order equation of motion, we start from the classical Euler-Lagrange equation

$$\partial_\mu(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}) - \frac{\delta\mathcal{L}}{\delta\phi} = 0 \tag{3.12}$$

This is derived by requiring the action to be invariant under $\phi \to \phi + \delta\phi$. In a similar way, we can change U as $U \to U + \delta U = U'$. We need to be careful with U since $U'U'^\dagger = U'^\dagger U = 1$ and $\det U' = 1$. Two ways of changing U are $U' = Ue^{i\epsilon}$ and $U' = ue^{i\epsilon}u$. Below we derive the EOM for both cases separately.

Solution 1 Insert $U' = Ue^{i\epsilon}$ into eq. (2.8). Then we have

$$\mathcal{L}_2 = \frac{F^2}{4}\langle\partial_\mu(Ue^{i\epsilon})\partial^\mu(e^{-i\epsilon}U^\dagger)\rangle \tag{3.13}$$

Expand this equation using the approximation $e^{i\epsilon} = 1 + i\epsilon, e^{-i\epsilon} = 1 - i\epsilon$. Only considering the first order of ϵ, we get

$$\delta\mathcal{L}_2 = \frac{F^2}{4}i\langle\partial_\mu\epsilon(-U^\dagger\partial^\mu U + \partial^\mu U^\dagger U)\rangle = -\frac{F^2}{4}i\langle\epsilon\partial_\mu(U^\dagger\partial^\mu U - \partial^\mu U^\dagger U)\rangle \tag{3.14}$$

Ignoring the overall $-\frac{F^2}{4}i$, this equation can be written as

$$\langle(\epsilon^a\tau^a)Y\rangle = 0 \tag{3.15}$$

Here $\epsilon = \epsilon^a\tau^a$ and Y is a matrix related to U through $Y = U^\dagger\partial^\mu U - \partial^\mu U^\dagger U$. We can write Y in the form $Y = y^b\tau^b + y^0 I$, where I is the identity matrix. Using $\langle\tau^a\tau^b\rangle = 2\delta^{ab}$ we thus get the EOM $y^b = 0$ or $y^b\tau^b = 0$. From this form of Y we know that $y^b\tau^b = Y - y^0 I$, but since $y^0 I = \frac{1}{N_f}\langle Y\rangle$ we have

$$Y - \frac{1}{N_f}\langle Y\rangle = 0 \tag{3.16}$$

Plugging in $Y = U^\dagger\partial^\mu U - \partial^\mu U^\dagger U$, we get the EOM

$$U^\dagger\partial^\mu U - \partial^\mu U^\dagger U - \frac{1}{N_f}\langle U^\dagger\partial^\mu U - \partial^\mu U^\dagger U\rangle = 0 \tag{3.17}$$

Solution 2 Let $U' = ue^{i\epsilon}u$, and insert it into eq. 2.8. This gives

$$\mathcal{L}_2 = \frac{F^2}{4}\langle\partial_\mu(ue^{i\epsilon}u)\partial^\mu(u^\dagger e^{-i\epsilon}u^\dagger)\rangle \tag{3.18}$$

Using the unitarity of u,

$$uu^\dagger = u^\dagger u = I; \partial(uu^\dagger) = \partial(u^\dagger u) = 0$$

the final result is

$$\langle 2\epsilon\nabla_\mu u^\mu\rangle = 0 \tag{3.19}$$

Therefore, the EOM is $\nabla_\mu u^\mu = 0$, where $\nabla_\mu u^\mu = \partial_\mu u_\mu + \Gamma_\mu u_\mu - u_\mu\Gamma_\mu$.

We need to prove the equivalence of the EOM and field transformations [6] since the EOM is from classical physics while field transformations can be performed in all quantum theories. From the definitions of U and u, we have $U = \exp(\frac{i}{2F_0}\Phi)$ and $u = \exp(\frac{i}{4F_0}\Phi)$, which satisfy $U = uu$. With the same U but a different Φ, the field transformation takes the form

$$\overline{U} = u\exp(iS)u \tag{3.20}$$

where S is a matrix. When we only consider the Lagrangian in order p^2, we can write $\overline{U}$ as $u\exp(iS_2)u$ using the approximation $\exp(iS) = 1 + iS_2$. Inserting eq. (3.20) into the equation of $\mathcal{L}_2$, eq. (2.8), we get

$$\mathcal{L}_2(\overline{U}) = \mathcal{L}_2(U) + \frac{F^2}{4}\langle S_2\nabla_\mu u^\mu\rangle + \mathcal{O}(S^2) = \mathcal{L}_2(U) + \Delta L_2(\overline{U}) \tag{3.21}$$

where

$$\Delta L_2(\overline{U}) = \frac{F^2}{4}\langle S_2\nabla_\mu u^\mu\rangle + \mathcal{O}(S^2) \tag{3.22}$$

Since $\nabla_\mu u^\mu \sim \mathcal{O}(p^2)$ and $S_2 \sim \mathcal{O}(p^2)$, the leading term in $\Delta L_2(\overline{U})$ is of order p^4. Therefore, in a quantum theory, the EOM $\nabla_\mu u^\mu = 0$ is equivalent to a field transformation, and can be used to remove terms from the Lagrangian at a certain order. A more detailed discussion is in [3, 4, 6].

