CHIRAL PERTURBATION THEORY
IN NEW SURROUNDINGS

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Various ChPT: http://www.thep.lu.se/~bijnens/chpt.html
Overview

Three new applications: i.e. Lund the last three years

- Hard Pion Chiral Perturbation Theory

- Leading Logarithms to five loop order and large $N$ (for $O(N)$)

- Chiral Extrapolation Formulas for Technicolor and QCDlike theories
  i.e. equal mass ChPT for
  - $SU(n) \times SU(n)/SU(n)$
  - $SU(2n)/SO(2n)$
  - $SU(2n)/Sp(2n)$

Hard pion ChPT?

- In Meson ChPT: the powercounting is from all lines in Feynman diagrams having soft momenta
- thus powercounting = (naive) dimensional counting
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Baryon and Heavy Meson ChPT: $p, n, \ldots B, B^* \text{ or } D, D^*$

$p = M_B v + k$

Everything else soft

Works because baryon or $b$ or $c$ number conserved
so the non soft line is continuous
In Meson ChPT: the powercounting is from all lines in Feynman diagrams having soft momenta
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Baryon and Heavy Meson ChPT: $p, n, \ldots B, B^* \text{ or } D, D^*$

- $p = M_B v + k$
- Everything else soft
- Works because baryon or $b$ or $c$ number conserved
  so the non soft line is continuous
- Decay constant works: takes away all heavy momentum
- General idea: $M_p$ dependence can always be reabsorbed in LECs, is analytic in the other parts $k$. 
Hard pion ChPT?

- (Heavy) (Vector or other) Meson ChPT:
  - (Vector) Meson: \( p = M_V v + k \)
  - Everyone else soft or \( p = M_V v + k \)
(Heavy) (Vector or other) Meson ChPT:

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Everyone else soft or \( p = M_V u + k \)

But (Heavy) (Vector) Meson ChPT decays strongly

\[ \begin{array}{c}
\rho \\
\pi \\
\rho
\end{array} \]
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Everyone else soft or $p = M_V v + k$

But (Heavy) (Vector) Meson ChPT decays strongly
First: keep diagrams where vectors always present
Applied to masses and decay constants
Decay constant works: takes away all heavy momentum

*It was argued that this could be done, the nonanalytic parts of diagrams with pions at large momenta are reproduced correctly* JB-Gosdzinsky-Talavera

Done both in relativistic and heavy meson formalism

General idea: $M_V$ dependence can always be reabsorbed in LECs, is analytic in the other parts $k$. 
Hard pion ChPT?

- **Heavy Kaon ChPT:**
  - \( p = M_K v + k \)
  - First: only keep diagrams where Kaon goes through
  - Applied to masses and \( \pi K \) scattering and decay constant Roessl, Allton et al.,…
  - Applied to \( K\ell_3 \) at \( q_{max}^2 \) Flynn-Sachrajda
  - Works like all the previous heavy ChPT
Hard pion ChPT?

- **Heavy Kaon ChPT:**
  - \( p = M_K v + k \)
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- Flynn-Sachrajda argued \( K\ell_3 \) also for \( q^2 \) away from \( q_{max}^2 \).

- **JB-Celis** Argument generalizes to other processes with hard/fast pions and applied to \( K \rightarrow \pi\pi \)

- **JB Jemos** \( B, D \rightarrow D, \pi, K, \eta \) vector formfactors and a two-loop check

- General idea: heavy/fast dependence can always be reabsorbed in LECs, is analytic in the other parts \( k \).
Hard pion ChPT?

- Nonanalyticities in the light masses come from soft lines.
- Soft pion couplings are constrained by current algebra.

\[
\lim_{q \to 0} \langle \pi^k(q) | O | \beta \rangle = - \frac{i}{F_\pi} \langle \alpha | \left[ Q_5^k, O \right] | \beta \rangle ,
\]
nonanalyticities in the light masses come from soft lines

soft pion couplings are constrained by current algebra

\[ \lim_{q \to 0} \langle \pi^k(q) | O | \beta \rangle = -\frac{i}{F_\pi} \langle \alpha | \left[ Q_5^k, O \right] | \beta \rangle, \]

Nothing prevents hard pions to be in the states \( \alpha \) or \( \beta \)

So by heavily using current algebra I should be able to get the light quark mass nonanalytic dependence
Field Theory: a process at given external momenta

- Take a diagram with a particular internal momentum configuration
- Identify the soft lines and cut them
- The result part is analytic in the soft stuff
- So should be describably by an effective Lagrangian with coupling constants dependent on the external given momenta
- If symmetries present, Lagrangian should respect them
- Lagrangian should be complete in *neighbourhood*
- Loop diagrams with this effective Lagrangian *should* reproduce the nonanalyticities in the light masses

Crucial part of the argument
This procedure works at one loop level, matching at tree level, nonanalytic dependence at one loop:

- Toy models and vector meson ChPT \cite{JB, Gosdzinsky, Talavera}
- Recent work on relativistic meson ChPT \cite{Gegelia, Scherer et al.}
- Extra terms kept in $K \to 2\pi$ and semileptonic: a one-loop check
- Some two-loop checks
$K \to \pi\pi$: Tree level

\[ A_{LO}^0 = \frac{\sqrt{3}i}{2F^2} \left[ -\frac{1}{2}E_1 + (E_2 - 4E_3) \overline{M}_K^2 + 2E_8 \overline{M}_K^4 + A_1 E_1 \right] \]

\[ A_{LO}^2 = \sqrt{\frac{3}{2}} \frac{i}{F^2} \left[ (-2D_1 + D_2) \overline{M}_K^2 \right] \]
$K \rightarrow \pi\pi$: One loop
$K \to \pi\pi$: One-loop

\[ A_{0}^{NLO} = A_{0}^{LO} \left( 1 + \frac{3}{8F^2} \overline{A}(M^2) \right) + \lambda_0 M^2 + O(M^4), \]

\[ A_{2}^{NLO} = A_{2}^{LO} \left( 1 + \frac{15}{8F^2} \overline{A}(M^2) \right) + \lambda_2 M^2 + O(M^4). \]

\[ \overline{A}(M^2) = -\frac{M^2}{16\pi^2} \log \frac{M^2}{\mu^2} \]
Similar arguments to JB-Celis, Flynn-Sachrajda work for the pion vector and scalar formfactor JB-Jemos.

Therefore at any $t$ the chiral log correction must go like the one-loop calculation.

But note the one-loop log chiral log is with $t >> m_{\pi}^2$.

Predicts:

$$F_V(t, M^2) = F_V(t, 0) \left( 1 - \frac{M^2}{16\pi^2 F^2} \ln \frac{M^2}{\mu^2} + \mathcal{O}(M^2) \right)$$

$$F_S(t, M^2) = F_S(t, 0) \left( 1 - \frac{5}{2} \frac{M^2}{16\pi^2 F^2} \ln \frac{M^2}{\mu^2} + \mathcal{O}(M^2) \right)$$

Note that $F_{V,S}(t, 0)$ is now a coupling constant and can be complex.
Hard Pion ChPT: A two-loop check

- Similar arguments to JB-Celis, Flynn-Sachrajda work for the pion vector and scalar formfactor JB-Jemos.
- Therefore at any $t$ the chiral log correction must go like the one-loop calculation.
- But note the one-loop log chiral log is with $t >> m^2_\pi$.
- Predicts
  
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- Note that $F_{V,S}(t, 0)$ is now a coupling constant and can be complex.
- Take the full two-loop ChPT calculation JB,Colangelo,Talavera and expand in $t >> m^2_\pi$. 
A two-loop check

Full two-loop ChPT \( \text{JB,Colangelo,Talavera} \), expand in \( t \gg m^2_\pi \):

\[
F_V(t, M^2) = F_V(t, 0) \left( 1 - \frac{M^2}{16\pi^2 F^2} \ln \frac{M^2}{\mu^2} + \mathcal{O}(M^2) \right)
\]

\[
F_S(t, M^2) = F_S(t, 0) \left( 1 - \frac{5}{2} \frac{M^2}{16\pi^2 F^2} \ln \frac{M^2}{\mu^2} + \mathcal{O}(M^2) \right)
\]

with

\[
F_V(t, 0) = 1 + \frac{t}{16\pi^2 F^2} \left( \frac{5}{18} - 16\pi^2 \ln^r_6 - \frac{1}{6} \ln \frac{t}{\mu^2} \right)
\]

\[
F_S(t, 0) = 1 + \frac{t}{16\pi^2 F^2} \left( 1 + 16\pi^2 \ln^r_4 + i\pi - \ln \frac{t}{\mu^2} \right)
\]

- The needed coupling constants are complex
- Both calculations have two-loop diagrams with overlapping divergences
- The chiral logs should be valid for any \( t \) where a pointlike interaction is a valid approximation
Electromagnetic formfactors

\[
F_{V}^{\pi}(s) = F_{V}^{\pi\chi}(s) \left( 1 + \frac{1}{F^2} \overline{A}(m_{\pi}^2) + \frac{1}{2F^2} \overline{A}(m_{K}^2) + \mathcal{O}(m_{L}^2) \right),
\]

\[
F_{V}^{K}(s) = F_{V}^{K\chi}(s) \left( 1 + \frac{1}{2F^2} \overline{A}(m_{\pi}^2) + \frac{1}{F^2} \overline{A}(m_{K}^2) + \mathcal{O}(m_{L}^2) \right).
\]
\[
\langle P_f(p_f) \bar{q}_i \gamma_\mu q_f | P_i(p_i) \rangle = (p_i + p_f)_\mu f_+(q^2) + (p_i - p_f)_\mu f_-(q^2)
\]

\[
f_{+B\to M}(t) = f_{+B\to M}^\chi(t) F_{B\to M}
\]

\[
f_{-B\to M}(t) = f_{-B\to M}^\chi(t) F_{B\to M}
\]

- \( F_{B\to M} \) always same for \( f_+ \), \( f_- \) and \( f_0 \)

- This is not heavy quark symmetry: not valid at endpoint and valid also for \( K \to \pi \).

