OLD AND NEW RESULTS FOR HADRONIC-LIGHT-BY-LIGHT

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Overview

1. Overview
2. Main contributions
   - QED
   - HO hadronic
3. HLbL
   - General properties
   - $\pi^0$-exchange
   - $\pi$-loop: new stuff is here
   - Quark-loop
   - Scalar
   - $a_1$-exchange
   - Summary
4. Future
   - Theory
   - Experiment
5. Conclusions
Final experimental paper:

Review 1:

Review 2:

Review 3:
Literature

- **Lectures:**

- **“Final” HLbL number:**

- **New stuff here:**
  JB, Mehran Zahiri Abyaneh, Johan Relefors
  HLbL pion loop contribution
Muon $g - 2$: measurement

LIFE OF A MUON: THE g-2 EXPERIMENT

Protons from AGS. → Hit Target. → Pions, weighing 1/6 proton, are created. → Pions decay to muons. → Muons are fed into a uniform, doughnut-shaped magnetic field and travel in a circle. After each circle, muon's spin axis changes by 12°, yet it keeps on traveling in the same direction. After circling the ring many times, muons spontaneously decay to electron, (plus neutrinos,) in the direction of the muon spin.

One of 24 detectors see an electron, giving the muon spin direction; $g - 2$ is this angle, divided by the magnetic field the muon is traveling through in the ring.
Muon $g - 2$: measurement
Muon $g - 2$: measurement

Muons $g - 2$: measurement


2001: $\mu^-$, others $\mu^+$
Muon $g - 2$: overview

- in terms of the anomaly $a_\mu = (g - 2)/2$
- Data dominated by BNL E821 (statistics)(systematic)
  \[ a^{\text{exp}}_{\mu^+} = 11659204(6)(5) \times 10^{-10} \]
  \[ a^{\text{exp}}_{\mu^-} = 11659215(8)(3) \times 10^{-10} \]
  \[ a^{\text{exp}}_{\mu} = 11659208.9(5.4)(3.3) \times 10^{-10} \]
- Theory is off somewhat (electroweak)(LO had)(HO had)
  \[ a^{\text{SM}}_{\mu} = 11659180.2(0.2)(4.2)(2.6) \times 10^{-10} \]
- $\Delta a_\mu = a^{\text{exp}}_\mu - a^{\text{SM}}_\mu = 28.7(6.3)(4.9) \times 10^{-10}$ (PDG)
- E821 goes to Fermilab, expect factor of four in precision
- Note: $g$ agrees to $3 \times 10^{-9}$ with theory
- Many BSM models CAN predict a value in this range (often a lot more or a lot less)
- Numbers taken from PDG2012, see references there
Summary of Muon $g - 2$ contributions

<table>
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<td>4.2</td>
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<td>HLbL</td>
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<td>2.6</td>
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<td>difference</td>
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- Error on LO had all $e^+ e^-$ based OK
- $\tau$ based 2 $\sigma$
- Error on HLbL
- Errors added quadratically
- 3.5 $\sigma$
- Difference: 4% of LO Had
- 270% of HLbL
- 1% of leptonic LbL

Generic SUSY: $12.3 \times 10^{-10} \left( \frac{100 \text{ GeV}}{M_{\text{SUSY}}} \right)^2 \tan \beta$

$M_{\text{SUSY}} \approx 66 \text{ GeV} \sqrt{\tan \beta}$
Muon $g - 2$: QED

$$a_{\mu}^{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857410(27) \left( \frac{\alpha}{\pi} \right)^2 + 24.05050964(43) \left( \frac{\alpha}{\pi} \right)^3$$

$$+ 130.8055(80) \left( \frac{\alpha}{\pi} \right)^4 + 663(20) \left( \frac{\alpha}{\pi} \right)^5 + \ldots$$

- First three loops known analytically
- Four-loops fully done numerically
- Five loops estimate
- Kinoshita, Laporta, Remiddi, Schwinger, ...
- $\alpha$ fixed from the electron $g - 2$: $\alpha = 1/137.035999084(51)$
- $a_{\mu}^{\text{QED}} = 11658471.809(0.015) \times 10^{-10}$
- Light-by-light surprisingly large: $2670 \times 10^{-10}$