3.2 Antisymmetry equations

The solutions in this subsection follow from the fact that partial derivatives commute. There are two antisymmetry relations that we use. The first we refer to as the antisymmetry equation:

$$\nabla^\mu u^\nu - \nabla^\nu u^\mu = 0 \tag{3.23}$$

The second is an antisymmetry equation of covariant derivatives:

$$[\nabla_\mu, \nabla_\nu]X = [\Gamma_{\mu\nu}, X] \tag{3.24}$$

This is similar to the commutation of covariant derivatives in gauge theory, where

$$[D_\mu, D_\nu] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu) = -ig\Gamma_{\mu\nu} \tag{3.25}$$

Here, the covariant derivative is defined as $D_\mu = \partial_\mu - igA_\mu$ for the gauge field A_μ and the parameter g called the coupling constant. We can confirm the two antisymmetry equations above by inserting the definitions of ∇_μ and u^μ.

3.3 Bianchi identity

We also use the so called Bianchi identity

$$\nabla_\mu\Gamma_{\nu\rho} + \nabla_\nu\Gamma_{\rho\mu} + \nabla_\rho\Gamma_{\mu\nu} = 0 \tag{3.26}$$

That this equation holds can be proven by plugging in the definitions of $\Gamma_{\mu\nu}$ and the covariant derivative given in the section introducing the u basis.

In fact, the Bianchi identity can also be derived in gauge theory [7]. The Jacobi identity for the covariant derivatives is a relation between three ternary operations, and can be written as

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0 \tag{3.27}$$

Acting with the first term on a field ψ gives

$$[D_\mu, [D_\nu, D_\rho]]\psi = -ig(D_\mu F_{\nu\rho})\psi \tag{3.28}$$

where $F_{\nu\rho} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]$. From $D_\mu F_{\nu\rho} = \partial_\mu F_{\nu\rho} - ig[A_\mu, F_{\nu\rho}]$, we thus get the Bianchi identity

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0 \tag{3.29}$$

3.4 Cayley-Hamilton theorem

Using the Cayley-Hamilton theorem, we can obtain the equations below that in turn allow for extra relations between terms in the Lagrangian. For $N_f = 2$ the relation between $N_f \times N_f$ matrices A and B is

$$AB + BA - A\langle B\rangle - B\langle A\rangle - \langle AB\rangle + \langle A\rangle\langle B\rangle = 0, \tag{3.30}$$

For $N_f = 3$, the relation, with C as another 3×3 matrix, is instead

$$\begin{aligned} ABC + ACB + BAC + BCA + CAB + CBA - AB\langle C\rangle - AC\langle B\rangle - BA\langle C\rangle \\ -BC\langle A\rangle - CA\langle B\rangle - CB\langle A\rangle - A\langle BC\rangle - B\langle AC\rangle - C\langle AB\rangle \\ -\langle ABC\rangle - \langle ACB\rangle + A\langle B\rangle\langle C\rangle + B\langle A\rangle\langle C\rangle + C\langle A\rangle\langle B\rangle \\ +\langle A\rangle\langle BC\rangle + \langle B\rangle\langle AC\rangle + \langle C\rangle\langle AB\rangle - \langle A\rangle\langle B\rangle\langle C\rangle = 0 \end{aligned} \tag{3.31}$$

It is these two relations that we use in our calculation. Below, we derive the equation for the $N_f = 2$ case.