- Not like Low’s theorem, not only dependence on external legs

- Check: heavy meson and relativistic formalism

- Endpoint also done, \( \eta \) final state new
\( B, D \rightarrow \pi, K, \eta \)

\[
F_{K \rightarrow \pi} = 1 + \frac{3}{8F^2} \bar{A}(m_{\pi}^2)
\]

\[
F_{B \rightarrow \pi} = 1 + \left( \frac{3}{8} + \frac{9}{8}g^2 \right) \frac{\bar{A}(m_{\pi}^2)}{F^2} + \left( \frac{1}{4} + \frac{3}{4}g^2 \right) \frac{\bar{A}(m_{K}^2)}{F^2} + \left( \frac{1}{24} + \frac{1}{8}g^2 \right) \frac{\bar{A}(m_{\eta}^2)}{F^2},
\]

\[
F_{B \rightarrow K} = 1 + \frac{9}{8}g^2 \frac{\bar{A}(m_{\pi}^2)}{F^2} + \left( \frac{1}{2} + \frac{3}{4}g^2 \right) \frac{\bar{A}(m_{K}^2)}{F^2} + \left( \frac{1}{6} + \frac{1}{8}g^2 \right) \frac{\bar{A}(m_{\eta}^2)}{F^2},
\]

\[
F_{B \rightarrow \eta} = 1 + \left( \frac{3}{8} + \frac{9}{8}g^2 \right) \frac{\bar{A}(m_{\pi}^2)}{F^2} + \left( \frac{1}{4} + \frac{3}{4}g^2 \right) \frac{\bar{A}(m_{K}^2)}{F^2} + \left( \frac{1}{24} + \frac{1}{8}g^2 \right) \frac{\bar{A}(m_{\eta}^2)}{F^2},
\]

\[
F_{B_s \rightarrow K} = 1 + \frac{3}{8} \frac{\bar{A}(m_{\pi}^2)}{F^2} + \left( \frac{1}{4} + \frac{3}{2}g^2 \right) \frac{\bar{A}(m_{K}^2)}{F^2} + \left( \frac{1}{24} + \frac{1}{2}g^2 \right) \frac{\bar{A}(m_{\eta}^2)}{F^2},
\]

\[
F_{B_s \rightarrow \eta} = 1 + \left( \frac{1}{2} + \frac{3}{2}g^2 \right) \frac{\bar{A}(m_{K}^2)}{F^2} + \left( \frac{1}{6} + \frac{1}{2}g^2 \right) \frac{\bar{A}(m_{\eta}^2)}{F^2}.
\]

\( F_{B_s \rightarrow \pi} \) vanishes at this order due to the possible flavour quantum numbers.
Experimental check

CLEO data on $f_+(q^2)|V_{cq}|$ for $D \rightarrow \pi$ and $D \rightarrow K$ with $|V_{cd}| = 0.2253$, $|V_{cs}| = 0.9743$
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CLEO data on $f_+ (q^2) |V_{cq}|$ for $D \rightarrow \pi$ and $D \rightarrow K$ with $|V_{cd}| = 0.2253$, $|V_{cs}| = 0.9743$.

$$f_+ D \rightarrow \pi = f_+ D \rightarrow K \frac{F_D \rightarrow \pi}{F_D \rightarrow K}$$
Hard Pion ChPT: summary

Why is this useful:

- Lattice works actually around the strange quark mass
- need only extrapolate in $m_u$ and $m_d$ for $K \rightarrow \pi$ and $K \rightarrow \pi\pi$
- Applicable in momentum regimes where usual ChPT might not work
- Three flavour case useful for $B$, $D$ decays
Take a quantity with a single scale: \( F(M) \)

The dependence on the scale in field theory is typically logarithmic

\[ L = \log \left( \frac{\mu}{M} \right) \]

\[ F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + F_3^3 L^3 + \cdots \]

Leading Logarithms: The terms \( F_m^m L^m \)

The \( F_m^m \) can be more easily calculated than the full result

\[ \mu \left( \frac{dF}{d\mu} \right) \equiv 0 \]

Ultraviolet divergences in Quantum Field Theory are always local
Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion

$$F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$$

- $f_{i}^{j}$ are pure numbers

- $\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$
Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion

$$F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$$

- $f_i^j$ are pure numbers

$$\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$$

$$\mu \frac{dF}{d\mu} = 0 \implies \beta_0 = -f_1^1 = f_2^2 = -f_3^3 = \cdots$$
Renormalization Group

- Can be extended to other operators as well
- Underlying argument always \( \mu \frac{dF}{d\mu} = 0 \).
- Gell-Mann–Low, Callan–Symanzik, Weinberg–’t Hooft
- In great detail: J.C. Collins, Renormalization
- Relies on the \( \alpha \) the same in all orders
- LL one-loop \( \beta_0 \)
- NLL two-loop \( \beta_1, f_0^1 \)
Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu \frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–’t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the $\alpha$ the same in all orders
- LL one-loop $\beta_0$
- NLL two-loop $\beta_1, f_0^1$
- In effective field theories: different Lagrangian at each order
- The recursive argument does not work
Weinberg’s argument

- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop
- Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- Proof at all orders using $\beta$-functions
  Büchler, Colangelo, hep-ph/0309049
Weinberg’s argument