$e = 20.95$, $\mu = 0.37$, $\tau = 0.002$
Muon $g - 2$: HO hadronic

- Two main types of contributions

- HO HVP is like LO Had, can be derived from $e^+ e^- \rightarrow \text{hadrons}$. $a_{\mu}^{\text{HO HVP}} = -9.84(0.06) \times 10^{-10}$

- HLbL is the real problem: best estimate now: $a_{\mu}^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$

- Note that the sum is very small: but not an indication of the error
HLbL: the main object to calculate

- Muon line and photons: well known
- The blob: **fill in with hadrons/QCD**
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
  - $p^4$, order 1: pion-loop
  - $p^8$, order $N_c$: quark-loop and heavier meson exchanges
  - $p^6$, order $N_c$: pion exchange

Does not fully solve the problem
only short-distance part of quark-loop is really $p^8$
but it’s a start
A separation proposal: a start


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  - $p^4$, order 1: pion-loop
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  - $p^6$, order $N_c$: pion exchange

Implemented by two groups in the 1990s:

- Hayakawa, Kinoshita, Sanda: meson models, pion loop using hidden local symmetry, quark-loop with VMD, calculation in Minkowski space
- JB, Pallante, Prades: Try using as much as possible a consistent model-approach, ENJL, calculation in Euclidean space
General properties

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) = \]

Actually we really need

\[ \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_3^{\lambda}} \bigg|_{p_3=0} \]
**General properties**

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3): \]

- In general 138 Lorentz structures (but only 28 contribute to \( g - 2 \))
- Using \( q_\rho \Pi^{\rho\nu\alpha\beta} = p_{1\nu} \Pi^{\rho\nu\alpha\beta} = p_{2\alpha} \Pi^{\rho\nu\alpha\beta} = p_{3\beta} \Pi^{\rho\nu\alpha\beta} = 0 \)
  - 43 gauge invariant structures
- Bose symmetry relates some of them
- All depend on \( p_1^2, p_2^2 \) and \( q^2 \), but before derivative and \( p_3 \to 0 \) also \( p_3^2, p_1 \cdot p_2, p_1 \cdot p_3 \)
- Compare HVP: one function, one variable
- General calculation from experiment difficult to see how
- In four photon measurement: lepton contribution
General properties

\[ \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \] plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
- 5 remain: \( p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu \)
- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.
- More recent: can do two more using Gegenbauer techniques
  - Knecht-Nyffeler,
  - Jegerlehner-Nyffeler, JB–Zahiri-Abyaneh–Relefors
- \( P_1^2, P_2^2 \) and \( Q^2 \) remain
- study \( a^{X}_{\mu} = \int dl_{P_1} dl_{P_2} a^{X}_{LL} = \int dl_{P_1} dl_{P_2} dl_{Q} a^{X}_{LLQ} \)
- \( l_P = \ln \left( \frac{P}{\text{GeV}} \right) \), to see where the contributions are

Study the dependence on the cut-off for the photons
General properties

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\( P_1^2, P_2^2 \) and \( Q^2 \) remain

\[ a^X_{\mu} = \int dl_1 P_1 dl_2 P_2 a^{X\text{L}L}\mu = \int dl_1 P_1 dl_2 P_2 dl_3 Q a^{X\text{L}L\text{Q}\mu} \]

\[ l_P = \ln \left( \frac{P}{\text{GeV}} \right) \], to see where the contributions are

Study the dependence on the cut-off for the photons
\( \pi^0 \) exchange

- \( \pi^0 = 1/(p^2 - m^2_{\pi}) \)
- The blobs need to be modelled, and in e.g. ENJL contain corrections also to the \( 1/(p^2 - m^2_{\pi}) \)
- Pointlike has a logarithmic divergence
- Numbers \( \pi^0 \), but also \( \eta, \eta' \)
\( \pi^0 \) exchange

- **BPP:** 
  \[ a^\pi^0_\mu = 5.9(0.9) \times 10^{-10} \]

- **Nonlocal quark model:** 
  \[ a^\pi^0_\mu = 6.27 \times 10^{-10} \]