In linear algebra, an $n \times n$ matrix A has n eigenvalues λ given by the equation

$$\det(A - \lambda I) = 0, \tag{3.32}$$

Each eigenvalue then satisfies

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} ... + a_0 \lambda^0 = 0, \tag{3.33}$$

where the a_i $(i = 0, 1, ...n)$ are numbers. The Cayley-Hamilton theorem now states that the matrix A satisfies this equation as well, i.e.

$$a_n A^n + a_{n-1}A^{n-1} ... + a_0 A^0 = 0 \tag{3.34}$$

Now consider $n = N_f = 2$. For a 2×2 matrix, eq. (3.34) then becomes

$$A^2 + \frac{1}{2}(\langle A \rangle^2 - \langle A^2 \rangle) - A\langle A \rangle = 0 \tag{3.35}$$

Decompose the matrix A as $A = bB + cC$, where b and c are numbers, and B as well as C are matrices. Plugging this into eq. (3.35) gives

$$\begin{aligned} b^2[B^2 + \frac{1}{2}(\langle B \rangle^2 - \langle B^2 \rangle) - B\langle B \rangle] + c^2[C^2 + \frac{1}{2}(\langle C \rangle^2 - \langle C^2 \rangle) - C\langle C \rangle] \\ + bc[BC + CB + \frac{1}{2}(2\langle BC \rangle - 2\langle B \rangle \langle C \rangle) - \langle B \rangle C - \langle C \rangle B] = 0 \end{aligned} \tag{3.36}$$

The first two terms for the matrices B and C are zero from eq. (3.35), so the last term which relates B and C is also zero, i.e.

$$BC + CB + \frac{1}{2}[2\langle BC \rangle - 2\langle B \rangle \langle C \rangle] - \langle B \rangle C - \langle C \rangle B = 0, \tag{3.37}$$

This is the same as eq. (3.30), so the relation has been proven. The derivation of the $N_f = 3$ relation is similar.

3.5 Schouten identity

First, we define full antisymmetry from which we can derive the Schouten identity. First consider the two dimensional case for simplicity. Then, a fully antisymmetric tensor $B_{\mu\nu\alpha}$ with three indices can be constructed from any other tensor with three indices, this via

$$B_{\mu\nu\alpha} = A_{\mu\nu\alpha} + A_{\nu\alpha\mu} + A_{\alpha\mu\nu} - A_{\nu\mu\alpha} - A_{\alpha\nu\mu} - A_{\mu\alpha\nu} \tag{3.38}$$

Here we have eight possibilities for $\mu\nu\alpha$:

$$(x,x,x), (x,x,y), (x,y,x), (y,x,x), (x,y,y), (y,x,y), (y,y,x), (y,y,y) \tag{3.39}$$

Choosing any one of the possibilities and plugging it into the definition $B_{\mu\nu\alpha}$, we get $B_{\mu\nu\alpha} = 0$. The general result can be stated as follows: if the number of indices is larger than the number of dimensions, any fully antisymmetric tensor is zero. Hence, for four dimensions, we have as consequence

$$\epsilon_{\mu\nu\alpha\beta}A_\gamma - \epsilon_{\gamma\nu\alpha\beta}A_\mu - \epsilon_{\mu\gamma\alpha\beta}A_\nu - \epsilon_{\mu\nu\gamma\beta}A_\alpha - \epsilon_{\mu\nu\alpha\gamma}A_\beta = 0 \tag{3.40}$$

Here the fully antisymmetric Levi-Civita tensor ϵ was used, and A_γ is just a tensor with another index (or indices). It is eq. (3.40) that is known as the Schouten identity and was first introduced in [8].

Choosing $A_\gamma = \epsilon_{\gamma\delta\rho\sigma}$ since we for order p^8 we have eight indices and only consider the even intrinsic parity, the equation above can be written

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\gamma\delta\rho\sigma} - \epsilon_{\gamma\nu\alpha\beta}\epsilon_{\mu\delta\rho\sigma} - \epsilon_{\mu\gamma\alpha\beta}\epsilon_{\nu\delta\rho\sigma} - \epsilon_{\mu\nu\gamma\beta}\epsilon_{\alpha\delta\rho\sigma} - \epsilon_{\mu\nu\alpha\gamma}\epsilon_{\beta\delta\rho\sigma} = 0 \tag{3.41}$$

We can get now get additional relations with the help of eq. (3.41), this by multiplying it with the basis of possible operators.

4 Construction of the chiral Lagrangian

The general form of the Lagrangian is

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots \tag{4.1}$$

The subscript indicates the order in the momentum expansion, and is an even number because of the contraction of Lorentz indices. Here we do not consider $\mathcal{L}_0$, which is a trivial constant term without derivatives. We will next present the construction of $\mathcal{L}_2$, $\mathcal{L}_4$, $\mathcal{L}_6$ and $\mathcal{L}_8$, in which we first list all the terms and then try to minimize that number by employing the relations presented in the previous section. At order p^4, we first derive the Lagrangian in the U basis, and then change to the u basis for orders p^4, p^6 and p^8.