- $\mu$: dimensional regularization scale
- $d = 4 - w$
- loop-expansion $\equiv \hbar$-expansion
- $L_{\text{bare}} = \sum_{n \geq 0} \hbar^n \mu^{-nw} L^{(n)} = \sum_{n \geq 0} \hbar^n \mu^{-nw} \sum_i \left( \sum_{k=0,n} c_{ki}^{(n)} \right) \mathcal{O}_i^{(n)}$
- $L_{l}^{n}$ $l$-loop contribution at order $\hbar^n$
- Expand in divergences from the loops (not from the counterterms) $L_{l}^{n} = \sum_{k=0,l} \frac{1}{w^k} L_{kl}^{n}$
- $L^{(n)} = \boxed{n}$
Weinberg’s argument

\[
\text{Mass} = 0 + 0 + 1 + 0 + 0 + 1 + 0 + 0 + 1 + 0 + 0 + 1 + 2 + 2 + 1 + \ldots
\]
Weinberg’s argument

\[ h^0: L_0^0 \]

\[ h^1: \frac{1}{w} (\mu^- w L_{00}^1 (\{c\}^1_1) + L_{11}^1) + \mu^- w L_{00}^1 (\{c\}^1_0) + L_{10}^1 \]
Weinberg’s argument

\[ h^0: L_0^0 \]

\[ h^1: \frac{1}{w} (\mu^{-w} L_{00}^1 \{c\}_1^1 + L_{11}^1) + \mu^{-w} L_{00}^1 \{c\}_0^1 + L_{10}^1 \]

Expand \( \mu^{-w} = 1 - w \log \mu + \frac{1}{2} w^2 \log^2 \mu + \cdots \)

1/w must cancel: \( L_{00}^1 \{c\}_1^1 + L_{11}^1 = 0 \)
this determines the \( c_{1i}^1 \)

Explicit \( \log \mu: - \log \mu L_{00}^1 \{c\}_0^1 \equiv \log \mu L_{11}^1 \)
All orders

\[ \hat{h}_n: \]

\[ \frac{1}{w^n} \left( \mu^{-nw} L^{n}_{00}(\{c\}_{n}^{\text{n}}) + \mu^{-(n-1)w} L^{n}_{11}(\{c\}_{n-1}^{\text{n}-\text{1}}) + \cdots \right. \]

\[ + \mu^{-w} L^{n}_{n-1 \ n-1}(\{c\}_{1}^{\text{n}-\text{1}} + L^{n}_{nn}) + \frac{1}{w^{n-1}} \cdots \]
All orders

\( h^n : \)
\[
\frac{1}{w^n} \left( \mu^{-nw} L_{00}^n \{c\}_n^n + \mu^{-(n-1)w} L_{11}^n \{c\}_{n-1}^{n-1} + \cdots \right. \\
+ \mu^{-w} L_{n-1}^n n^{-1} \{c\}_{1}^{1} + L_{nn}^n \left) + \frac{1}{w^{n-1}} \cdots \right.
\]

\( 1/w^n, \log \mu/w^{n-1}, \ldots, \log^{n-1} \mu/w \) cancel:

\[
\sum_{i=0}^{n} i^j L_{n-i}^n n^{-i} \{c\}_{i}^i = 0 \quad j = 0, \ldots, n - 1.
\]
\[ \frac{1}{w^n} \left( \mu^{-n} L_{00}^n(\{c\}^n_n) + \mu^{-(n-1)} L_{11}^n(\{c\}^n_{n-1}) + \cdots \right) \\
+ \mu^{-w} L_{n-1}^n n-1(\{c\}^1_1) + L_{nn}^n + \frac{1}{w^{n-1}} \cdots \]

1/\( w^n \), \( \log \mu/w^{n-1} \), \ldots, \( \log^{n-1} \mu/w \) cancel:

\[ \sum_{i=0}^{n} i^j L_{n-i}^n n-i(\{c\}^i_i) = 0 \quad j = 0, \ldots, n-1. \]

Solution: \( L_{n-i}^n n-i(\{c\}^i_i) = (-1)^i \binom{n}{i} L_{nn}^n \)

explicit leading \( \log \mu \) dependence and divergence

\[ \log^n \mu \frac{(-1)^{n-1}}{n} L_{11}^n(\{c\}^n_{n-1}) \quad L_{00}^n(\{c\}^n_n) = -\frac{1}{n} L_{11}^n(\{c\}^n_{n-1}) \]
Mass to $\hbar^2$

\[ h^1: \quad 0 \quad \Rightarrow \quad 1 \]
Mass to $\hbar^2$

$h^1$: $0 \rightarrow 1$

$h^2$: $1 \rightarrow 0 \rightarrow 2$
Mass to $\hbar^2$

$h^1$: 0 $\Rightarrow$ 1

$h^2$: 1 $\Rightarrow$ 0 $\Rightarrow$ 2

but also needs $h^1$: 0 $\Rightarrow$ 0 $\Rightarrow$ 1
Mass to order $\hbar^3$
Mass+decay to $\bar{h}^5$

- $\bar{h}^1: 18 + 27$
- $\bar{h}^2: 26 + 45$
- $\bar{h}^3: 33 + 51$
- $\bar{h}^4: 26 + 33$
- $\bar{h}^5: 13 + 13$

Calculate the divergence

rewrite it in terms of a local Lagrangian

Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form

We keep all terms to have all 1PI (one particle irreducible) diagrams finite
Massive $O(N)$ sigma model

- $O(N + 1)/O(N)$ nonlinear sigma model

\[ \mathcal{L}_{n\sigma} = \frac{F^2}{2} \partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi . \]

- $\Phi$ is a real $N + 1$ vector; $\Phi \to O\Phi$; $\Phi^T \Phi = 1$.