- **DSE model:** 
  \[ a^\pi^0_\mu = 5.75 \times 10^{-10} \]
  Goecke, Fischer and Williams, Phys. Rev. D83 (2011)094006 [1012.3886]

- **LMD+V:** 
  \[ a^\pi^0_\mu = (5.8 - 6.3) \times 10^{-10} \]

- **Formfactor inspired by AdS/QCD:** 
  \[ a^\pi^0_\mu = 6.54 \times 10^{-10} \]
  Cappiello, Cata and D’Ambrosio, Phys. Rev. D83 (2011)093006 [1009.1161]

- **Chiral Quark Model:** 
  \[ a^\pi^0_\mu = 6.8 \times 10^{-10} \]

- **Constraint via magnetic susceptibility:** 
  \[ a^\pi^0_\mu = 7.2 \times 10^{-10} \]

- All in reasonable agreement
MV short-distance: $\pi^0$ exchange


- take $P_1^2 \approx P_2^2 \gg Q^2$: Leading term in OPE of two vector currents is proportional to axial current
  \[ \Pi_{\rho\nu\alpha\beta} \propto \frac{P_\rho}{P_1^2} \langle 0 | T (J_\rho J_\nu J_\alpha J_\nu) | 0 \rangle \]

- $J_A$ comes from

- AVV triangle anomaly: extra info

- Implemented via setting one blob $= 1$

- $a_{\mu}^{\pi^0} = 7.7 \times 10^{-10}$
The pointlike vertex implements short-distance part, not only $\pi^0$-exchange.

Are these part of the quark-loop? See also in Dorokhov, Broniowski, Phys. Rev. D78(2008)07301

BPP quarkloop $+$ $\pi^0$-exchange $\approx$ MV $\pi^0$-exchange
\( \pi^0 \) exchange


\[ a_\mu = \int dl_1 dl_2 a^{LL}_\mu \text{ with } l_i = \log(P_i/\text{GeV}) \]

Which momentum regions do what: volume under the plot \( \propto a_\mu \)
Pseudoscalar exchange

- Point-like VMD: $\pi^0$, $\eta$ and $\eta'$ give 5.58, 1.38, 1.04.
- Models that include $U(1)_A$ breaking give similar ratios.
- Pure large $N_c$ models use this ratio.
- The MV argument should give some enhancement over the full VMD like models.
- Total pseudo-scalar exchange is about
  \[ a_{\mu}^{PS} = 8 - 10 \times 10^{-10} \]
- AdS/QCD estimate (includes excited pseudo-scalars)
  \[ a_{\mu}^{PS} = 10.7 \times 10^{-10} \]

A bare $\pi$-loop (sQED) give about $-4 \cdot 10^{-10}$

The $\pi\pi\gamma^*$ vertex is always done using VMD

$\pi\pi\gamma^*\gamma^*$ vertex two choices:
- Hidden local symmetry model: only one $\gamma$ has VMD
- Full VMD
- Both are chirally symmetric
- The HLS model used has problems with $\pi^+ - \pi^0$ mass difference (due to not having an $a_1$)

Final numbers quite different: $-0.45$ and $-1.9 \times 10^{-10}$

For BPP stopped at 1 GeV but within 10% of higher $\Lambda$
\[ \pi \text{ loop: Bare vs VMD} \]

- plotted \( a_{LLQ}^{LQ} \) for \( P_1 = P_2 \)
- \( a_\mu = \int dl_{P_1} dl_{P_2} dl_Q a_{LLQ}^{LQ} \)
- \( l_Q = \log(Q/1 \text{ GeV}) \)
π loop: VMD vs HLS

Usual HLS, $a = 2$
HLS with $a = 1$, satisfies more short-distance constraints
\( \pi \) loop

- \( \pi \pi \gamma^* \gamma^* \) for \( q_1^2 = q_2^2 \) has a short-distance constraint from the OPE as well.
- HLS does not satisfy it
- full VMD does: so probably better estimate
- Ramsey-Musolf suggested to do pure ChPT for the \( \pi \) loop
  
  
  So far ChPT at \( p^4 \) done for four-point function in limit \( p_1, p_2, q \ll m_\pi \) (Euler-Heisenberg plus next order)
- Polarizability \( (L_9 + L_{10}) \) up to 10%, charge radius 30%
- Both HLS and VMD have charge radius effect but not polarizability
π loop