4.1 $\mathcal{L}_4$

Similar to the earlier derivation of $\mathcal{L}_2$, but with 4 derivatives. We divide the structure of derivatives into five types:

$$(\partial,\partial,\partial,\partial), (\partial^2,\partial,\partial), (\partial^2,\partial^2), (\partial^3,\partial), (\partial^4)$$

Here $\partial = \partial_\mu = \frac{\partial}{\partial x^\mu}$. ∂^2 represents two such derivatives acting on the same object u_μ. Similarly we define ∂^3 and ∂^4.

Considering the possible permutations of the indices μ and ν, we get the total list of terms. Taking $(\partial,\partial,\partial,\partial)$ and a single trace as a simple example, there are three terms considering permutations and cyclicity. These are

$$\langle\partial_\mu U\partial^\mu U^\dagger\partial_\nu U\partial^\nu U^\dagger\rangle, \langle\partial_\mu U\partial^\nu U^\dagger\partial_\mu U\partial^\nu U^\dagger\rangle, \langle\partial_\mu U\partial^\nu U^\dagger\partial_\nu U\partial^\mu U^\dagger\rangle$$

Remembering the unitary of U and the invariance of the action under the addition of total derivatives,

$$UU^\dagger = 1; \partial_\mu(\langle U\partial^\mu U^\dagger\rangle) = 0;$$

$$\partial_\mu(UU^\dagger) = 0 = \partial_\mu(U^\dagger U)$$

$$\partial_\alpha(\partial_\beta UU^\dagger) = 0; \partial_\beta(U\partial_\alpha U^\dagger) = 0$$

we get the final four independent terms (i.e. operators)

$$\begin{aligned}\mathcal{L}_4 = L_0\langle\partial_\mu U\partial^\nu U^\dagger\partial_\mu U\partial^\nu U^\dagger\rangle + L_1\langle\partial_\mu U\partial^\mu U^\dagger\rangle\langle\partial_\nu U\partial^\nu U^\dagger\rangle + \\ L_2\langle\partial_\mu U\partial^\nu U^\dagger\rangle\langle\partial_\mu U\partial^\nu U^\dagger\rangle + L_3\langle\partial_\mu U\partial^\mu U^\dagger\partial_\nu U\partial^\nu U^\dagger\rangle\end{aligned} \tag{4.2}$$

were the L_i are low energy constants.

Now we switch to the u basis, and for this basis we use the computer. As explained earlier, we replace ∂ with ∇, change U into u_μ which is defined by u and contains one derivative. We use the software FORM [9] to form a matrix of all the relations between the operators, R, and `C++` with exact rational arithmetic [10] to calculate the rank of R. In particular, we want to have $RA = 0$, where R is an $n \times m$ matrix. n is the number of equations we get from relations and m is the number of possible operators after removing equivalent ones under cyclicity of the trace. A is a vector of length m containing this generated fundamental basis of operators. The general form of $RA = 0$ is

$$\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{1m} \end{bmatrix} \begin{bmatrix} ope_1 \\ \vdots \\ ope_m \end{bmatrix} = 0 \tag{4.3}$$

where ope_i is one of the operators in the basis.

Next we discuss the three main procedures we follow in the FORM program (details are shown in appendix A). First, we generate a list of allowed operators identified by $ope(aa)$, where aa is a number. There are four main operator structures, $\langle u_\alpha u_\beta u_\gamma u_\delta\rangle$, $\langle\nabla_\alpha u_\beta\nabla_\gamma u_\delta\rangle$, $\langle\nabla_\alpha\nabla_\beta u_\gamma u_\delta\rangle$ and $\langle u_\alpha u_\beta\rangle\langle u_\gamma u_\delta\rangle$. The structures with an odd number of u's, like$\langle\nabla_\alpha u_\beta u_\gamma u_\delta\rangle$, cannot exist since we only consider even parity which requires even number of u_μ. After considering the permutations of two indices μ and ν, each operator structure has three possible permutations, and we have twelve operator terms. Applying the equation of motion and cyclicity of the trace some operators will be equal. We can thus reduce the number of terms and build a simpler operator basis, each operator now related to a new $ope(aa)$.

We also need to generate equations from the relations discussed earlier. The equations have the form $R_{i1}ope(1) + R_{i2}ope(2) + ... + R_{ij}ope(j) + ... + R_{im}ope(m) = 0$. To form the matrix R, we need to set a number for each equation, which here is i and is given by a row in R. For order p^4 the relevant relations that we use are total derivatives, the antisymmetry equations, the Bianchi identity and Cayley-Hamilton theorem, all mentioned in the previous section. We get the final minimized number of terms to be four, exactly the same number that we derived in the U basis.