- Vacuum expectation value $\langle \Phi^T \rangle = (1 \ 0 \ldots 0)$

- Explicit symmetry breaking: $\chi^T = (M^2 \ 0 \ldots 0)$

- Both spontaneous and explicit symmetry breaking

- $N$-vector $\phi$

- $N$ (pseudo-)Nambu-Goldstone Bosons

- $N = 3$ is two-flavour Chiral Perturbation Theory
Massive $O(N)$ sigma model: $\Phi$ vs $\phi$

$\Phi_1 = \begin{pmatrix} \sqrt{1 - \frac{\phi^T \phi}{F^2}} \\ \frac{\phi_1}{F} \\ \vdots \\ \frac{\phi_N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \frac{\phi^T \phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}$

Gasser, Leutwyler

$\Phi_2 = \frac{1}{\sqrt{1 + \frac{\phi^T \phi}{F^2}}} \begin{pmatrix} 1 \\ \frac{\phi}{F} \end{pmatrix}$

similar to Weinberg

$\Phi_3 = \begin{pmatrix} 1 - \frac{1}{2} \frac{\phi^T \phi}{F^2} \\ \sqrt{1 - \frac{1}{4} \frac{\phi^T \phi \phi}{F^2}} \end{pmatrix}$

only mass term

$\Phi_4 = \begin{pmatrix} \cos \sqrt{\frac{\phi^T \phi}{F^2}} \\ \sin \sqrt{\frac{\phi^T \phi}{F^2}} \frac{\phi}{\sqrt{\phi^T \phi}} \end{pmatrix}$

CCWZ
Massive $O(N)$ sigma model: Checks

Need (many) checks:
- use the four different parametrizations
- compare with known results:

\[ M_{phys}^2 = M^2 \left( 1 - \frac{1}{2}L_M + \frac{17}{8}L_M^2 + \cdots \right), \]

\[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}. \]

Usual choice $M = M$.

- large $N$ (but known results only for massless case)
  Coleman, Jackiw, Politzer 1974

- large $N$ massive later found partly in appendix of Kivel, Polyakov, Vladimirov on distribution functions.
\[ M_{\text{phys}}^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + ...) \]

\[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \]

<table>
<thead>
<tr>
<th>i</th>
<th>( a_i, N = 3 )</th>
<th>( a_i ) for general ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\frac{1}{2})</td>
<td>(1 - \frac{N}{2})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{17}{8})</td>
<td>(\frac{7}{4} - \frac{7N}{4} + \frac{5N^2}{8})</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{103}{24})</td>
<td>(\frac{37}{12} - \frac{113N}{24} + \frac{15N^2}{4} - N^3)</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{24367}{1152})</td>
<td>(\frac{839}{144} - \frac{1601N}{144} + \frac{695N^2}{48} - \frac{135N^3}{16} + \frac{231N^4}{128})</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{8821}{144})</td>
<td>(\frac{33661}{2400} - \frac{1151407N}{43200} + \frac{197587N^2}{4320} - \frac{12709N^3}{300} + \frac{6271N^4}{320} - \frac{7N^5}{2})</td>
</tr>
</tbody>
</table>

\( F_{\text{phys}}, \langle \bar{q}iq_i \rangle \) as well done

Anyone recognize any funny functions?
Large N

Power counting: pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N, M^2 \sim 1$

$\Rightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$

$\Leftrightarrow N$

1PI diagrams:

\[
\begin{align*}
N_L &= N_I - \sum_n N_{2n} + 1 \\
2N_I + N_E &= \sum_n 2nN_{2n}
\end{align*}
\]

\[\Rightarrow N_L = \sum_n (n - 1)N_{2n} - \frac{1}{2}N_E + 1\]

diagram suppression factor: $\frac{N^{N_L}}{N^{N_E}/2-1}$
Large $N$

- diagrams with shared lines are suppressed

- each new loop needs also a new flavour loop

- in the large $N$ limit only “cactus” diagrams survive:
large N: propagator

Generate recursively via a Gap equation

\[(\_\_\_\_\_\_\_\_\_\_)^{-1} = (\_\_\_\_\_\_\_\_\_\_)^{-1} + \_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_ + \ldots \]

⇒ resum the series and look for the pole

\[M^2 = M_{\text{phys}}^2 \sqrt{1 + \frac{N}{F^2} \overline{A}(M_{\text{phys}}^2)}\]

\[\overline{A}(m^2) = \frac{m^2}{16\pi^2} \log \frac{\mu^2}{m^2} . \]

Solve recursively, agrees with other result

Note: can be done for all parametrizations
large $N$

\[
F_{\text{phys}} = F \sqrt{1 + \frac{N}{F^2} \bar{A}(M_{\text{phys}}^2)}
\]

\[
\langle \bar{q}q \rangle_{\text{phys}} = \langle \bar{q}q \rangle_0 \sqrt{1 + \frac{N}{F^2} \bar{A}(M_{\text{phys}}^2)}
\]

Comments:

- These are the full* leading $N$ results, not just leading log
- But depends on the choice of $N$-dependence of higher order coefficients
- Assumes higher LECs zero (\(< N^{n+1} \) for $\hbar^n$)
- Large $N$ as in $O(N)$ not large $N_c$
Large N: Checking expansions

\[ M^2 = M_{\text{phys}}^2 \sqrt{1 + \frac{N}{F^2} A(M_{\text{phys}}^2)} \]

much smaller expansion coefficients than the table, try

\[ M^2 = M_{\text{phys}}^2 \left(1 + d_1 L M_{\text{phys}} + d_2 L^2 M_{\text{phys}} + d_3 L^3 M_{\text{phys}} + \ldots\right) \]
Numerical results

Left: \[ \frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L^2_M + a_3 L^3_M + \cdots \]

\[ F = 90 \text{ MeV}, \quad \mu = 0.77 \text{ GeV} \]
**Numerical results**

Left: \[
\frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L^2_M + a_3 L^3_M + \cdots
\]

Right: \[
\frac{M^2}{M^2_{\text{phys}}} = 1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + \cdots
\]

\[F = 90 \text{ MeV}, \quad \mu = 0.77 \text{ GeV}\]
Large $N$: $\pi\pi$-scattering

- Cactus diagrams for $A(s, t, u)$
- Branch with no momentum: resummed by
- Branch starting at vertex: resum by

\[
\begin{align*}
\text{\includegraphics[width=1cm]{cactus1}} & = \text{\includegraphics[width=1cm]{cactus2}} + \text{\includegraphics[width=1cm]{cactus3}} + \text{\includegraphics[width=1cm]{cactus4}} + \text{\includegraphics[width=1cm]{cactus5}} + \text{\includegraphics[width=1cm]{cactus6}} + \cdots \\
\end{align*}
\]

- The full result is then

\[
\begin{align*}
\text{\includegraphics[width=2cm]{full_result1}} & + \text{\includegraphics[width=2cm]{full_result2}} + \text{\includegraphics[width=2cm]{full_result3}} + \cdots \\
\end{align*}
\]

- Can be summarized by a recursive equation

\[
\begin{align*}
\text{\includegraphics[width=2cm]{recursive1}} & = \text{\includegraphics[width=1cm]{recursive2}} + \text{\includegraphics[width=1cm]{recursive3}} \\
\end{align*}
\]
Large $N$: $\pi\pi$ scattering

$$y = \frac{N}{F^2} \bar{A}(M^2_{\text{phys}})$$

$$A(s, t, u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{N}{2} \left( \frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}} \right) \bar{B}(M^2_{\text{phys}}, M^2_{\text{phys}}, s)}$$

or

$$A(s, t, u) = \frac{s - M^2_{\text{phys}}}{F^2_{\text{phys}}} \frac{1}{1 - \frac{N}{2} \frac{s - M^2_{\text{phys}}}{F^2_{\text{phys}}} \bar{B}(M^2_{\text{phys}}, M^2_{\text{phys}}, s)}$$

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results
Other results

- Bissegger, Fuhrer, hep-ph/0612096 Dispersive methods, massless $\Pi_S$ to five loops
- Kivel, Polyakov, Vladimirov, 0809.3236, 0904.3008, 1004.2197
  - In the massless case tadpoles vanish
  - hence the number of external legs needed does not grow
  - All 4-meson vertices via Legendre polynomials
  - can do divergence of all one-loop diagrams analytically
  - algebraic (but quadratic) recursion relations
  - massless $\pi\pi$, $F_V$ and $F_S$ to arbitrarily high order
- large $N$ agrees with Coleman, Wess, Zumino
- large $N$ is not a good approximation
JB, Carloni, arXiv:1008.3499

- **massive case**: $\pi\pi$, $F_V$ and $F_S$ to 4-loop order
- large $N$ for these cases also for massive $O(N)$.
- done using bubble resummations or recursion equation which can be solved analytically.
Conclusions Leading Logs

- Several quantities in massive $O(N)$ LL known to high loop order
- Large $N$ in massive $O(N)$ model solved
- Had hoped: recognize the series also for general $N$
- Limited essentially by CPU time and size of intermediate files
- Some first studies on convergence etc.
- $\pi\pi$, $F_V$ and $F_S$ to four-loop order
- The technique can be generalized to other models/theories
  - $SU(N) \times SU(N)/SU(N)$
  - One nucleon sector
QCDlike and/or technicolor theories

A typical gauge group and $N_F$ fermions:

- **QCD or complex**: $q^T = (q_1 \ q_2 \ldots \ q_{N_F})$

- **Global** $G = SU(N_F)_L \times SU(N_F)_R$
  
  $q_L \rightarrow g_L q_L \text{ and } g_R \rightarrow g_R q_R$

- **Vacuum condensate** $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$

- **Conserved** $H = SU(N_F) \ g_L = g_R \ \Sigma_{ij} \rightarrow \Sigma_{ij}$

- $q$ in complex prepresentation of gauge group
QCD-like and/or technicolor theories

A typical gauge group and $N_F$ fermions:

- Real (e.g. adjoint):
  - $\tilde{q}_{Ri} \equiv C\bar{q}^{T}_{Lj}$ is in the same gauge group representation as $q_{Ri}$
  - $\hat{q}^T = (q_{R1} \ldots q_{RN_F} \tilde{q}_{R1} \ldots \tilde{q}_{RN_F})$
  - Global $G = SU(2N_F)$ and $\hat{q} \rightarrow g\hat{q}$
  - Vacuum condensate $\langle \bar{q}_j q_i \rangle$ is really $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto J_{Sij}$
  - $J_S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$
  - Conserved symmetry part has $gJ_S g^T = J_S$
  - $H = SO(2N_F)$
  - some Goldstone bosons have baryon number
QCDlike and/or technicolor theories

A typical gauge group and $N_F$ fermions:

- Pseudoreal (e.g. two-colours):
  \[ \tilde{q}_{R\alpha i} \equiv \epsilon_{\alpha\beta} C \tilde{q}_{L\beta i} \] is in the same gauge group representation as $q_{R\alpha i}$
  \[ \hat{q}^T = (q_{R1} \ldots q_{RN_F} \tilde{q}_{R1} \ldots \tilde{q}_{RN_F}) \]
  Global $G = SU(2N_F)$ and $\hat{q} \rightarrow g\hat{q}$
  Vacuum condensate $\langle \bar{q}_j q_i \rangle$ is really
  \[ \epsilon_{\alpha\beta} \langle (\hat{q}_{\alpha j})^T C \hat{q}_{\beta i} \rangle \propto J_{Ai j} J_A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]

- Conserved symmetry part has $gJ_A g^T = J_A$
- $H = Sp(2N_F)$
- some Goldstone bosons have baryon number
In arXiv:0910.5424 we showed that there is a very similar way of phrasing the two theories using \[ u = \exp \left( \frac{i}{\sqrt{2F}} \phi^a X^a \right) \]

But the matrices \( X^a \) are:

- Complex or \( SU(N) \times SU(N)/SU(N) \): all \( SU(N) \) generators

- Real or \( SU(2N)/SO(2N) \): \( SU(2N) \) generator with \( X^a J_S = J_S X^a T \)

- Pseudoreal or \( SU(2N)/Sp(2N) \): \( SU(2N) \) generator with \( X^a J_A = J_A X^a T \)

Note that the latter are not the usual ways of parametrizing \( SO(2N) \) and \( Sp(2N) \) matrices.
Calculating for equal mass case goes through using:

\[ \begin{align*}
\langle X^a AX^a B \rangle &= \langle A \rangle \langle B \rangle - \frac{1}{N_F} \langle AB \rangle , \\
\langle X^a A \rangle \langle X^a B \rangle &= \langle AB \rangle - \frac{1}{N_F} \langle A \rangle \langle B \rangle .
\end{align*} \]

\[ \begin{align*}
\langle X^a AX^a B \rangle &= \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \langle AJ_S B^T J_S \rangle - \frac{1}{2N_F} \langle AB \rangle , \\
\langle X^a A \rangle \langle X^a B \rangle &= \frac{1}{2} \langle AB \rangle + \frac{1}{2} \langle AJ_S B^T J_S \rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle .
\end{align*} \]

\[ \begin{align*}
\langle X^a AX^a B \rangle &= \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \langle AJ_A B^T J_A \rangle - \frac{1}{2N_F} \langle AB \rangle , \\
\langle X^a A \rangle \langle X^a B \rangle &= \frac{1}{2} \langle AB \rangle - \frac{1}{2} \langle AJ_A B^T J_A \rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle .
\end{align*} \]

So can do the calculations for all cases
Vacuum expectation value

All cases: $\langle \overline{q}q \rangle_{\text{LO}} \equiv \sum_{i=1,N_F} \langle \overline{q}_R i q_L i + \overline{q}_L i q_R i \rangle_{\text{LO}} = -N_F B_0 F^2$

$M^2 = 2B_0 \hat{m} \text{ and } \overline{A}(M^2) = -\frac{M^2}{16\pi^2} \log \frac{M^2}{\mu^2}.$

$\langle \overline{q}q \rangle = \langle \overline{q}q \rangle_{\text{LO}} + \langle \overline{q}q \rangle_{\text{NLO}} + \langle \overline{q}q \rangle_{\text{NNLO}}.$

$$
\langle \overline{q}q \rangle_{\text{NLO}} = \langle \overline{q}q \rangle_{\text{LO}} \left( a_V \frac{\overline{A}(M^2)}{F^2} + b_V \frac{M^2}{F^2} \right),
$$

$$
\langle \overline{q}q \rangle_{\text{NNLO}} = \langle \overline{q}q \rangle_{\text{LO}} \left( c_V \frac{\overline{A}(M^2)^2}{F^4} + \frac{M^2\overline{A}(M^2)}{F^4} \left( d_V + \frac{e_V}{16\pi^2} \right) + \frac{M^4}{F^4} \left( f_V + \frac{g_V}{16\pi^2} \right) \right).
$$