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π loop: $L_9, L_{10}$

- ChPT for muon $g - 2$ at order $p^6$ is not powercounting finite so no prediction for $a_\mu$ exists.
- But can be used to study the low momentum end of the integral over $P_1, P_2, Q$
- The four-photon amplitude is finite still at two-loop order (counterterms start at order $p^8$)
- Add $L_9$ and $L_{10}$ vertices to the bare pion loop
- Program the Euler-Heisenberg plus NLO result of Ramsey-Musolf et al. into our programs for $a_\mu$
- Bare pion-loop and $L_9, L_{10}$ part in limit $p_1, p_2, q \ll m_\pi$ agree with Euler-Heisenberg plus next order analytically
\(\pi\) loop: \(L_9, L_{10}\)

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- Bare pion-loop and \(L_9, L_{10}\) part in limit \(p_1, p_2, q \ll m_\pi\) agree with Euler-Heisenberg plus next order analytically.
low scale, charge radius effect well reproduced
$\pi$ loop: VMD vs $L_9$ and $L_{10}$

- $L_9 + L_{10} \neq 0$ gives an enhancement of 10-15%
- To do it fully need to get a model: include $a_1$
Include $a_1$

- $L_9 + L_{10}$ effect is from

- But to get gauge invariance correctly need
Include $a_1$

- Consistency problem: full $a_1$-loop?
- Treat $a_1$ and $\rho$ classical and $\pi$ quantum: there must be a $\pi$ that closes the loop
  - Argument: integrate out $\rho$ and $a_1$ classically, then do pion loops with the resulting Lagrangian
- To avoid problems: representation without $a_1-\pi$ mixing
- Check for curiosity what happens if we add $a_1$-loop
Include $a_1$

- Use antisymmetric vector representation for $a_1$ and $\rho$
- Fields $A_{\mu\nu}$, $V_{\mu\nu}$ (nonets)
- Kinetic terms: $-\frac{1}{2} \left< \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} \right> + \frac{1}{2} \left< V_{\mu\nu} V^{\mu\nu} \right>$
  $-\frac{1}{2} \left< \nabla^\lambda A_{\lambda\mu} \nabla_\nu A^{\nu\mu} \right> - \frac{1}{2} \left< A_{\mu\nu} A^{\mu\nu} \right>$
- Terms that give contributions to the $L_i^r$:
  $\frac{F_V}{2\sqrt{2}} \left< f_{+\mu\nu} V^{\mu\nu} \right> + \frac{iG_V}{\sqrt{2}} \left< V^{\mu\nu} u_\mu u_\nu \right> + \frac{F_A}{2\sqrt{2}} \left< f_{-\mu\nu} A^{\mu\nu} \right>$
- $L_9 = \frac{F_V G_V}{2M_V^2}$, $L_{10} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}$
- Weinberg sum rules: (Chiral limit)
  $F_V^2 = F_A^2 + F_\pi^2$  \hspace{1cm} $F_V^2 M_V^2 = F_A^2 M_A^2$
- VMD for $\pi\pi\gamma$:
  $F_V G_V = F_\pi^2$
\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \] is not finite
(but was also not finite for HLS)

But
\[ \left. \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \right|_{p_3=0} \]
also not finite
(but was finite for HLS)

Derivative one finite for \( G_V = F_V / 2 \)

Surprise: \( g - 2 \) identical to HLS with \( a = \frac{F_V^2}{F_\pi^2} \)

Yes I know, different representations are identical BUT they do differ in higher order terms and even in what is higher order

Same comments as for HLS numerics
\( V_{\mu \nu} \) only

- \( \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) \) is not finite
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- But \( \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \) also not finite
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- Same comments as for HLS numerics
$V_{\mu\nu}$ and $A_{\mu\nu}$