4.2 $\mathcal{L}_6$

From the example of $\mathcal{L}_4$ in two different bases, we could see that there are two main differences when we switch to the u basis. Firstly, the combinations of u_μ and ∇_μ are slightly different, since we know that each u_μ contains a derivative. The second is that the calculation is done in FORM and `C++` [10] where the process is more mathematical in the sense of constructing the matrix R and finding its rank.

For order p^6, the number of independent operators is 18 after applying the equation of motion, total derivatives, the antisymmetry relations and the Bianchi identity. All the details will be discussed below for $\mathcal{L}_8$, since this is the main part that we did.

4.3 $\mathcal{L}_8$

There are four ways of distributing eight derivatives, i.e. choosing how many u_μ's and ∇_μ's to have: $(8u, 0\nabla)$, $(6u, 2\nabla)$, $(4u, 4\nabla)$ and $(2u, 6\nabla)$. As an example of the notation, 4∇ could be of the four forms $(\nabla\nabla\nabla\nabla)$, $(\nabla\nabla\nabla, \nabla)$, $(\nabla\nabla, \nabla\nabla)$, $(\nabla, \nabla, \nabla, \nabla)$. Each derivative operator should then be combined with one u and the remaining u's then added, so that e.g. $\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta u_\gamma u_\delta u_\rho u_\sigma$ is the first one and $\nabla_\mu\nabla_\nu\nabla_\alpha u_\beta\nabla_\gamma u_\delta u_\rho u_\sigma$ is the second (Lorentz invariance then requires all the indices to be pairwise contracted). Also involving the possibility of multiple traces (like $\langle uuuu\rangle\langle uuuu\rangle$), we have a set of 45 possible terms.

Programming is required since each of those 45 terms has 105 ways of permuting the Lorentz indices γ, δ, ρ and σ, which greatly increases the amount of computations needed. As mentioned for $\mathcal{L}_4$, the first step is to make all the basic set of operators into a single file and then generate the matrix R by using total derivatives, the antisymmetry equations, the Bianchi identity and the Cayley-Hamilton theorem. The difference from before is that we at order p^8 also must add the Schouten identity. After forming the R matrix, we use Gaussian elimination to find the its rank. The number of independent operators is equal to the number of operators in the basis minus the rank of R, this since the rank indicates the number of linearly independent rows and each independent row can be used to subtract one of the basis operators.

For p^8 and a general N_f, before adding Schouten identity, the number of independent operators is 115. When adding the Schouten identity it reduces to 99.

Excluding the Schouten identity but using the Cayley-Hamilton theorem for $N_f = 2$, gives the least independent operators as 11. After adding the Schouten identity, it is reduced to 9. Similarly, for $N_f = 3$, the least independent operators is 53 before adding the Schouten identity, and 41 once it has been used.

5 Conclusion

In conclusion, what we mainly do is construct a chiral Lagrangian in the u basis considering the chiral symmetry, then, by using certain relations, reduce the total number of operators allowed by this symmetry to a smaller number. We follow this procedure for orders p^2, p^4, p^6 and p^8. The difference between this thesis and previous articles is we calculate the

minimized number of operators at order p^8, which can provide a clue for experimental physicists of how many parameters of order p^8 they should measure in particle physics experiments.

A Code in FORM

FORM, that we used to generate the matrix R, is a programming software for analytic computations in e.g. particle physics.

A.1 Generate operators

There are two main parts of producing operators – creating the simple operator basis and making a whole list of the operator terms. For simplicity, we choose the p^4 code as an example.

```
L X = f(j1,j1,j2,j2)+f(j1,j2,j1,j2)+f(j1,j2,j2,j1);

G LP4x1 = X*(
         tr(u(i1),u(i2),u(i3),u(i4))
        +tr(u(i1,i2,i3),u(i4))
        +tr(u(i1,i2),u(i3,i4))
        +tr(u(i1),u(i2))*tr(u(i3),u(i4)));

id f(<j1?>,...,<j4?>) = replace_(<i1,j1>,...,<i6,j6>);
```

This part generates the operator basis by adding permutations of indices $j1$ and $i2$, which represent the Lorentz indices μ and ν in the theory. X generates all permutations, and $tr(u(i1),u(i2),u(i3),u(i4))+tr(u(i1,i2,i3),u(i4))+tr(u(i1,i2),u(i3,i4))+tr(u(i1),u(i2))*tr(u(i3),u(i4)))$ are the possible operator structures. Here $u(i1)$ represents u_μ, and $u(i1,i2)$ is the derivative of u_ν, i.e. $\nabla_\mu u_\nu$.

```
#$opecount = 0;
multiply ope;
$opecount = $opecount+1;
id ope = ope($opecount);
```