- Add $a_1$
- Calculate a lot
  \[
  \left. \delta\Pi^{\rho\nu\alpha\beta}_{\mu\lambda}(p_1, p_2, p_3) \right|_{\delta p_3 \lambda} \quad \text{finite for:} \quad p_3 = 0
  \]
  \begin{itemize}
  \item $G_V = F_V = 0$ and $F_A^2 = -2F_\pi^2$
  \item If adding full $a_1$-loop $G_V = F_V = 0$ and $F_A^2 = -F_\pi^2$
  \end{itemize}
- Clearly unphysical (but will show some numerics anyway)
$V_{\mu\nu}$ and $A_{\mu\nu}$

- Add $a_1$
- Calculate a lot

\[
\left. \delta \Pi^{\rho \nu \alpha \beta} (p_1, p_2, p_3) \right|_{\delta p_3\lambda}^{p_3=0} \text{ finite for:}
\]

- $G_V = F_V = 0$ and $F_A^2 = -2F_{\pi}^2$
- If adding full $a_1$-loop $G_V = F_V = 0$ and $F_A^2 = -F_{\pi}^2$

- Clearly unphysical (but will show some numerics anyway)
\( V_{\mu\nu} \) and \( A_{\mu\nu} \)

- Start by adding \( \rho a_1 \pi \) vertices

\[
\lambda_1 \left< [V_{\mu\nu}, A_{\mu\nu}] \chi_- \right> + \lambda_2 \left< [V_{\mu\nu}, A_{\nu\alpha}] h_{\mu\nu} \right> \\
+ \lambda_3 \left< i [\nabla_{\mu} V_{\mu\nu}, A_{\nu\alpha}] u_\alpha \right> + \lambda_4 \left< i [\nabla_{\alpha} V_{\mu\nu}, A_{\alpha\nu}] u^\mu \right> \\
+ \lambda_5 \left< i [\nabla^\alpha V_{\mu\nu}, A_{\mu\nu}] u_\alpha \right> + \lambda_6 \left< i [V_{\mu\nu}, A_{\mu\nu}] f_{-\alpha} \right> \\
+ \lambda_7 \left< iV_{\mu\nu} A^{\mu\rho} A^\nu_{\rho} \right>
\]

- All lowest dimensional vertices of their respective type
- Not all independent, there are three relations
- Follow from the constraints on \( V_{\mu\nu} \) and \( A_{\mu\nu} \) (thanks to Stefan Leupold)
$V_{\mu\nu}$ and $A_{\mu\nu}$: big disappointment

- Work a whole lot
  \[
  \left. \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \right|_{p_3=0}
  \]
  not obviously finite

- Work a lot more

- Prove that
  \[
  \left. \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \right|_{p_3=0}
  \]
  finite, only same solutions as before

- Try the combination that show up in $g - 2$ only

- Work a lot

- Again, only same solutions as before

- Small loophole left: after the integration for $g - 2$ could be finite but many funny functions of $m_\pi$, $m_\mu$, $M_V$ and $M_A$ show up.
π loop: add $a_1$ and $F_A^2 = -2F_\pi^2$

- Lowers at low energies, $L_9 + L_{10} < 0$ here
- Funny peak at $a_1$ mass
\( \pi \) loop: add \( a_1 \) and \( F_A^2 = -F_{\pi}^2 \) plus \( a_1 \)-loop

- Lowers at low energies, \( L_9 + L_{10} < 0 \) here
- Funny peak at \( a_1 \) mass canceled
- Still unphysical case
$a_1$-loop: cases with good $L_9$ and $L_{10}$

- Add $F_V$, $G_V$ and $F_A$
- Fix values by Weinberg sum rules and VMD in $\gamma^* \pi \pi$
- no $a_1$-loop
Add $F_V$, $G_V$ and $F_A$

Fix values by Weinberg sum rules and VMD in $\gamma^* \pi \pi$

With $a_1$-loop (is different plot!!)
Add $a_1$ with $F_A^2 = +F_\pi^2$

Add the full VMD as done earlier for the bare pion loop
Add $a_1$ with $F_A^2 = +F_\pi^2$ and $a_1$-loop

Add the full VMD as done earlier for the bare pion loop
Integration results

\[
\begin{align*}
-a_\mu^\Lambda &\quad 5\times 10^{-11} &\quad 5\times 10^{-10} &\quad 2.5\times 10^{-10} &\quad 1.5\times 10^{-10} &\quad 1\times 10^{-10} &\quad 5\times 10^{-11} &\quad 0 \\
\Lambda &\quad 0.1 &\quad 1 &\quad 10
\end{align*}
\]

- sQED
- sQED \(\pi^0\)
- VMD
- HLS
- HLS \(a=1\)
- ENJL
Integration results

\[ a_1 F_A^2 = -2F^2 \]
\[ a_1 F_A^2 = -1 a_1\text{-loop} \]

HLS
HLS \( a=1 \)
VMD
\( a_1 \) VMD
\( a_1 \) Weinberg

\( a_\mu \)

\[ P_1, P_2, Q \leq \Lambda \]
Integration results with $a_1$

- Problem: get high energy behaviour good enough
- But all models with reasonable $L_9$ and $L_{10}$ fall way inside the error quoted earlier ($-1.9 \pm 1.3 \times 10^{-10}$)
- Tentative conclusion: Use hadrons only below about 1 GeV: $a_{\mu}^{\pi-\text{loop}} = (-2.0 \pm 0.5) \times 10^{-10}$
- Note that Engel and Ramsey-Musolf, arXiv:1309.2225 is a bit more pessimistic quoting numbers from ($-1.1$ to $-7.1$) $10^{-10}$
Pure quark loop

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^7$ Electron Loop</th>
<th>$a_\mu \times 10^9$ Muon Loop</th>
<th>$a_\mu \times 10^9$ Constituent Quark Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.41(8)</td>
<td>2.41(3)</td>
<td>0.395(4)</td>
</tr>
<tr>
<td>0.7</td>
<td>2.60(10)</td>
<td>3.09(7)</td>
<td>0.705(9)</td>
</tr>
<tr>
<td>1.0</td>
<td>2.59(7)</td>
<td>3.76(9)</td>
<td>1.10(2)</td>
</tr>
<tr>
<td>2.0</td>
<td>2.60(6)</td>
<td>4.54(9)</td>
<td>1.81(5)</td>
</tr>
<tr>
<td>4.0</td>
<td>2.75(9)</td>
<td>4.60(11)</td>
<td>2.27(7)</td>
</tr>
<tr>
<td>8.0</td>
<td>2.57(6)</td>
<td>4.84(13)</td>
<td>2.58(7)</td>
</tr>
<tr>
<td>Known Results</td>
<td>2.6252(4)</td>
<td>4.65</td>
<td>2.37(16)</td>
</tr>
</tbody>
</table>

- $M_Q : 300$ MeV
- now known fully analytically
- Us: 5+(3-1) integrals extra are Feynman parameters
- Slow convergence:
  - electron: all at 500 MeV
  - Muon: only half at 500 MeV, at 1 GeV still 20% missing
  - 300 MeV quark: at 2 GeV still 25% missing
Pure quark loop: momentum area

quark loop $m_Q = 0.3$ GeV

Most from $P_1 \approx P_2 \approx Q$, sizable large momentum part
**ENJL quark-loop**

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ VMD</th>
<th>$a_\mu \times 10^{10}$ ENJL</th>
<th>$a_\mu \times 10^{10}$ masscut</th>
<th>$a_\mu \times 10^{10}$ ENJL + masscut</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>2.46</td>
<td>3.2</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>1.13</td>
<td>2.3</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>0.59</td>
<td>2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>0.13</td>
<td>1.9</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>0.03</td>
<td>2.0</td>
</tr>
<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>.005</td>
<td>2.0</td>
</tr>
</tbody>
</table>

- **Very stable**
- ENJL cuts off slower than pure VMD
- masscut: $M_Q = \Lambda$ to have short-distance and no problem with momentum regions
- Quite stable in region 1-4 GeV
ENJL: scalar

\[
\Pi^{\rho\nu\alpha\beta} = \prod_{ab}^{\mathbb{V}V\mathbb{S}} (p_1, r) g_s \left(1 + g_s \Pi^S(r)\right) \prod_{cd}^{\mathbb{S}V\mathbb{V}} (p_2, p_3) \nu^{abcd\rho\nu\alpha\beta}
\]