After using simple relations, we have reduced the simple operator basis. It is important to identify each operator by an $ope(aa)$. Here we do it by multiplying ope. We use the same trick when we count the number of relations.

```
id tr(?a) = tr(aa,?a);
repeat;
  id tr(?a,aa,u(?b),?c) = tr(u(?b),?c,?a)+tr(?a,u(?b),aa,?c);
  id tr(?a,aa) = 0;
endrepeat;
if match(tr(?a,u?(?b,j2,?c),?d)) multiply (perm_(f,j1,j2));
id f(<i1?>,...,<i2?>) = replace_(<j1,i1>,...,<j2,i2>);
```

This part produces the whole list of operator terms using the cyclicity of the trace and adding all permutations. Here we add *aa* as a trick to cyclically permute traces and multiply by *perm* to obtain all permutations.

A.2 Generating linear equations from relations

We reduce the number of operators by using relations. The tracelessness of u_μ and the equation of motion are relations that simply set certain individual terms to zero. The relations from total derivatives, the antisymmetry equations, the Bianchi identity, the Schouten identity and the Cayley-Hamilton theorem produce the linear equations that form the matrix R. It is these relations that we now wish to obtain, but first consider the implementation of the EOM and the tracelessness:

```
id tr(u(?a)) = 0;* u_mu is traceless
id tr(?a,u(?b,i1?,i1?),?c) = 0;* removes terms zero from EOM
```

Here, traces with only one u_μ or containing EOM $\nabla_\mu u_\mu$ are removed. These two relations are used repeatedly thoughout the whole program to simplify as much as possible in every step.

Next, consider the total derivative relations obtained from the following pieces of code.:

```
L X =
       + f(j1,j2,j2)
       + f(j2,j1,j2)
       + f(j2,j2,j1)
      ;
L LP4derivs = X*(
        +tr(u(i1,i2),u(i3))
        );
id f(<j1?>,...,<j3?>) = replace_(<i1,j1>,...,<i3,j3>);
```

We first produce the structures missing one ∇ from the possible operator structure, then find all permutations of Lorentz indices.

```
multiply deriv(j1);
repeat;
  id deriv(?a)*tr(?b) = deriv(?a,tr(?b));
```

```
endrepeat;

#$relcount = 0;
multiply rel;
$relcount = $relcount+1;
id rel = rel($relcount);

repeat;
  id deriv(j1?,tr(?a),?b) = deriv(j1,?b)*tr(?a)+deriv(?b)*tr(j1,?a);
  id deriv(tr(?a),?b) = deriv(?b)*tr(?a);
  id deriv(j1?) = 0;
  id deriv = 1;
endrepeat;
repeat;
  id tr(?c,j1?,u(?a),?b) = tr(?c,u(j1,?a),?b)+tr(?c,u(?a),j1,?b);
  id tr(?a,j1?) = 0;
endrepeat;
```

Here are three main parts. In the first part we multiply $deriv(?a)$, which means ∇_α. In the second part we identify the relations with rel. Finally we do the derivative in steps.

For the antisymmetry relations we use the following code:

```
#do i=1,`MAXINDEX'
  id,once tr(?a,u(?b,i1?,i2?),?c) = aa*tr(?a,u`i'(?b,i1,i2),?c);
#enddo
if (match(aa)==0) discard;

#do i=1,`MAXINDEX'
  id,only aa^`i' = aa1+...+aa`i';
#enddo

$relcount = $relcount+1;
multiply rel($relcount);

#do i=1,`MAXINDEX'
  id aa`i'*tr(?a,u'i'(?b,i1?,i2?),?c) =
         tr(?a,u`i'(?b,i1,i2),?c)-tr(?a,u`i'(?b,i2,i1),?c);
#enddo
repeat;
  id tr(?a,u1?ui(?b),?c) = tr(?a,u(?b),?c);
endrepeat;
```

We first use aa to whenever a structure contains a $\nabla_\mu u_\nu$ and its derivatives. Then we expand the structure using antisymmetry relations into equations. For the antisymmetry of derivatives we instead have

```
#do i=1,‘MAXINDEX’
  id,once tr(?a,u(i1?,i2?,?b,i3?),?c) = aa*tr(?a,u‘i’(i1,i2,?b,i3),?c);
#enddo
if (match(aa)==0) discard;

#do i=1,‘MAXINDEX’
  id,only aa^‘i’ = aa1+...+aa‘i’;
#enddo
.sort
$relcount = $relcount+1;
multiply rel($relcount);
.sort
#do i=1,‘MAXINDEX’
  id aa‘i’*tr(?a,u’i’(i1?,i2?,?b),?c) =
         tr(?a,u‘i’(i1,i2,?b),?c)-tr(?a,u‘i’(i2,i1,?b),?c)
         -1/4*tr(?a,u(i1),u(i2),u‘i’(?b),?c)+1/4*tr(?a,u(i2),u(i1),u‘i’(?b),?c)
         +1/4*tr(?a,u‘i’(?b),u(i1),u(i2),?c)-1/4*tr(?a,u‘i’(?b),u(i2),u(i1),?c);
#enddo
repeat;
  id tr(?a,u1?ui(?b),?c) = tr(?a,u(?b),?c);
endrepeat;
```