+ permutations

\[
g_s \left(1 + g_s \Pi^S\right) = \frac{g_A(r^2)(2M_Q)^2}{2f^2(r^2)} \frac{1}{M_S^2(r^2) - r^2}
\]

\[
\nu^{abcd\rho\nu\alpha\beta}. \text{ ENJL VMD legs}
\]

In ENJL only scalar+quark-loop properly chiral invariant
ENJL: scalar/QL

<table>
<thead>
<tr>
<th>Cut-off Λ GeV</th>
<th>$a_\mu \times 10^{10}$ Quark-loop VMD</th>
<th>$a_\mu \times 10^{10}$ Quark-loop ENJL</th>
<th>$a_\mu \times 10^{10}$ Scalar Exchange</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>$-0.22$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>$-0.46$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>$-0.60$</td>
</tr>
<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>$-0.68$</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>$-0.68$</td>
</tr>
<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>$-0.68$</td>
</tr>
</tbody>
</table>

- ENJL only scalar+quark-loop properly chiral invariant
- Note: ENJL+scalar (BPP) ≈ Quark-loop VMD (HKS)
- $M_S \approx 620$ MeV certainly an overestimate for real scalars
- If scalar is $\sigma$: related to pion loop part?
- Quark-loop: $a_\mu^{ql} \approx 1 \times 10^{-10}$ bare $a_\mu^{ql} = 2.37 \times 10^{-9}$
Quark loop DSE

- DSE model: \( a_{\mu}^{qI} = 13.6(5.9) \times 10^{-10} \) T. Goecke, C. S. Fischer and R. Williams, Phys. Rev. D 83 (2011) 094006 [arXiv:1012.3886 [hep-ph]]

- Not a full calculation (yet) but includes an estimate of some of the missing parts

- A lot larger than bare quark loop with constituent mass

- I am puzzled: this DSE model (Maris-Roberts) does reproduce a lot of low-energy phenomenology. I would have guessed that it would give numbers very similar to ENJL.

- Can one find something in between full DSE and ENJL that is easier to handle?

- Error found in calculation, still not finalized: preliminary \( a_{\mu}^{qI} = 10.7(0.2) \times 10^{-10} \) T. Goecke, C. S. Fischer and R. Williams, arXiv:1210.1759
Other quark loop

- de Rafael-Greynat 1210.3029 (7.6 – 8.9) \(10^{-10}\)
- Boughezal-Melnikov 1104.4510 (11.8 – 14.8) \(10^{-10}\)
- Masjuan-Vanderhaeghen 1212.0357 (7.6 – 12.5) \(10^{-10}\)

Various interpretations: the full calculation or not

All (even DSE) have in common that a low quark mass is used for a large part of the integration range
Axial-vector exchange exchange

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ from Axial-Vector Exchange $\mathcal{O}(N_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.05(0.01)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.07(0.01)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.13(0.01)</td>
</tr>
<tr>
<td>2.0</td>
<td>0.24(0.02)</td>
</tr>
<tr>
<td>4.0</td>
<td>0.59(0.07)</td>
</tr>
</tbody>
</table>

There is some pseudo-scalar exchange piece here as well, off-shell not quite clear what is what.

- $a_{\mu}^{\text{axial}} = 0.6 \times 10^{-10}$
- MV: short distance enhancement + mixing (both enhance about the same)
  - $a_{\mu}^{\text{axial}} = 2.2 \times 10^{-10}$
## Summary: ENJL vs PdRV

<table>
<thead>
<tr>
<th></th>
<th>BPP</th>
<th>PdRV arXiv:0901.0306</th>
</tr>
</thead>
<tbody>
<tr>
<td>quark-loop</td>
<td>$(2.1 \pm 0.3) \cdot 10^{-10}$</td>
<td>(11.4 ± 1.3) · 10^{-10}</td>
</tr>
<tr>
<td>pseudo-scalar</td>
<td>$(8.5 \pm 1.3) \cdot 10^{-10}$</td>
<td>(1.5 ± 1.0) · 10^{-10}</td>
</tr>
<tr>
<td>axial-vector</td>
<td>$(0.25 \pm 0.1) \cdot 10^{-10}$</td>
<td>(−0.7 ± 0.7) · 10^{-10}</td>
</tr>
<tr>
<td>scalar</td>
<td>$(−0.68 \pm 0.2) \cdot 10^{-10}$</td>
<td>$(−1.9 \pm 1.9) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>$\pi K$-loop</td>
<td>$(−1.9 \pm 1.3) \cdot 10^{-10}$</td>
<td></td>
</tr>
<tr>
<td>errors</td>
<td>linearly</td>
<td>quadratically</td>
</tr>
<tr>
<td>sum</td>
<td>$(8.3 \pm 3.2) \cdot 10^{-10}$</td>
<td>$(10.5 \pm 2.6) \cdot 10^{-10}$</td>
</tr>
</tbody>
</table>
What can we do more?