As can be seen, this builds on the same idea but uses the equation for antisymmetry in derivatives instead. This again yields linear equations for the matrix R, but with more complex structure.

Below, the Bianchi identity is implemented:

```
L LP4Bianchi =
  tr(aa,u(j1,j2),u(j3))+tr(aa,u(j1),u(j2,j3))
 +tr(aa,u(j2,j1),u(j3))+tr(aa,u(j1),u(j3,j2));

$relcount = $relcount+1;
multiply rel($relcount);

id tr(aa,?a) =tr(u(j1,j2),u(j3),?a)-tr(u(j2),u(j1,j3),?a)+tr(u(j2,j3),u(j1),?a)
             -tr(u(j3),u(j2,j1),?a)+tr(u(j3,j1),u(j2),?a)-tr(u(j1),u(j3,j2),?a);
```

We first list all the possible operator terms that can contain structure $\nabla_\mu \Gamma_{\nu\rho}$, and then apply the Bianchi identity in eq. (3.26).

Next consider the Cayley-Hamilton relations.

```
* SU(2) or two flavour
#if ‘CH’=="NF2"
```

```
id tr(u(?a),u(?b),u(?c),?d) = tr(aa,u(?a),u(?b),u(?c),?d);

#do i=1,`MAXINDEX'
  id,once tr(aa,?a,u(?b),?c) = aa*tr(aa,?a,u`i'(?b),?c);
#enddo
if (match(aa)==0) discard;

#do i=1,`MAXINDEX'
  id,only aa^`i' = aa1+...+aa`i';
#enddo
id tr(aa,?a) = tr(?a);

$relcount = $relcount+1;
multiply rel($relcount);

#do i=1,`MAXINDEX'
  id aa`i'*tr(?a,u'i'(?b),?c) = aa`i'*tr(u`i'(?b),?c,?a);
  id aa`i'*tr(u'i'(?a),uu1?(?b),?c) =
      tr(u`i'(?a),uu1(?b),?c)+tr(uu1(?b),u`i'(?a),?c)
     -tr(u`i'(?a),?c)*tr(uu1(?b))-tr(uu1(?b),?c)*tr(u`i'(?a))
     -tr(?c)*tr(u`i'(?a),uu1(?b))+tr(?c)*tr(u`i'(?a))*tr(uu1(?b));
  id tr(?a,u`i'(?b),?c) = tr(?a,u(?b),?c);
#enddo
#endif
```

Above we have taken the two flavour case as an example. Similar to before, we first find the operator terms that contains at least three u_μ, this by using aa. After this, we expand them into equations using the Cayley-Hamilton theorem in eq. (3.30).