- The ENJL model can certainly be improved:
  - Chiral nonlocal quark-model (like nonlocal ENJL): so far only $\pi^0$-exchange done
  - DSE: $\pi^0$-exchange similar to everyone else, quark-loop very different, looking forward to final results

- More resonances models should be tried, AdS/QCD is one approach, $R_\chi T$ (Valencia et al.) possible, . . .

- Note short-distance matching must be done in many channels, there are theorems JB, Gamiz, Lipartia, Prades that with only a few resonances this requires compromises

- $\pi$-loop: HLS smaller than double VMD (understood) models with $\rho$ and $a_1$: difficulties with infinities
What can we do more?

- **Constraints from experiment:**
  
  
  Studying three formfactors \( P\gamma^*\gamma^* \) in \( P \rightarrow \ell^+\ell^-\ell'^+\ell'^- \), \( e^+e^- \rightarrow e^+e^− \) \( P \) exact tree level and for \( g − 2 \) (but beware sign):
  
  - **Conclusion:** possible but VERY difficult
  - Two \( \gamma^* \) off-shell not so important for our choice of form-factor

- All information on hadrons and 1-2-3-4 off-shell photons is welcome: constrain the models

- More short-distance constraints: MV, Nyffeler integrate with all contributions, not just \( \pi^0 \)-exchange

- Need a new overall evaluation with consistent approach.

- Lattice has done first steps

- Some tentative steps from dispersion theory

  Pauk-Vanderhaeghen
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Pauk-Vanderhaeghen
BNL magnet has moved to Fermilab

Goal $\pm 1.6 \times 10^{-10}$

Credit: Brookhaven National Laboratory
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Ultracold muons at low energy (Credit: JPARC)

- 3 GeV proton beam (333 µA)
- Graphite target (20 mm)
- Surface muon beam (28 MeV/c, 4x10^8/s)
- Muonium Production (300 K ~ 25 meV)
- Super Precision Magnetic Field (3T, ~1ppm local precision)
- Silicon Tracker 66 cm diameter
- Resonant Laser Ionization of Muonium (~10^6 µ+/s)
- Muon LINAC (300 MeV/c)

New Muon g-2/EDM Experiment at J-PARC with Ultra-Cold Muon Beam
## Summary of Muon $g - 2$ contributions

<table>
<thead>
<tr>
<th></th>
<th>$10^{10} a_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exp</td>
<td>11 659 208.9</td>
</tr>
<tr>
<td>theory</td>
<td>11 659 180.2</td>
</tr>
<tr>
<td>QED</td>
<td>11 658 471.8</td>
</tr>
<tr>
<td>EW</td>
<td>15.4</td>
</tr>
<tr>
<td>LO Had</td>
<td>692.3</td>
</tr>
<tr>
<td>HO HVP</td>
<td>-9.8</td>
</tr>
<tr>
<td>HLbL</td>
<td>10.5</td>
</tr>
<tr>
<td>difference</td>
<td>28.7</td>
</tr>
</tbody>
</table>

- Error on LO had all $e^+ e^-$ based OK
- $\tau$ based 2 $\sigma$
- Error on HLbL
- Errors added quadratically
- 3.5 $\sigma$
- Difference:
  - 4% of LO Had
  - 270% of HLbL
  - 1% of leptonic LbL

**Generic SUSY:** $12.3 \times 10^{-10} \left( \frac{100 \text{ GeV}}{M_{\text{SUSY}}} \right)^2 \tan \beta$

$M_{\text{SUSY}} \approx 66 \text{ GeV} \sqrt{\tan \beta}$