For the Schouten identity we have

```
L XS = distrib_(1,5,f1,f2,i1,...,i8);
L LP8Schouten = XS*(
         tr(u(i1),u(i2),u(i3),u(i4),u(i5),u(i6),u(i7),u(i8))
        +tr(u(i1),u(i2),u(i3),u(i4),u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1),u(i2),u(i3),u(i4),u(i5))*tr(u(i6),u(i7),u(i8))
        +tr(u(i1),u(i2),u(i3),u(i4))*tr(u(i5),u(i6),u(i7),u(i8))
        +tr(u(i1),u(i2),u(i3),u(i4))*tr(u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1),u(i2),u(i3))*tr(u(i4),u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1),u(i2))*tr(u(i3),u(i4))*tr(u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2,i3),u(i4),u(i5),u(i6),u(i7),u(i8))
        +tr(u(i1,i2),u(i3,i4),u(i5),u(i6),u(i7),u(i8))
        +tr(u(i1,i2),u(i3),u(i4,i5),u(i6),u(i7),u(i8))
        +tr(u(i1,i2),u(i3),u(i4),u(i5,i6),u(i7),u(i8))
        +tr(u(i1,i2,i3),u(i4),u(i5),u(i6))*tr(u(i7),u(i8))
```

```
        +tr(u(i1,i2),u(i3,i4),u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2),u(i3),u(i4,i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2),u(i3),u(i4),u(i5))*tr(u(i6,i7),u(i8))
        +tr(u(i1),u(i2),u(i3),u(i4))*tr(u(i5,i6,i7),u(i8))
        +tr(u(i1),u(i2),u(i3),u(i4))*tr(u(i5,i6),u(i7,i8))
        +tr(u(i1,i2,i3),u(i4),u(i5))*tr(u(i6),u(i7),u(i8))
        +tr(u(i1,i2),u(i3,i4),u(i5))*tr(u(i6),u(i7),u(i8))
        +tr(u(i1,i2),u(i3),u(i4))*tr(u(i5,i6),u(i7),u(i8))
        +tr(u(i1,i2,i3),u(i4))*tr(u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2),u(i3,i4))*tr(u(i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2),u(i3))*tr(u(i4,i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2,i3,i4,i5),u(i6),u(i7),u(i8))
        +tr(u(i1,i2,i3,i4),u(i5,i6),u(i7),u(i8))
        +tr(u(i1,i2,i3,i4),u(i5),u(i6,i7),u(i8))
        +tr(u(i1,i2,i3,i4),u(i5),u(i6),u(i7,i8))
        +tr(u(i1,i2,i3),u(i4,i5,i6),u(i7),u(i8))
        +tr(u(i1,i2,i3),u(i4),u(i5,i6,i7),u(i8))
        +tr(u(i1,i2,i3),u(i4,i5),u(i6,i7),u(i8))
        +tr(u(i1,i2,i3),u(i4,i5),u(i6),u(i7,i8))
        +tr(u(i1,i2,i3),u(i4),u(i5,i6),u(i7,i8))
        +tr(u(i1,i2),u(i3,i4),u(i5,i6),u(i7,i8))
        +tr(u(i1,i2,i3,i4,i5),u(i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2,i3,i4),u(i5,i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2,i3,i4),u(i5))*tr(u(i6,i7),u(i8))
        +tr(u(i1,i2,i3),u(i4,i5,i6))*tr(u(i7),u(i8))
        +tr(u(i1,i2,i3),u(i4,i5))*tr(u(i6,i7),u(i8))
        +tr(u(i1,i2,i3),u(i4))*tr(u(i5,i6,i7),u(i8))
        +tr(u(i1,i2,i3),u(i4))*tr(u(i5,i6),u(i7,i8))
        +tr(u(i1,i2),u(i3,i4))*tr(u(i5,i6),u(i7,i8))
        +tr(u(i1,i2,i3,i4,i5,i6,i7),u(i8))
        +tr(u(i1,i2,i3,i4,i5,i6),u(i7,i8))
        +tr(u(i1,i2,i3,i4,i5),u(i6,i7,i8))
        +tr(u(i1,i2,i3,i4),u(i5,i6,i7,i8))
       );

$relcount = $relcount+1;
multiply rel($relcount);

id f1(i1?,i2?,i3?,i4?,i5?)*f2(i6?,i7?,i8?)
  = e_(i1,i2,i3,i4)*e_(i5,i6,i7,i8)
   -e_(i8,i2,i3,i4)*e_(i1,i6,i7,i8)
   -e_(i1,i8,i3,i4)*e_(i2,i6,i7,i8)
   -e_(i1,i2,i8,i4)*e_(i3,i6,i7,i8)
```

```
   -e_(i1,i2,i3,i8)*e_(i4,i6,i7,i8);
contract;
id d_(i1?,i2?) = replace_(i2,i1);
```

This is used only for the p^8 Lagrangian as there are eight indices. First we do the permutations of the in total 45 possible operator structures. Then, all indices are contracted and the extra relations can be found for R.

References

[1] S. Weinberg, Physica **A96** (1979) 327-340.

[2] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

[3] J. Gasser and H. Leutwyler, Nucl. Phys. **B250** (1985) 456-516.

[4] J. Bijnens, G. Colangelo and G. Ecker, JHEP **02** (1999) 020.

[5] H.W. Fearing and S. Scherer, Phys. Rev. D **53** (1996) 315-348.

[6] S. Scherer and H.W. Fearing, Phys. Rev. D **52** (1995) 6445-6450.

[7] B. de Wit, "Introduction to gauge theories and the Standard Model," (1995) `https://cds.cern.ch/record/292286`

[8] J.A. Schouten, Proc. Kon. Ned. Akad. v. Wet. **41** (1938) 709-716.

[9] J.A.M. Vermaseren, "New features of FORM", `arXiv/math-ph/0010025`

[10] "GNU multiple precision arithmetic library `GMP` 6.1.2," `https://gmplib.org/`