Diagrams:
### Vacuum expectation value

<table>
<thead>
<tr>
<th></th>
<th>QCD</th>
<th>Adjoint</th>
<th>2-colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_V$</td>
<td>$n - \frac{1}{n}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_V$</td>
<td>$16nL_6^r + 8L_8^r + 4H_2^r$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_V$</td>
<td>$\frac{3}{2} \left( -1 + \frac{1}{n^2} \right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_V$</td>
<td>$-24 (n^2 - 1) \left( L_A + \frac{1}{n} L_B \right)$</td>
<td>$L_A = L_4^r - 2L_6^r$</td>
<td>$32nL_6^r + 8L_8^r + 4H_2^r$</td>
</tr>
<tr>
<td>$e_V$</td>
<td>$1 - \frac{1}{n^2}$</td>
<td></td>
<td>$\frac{3}{8} \left( -1 + \frac{1}{n^2} + \frac{2}{n} - 2n \right)$</td>
</tr>
<tr>
<td>$f_V$</td>
<td>$48 \left( K_{25}^r + nK_{26}^r + n^2K_{27}^r \right)$</td>
<td></td>
<td>$-12 \left( 2n^2 - n - 1 \right) \left( 2L_A + \frac{1}{n} L_B \right)$</td>
</tr>
<tr>
<td>$g_V$</td>
<td>$8 \left( n^2 - 1 \right) \left( L_A + \frac{1}{n} L_B \right)$</td>
<td></td>
<td>$\frac{1}{4} \left( 1 - \frac{1}{n^2} - \frac{2}{n} + 2n \right)$</td>
</tr>
</tbody>
</table>

**Adjoint**

**2-colour**
\[ \phi \phi \rightarrow \phi \phi \]

\[ \pi \pi \] scattering

Amplitude in terms of \( A(s, t, u) \)

\[ M_{\pi \pi}(s, t, u) = \delta_{ab} \delta_{cd} A(s, t, u) + \delta_{ac} \delta_{bd} A(t, u, s) + \delta_{ad} \delta_{bc} A(u, s, t) . \]

Three intermediate states \( I = 0, 1, 2 \)

Our three cases

Two amplitudes needed \( B(s, t, u) \) and \( C(s, t, u) \)

\[ M(s, t, u) = \left[ \langle X^a X^b X^c X^d \rangle + \langle X^a X^d X^c X^b \rangle \right] B(s, t, u) \]
\[ + \left[ \langle X^a X^c X^d X^b \rangle + \langle X^a X^b X^d X^c \rangle \right] B(t, u, s) \]
\[ + \left[ \langle X^a X^d X^b X^c \rangle + \langle X^a X^c X^b X^d \rangle \right] B(u, s, t) \]
\[ + \delta_{ab} \delta_{cd} C(s, t, u) + \delta_{ac} \delta_{bd} C(t, u, s) + \delta_{ad} \delta_{bc} C(u, s, t) . \]

\[ B(s, t, u) = B(u, t, s) \quad C(s, t, u) = C(s, u, t) . \]

7, 6 and 6 possible intermediate states
\[ \phi \phi \rightarrow \phi \phi \]

- calculate all the diagrams
- Do all integrals, renormalize,…
- Construct states for all the presentations and their projection operators
- Get the amplitudes for all intermediate states
- Get all scattering lengths
- All formulas similar length to \( \pi \pi \) cases but there are so many of them
- arXiv:1102.0172:
  - Very long appendix part
  - References for the Young diagrams, tensor algebra we did ourselves but probably exists (e.g. Cvitanovic group theory book)
Some curious large $N_F = n$ relations

Leading in $n$:

\[
\begin{align*}
    a_0^I |_{\text{complex}} &= a_0^I |_{\text{real}} = a_0^I |_{\text{pseudoreal}} = \text{LO} \frac{x_2}{\pi} \frac{n}{8}, \\
    a_0^S |_{\text{complex}} &= a_0^S |_{\text{real}} = a_0^A |_{\text{pseudoreal}} = \text{LO} \frac{x_2}{\pi} \frac{n}{16}, \\
    a_1^A |_{\text{complex}} &= a_1^A |_{\text{real}} = a_1^S |_{\text{pseudoreal}} = \text{LO} \frac{x_2}{\pi} \frac{n}{48}, \\
\end{align*}
\]

Subleading:

\[
\begin{align*}
    a_0^{SS} |_{\text{complex}} &= a_0^{FS} |_{\text{real}} = 2a_0^{MS} |_{\text{pseudoreal}} = \text{LO} \frac{x_2}{\pi} \frac{-1}{16}, \\
    a_0^{AA} |_{\text{complex}} &= 2a_0^{MS} |_{\text{real}} = a_0^{FA} |_{\text{pseudoreal}} = \text{LO} \frac{x_2}{\pi} \frac{1}{16}. \\
\end{align*}
\]

Subsubleading:

\[
\begin{align*}
    a_1^{SA} |_{\text{complex}} &= a_1^{AS} |_{\text{complex}} = 2a_1^{MA} |_{\text{real}} = 2a_1^{MA} |_{\text{pseudoreal}} = \text{LO} 0. \\
\end{align*}
\]

At NNLO here violated by an $L_4^r L_6^r$ term
$\phi\phi \rightarrow \phi\phi: a_0^I/n$
Other results: fully to NNLO

- $M^2_{\text{phys}}$
- $F_{\text{phys}}$
- Meson-meson scattering
- Equal mass case: allows to get fully analytical result just as for 2-flavour ChPT
- Note: large $N_F$ here not cactus but planar diagrams (in flavour lines)
QCDlike: conclusions

- Different symmetry patterns can appear for different gauge groups and fermion representations
- Nonperturbative: lattice needs extrapolation formulae
- Masses, decay constant and VEV: done to NNLO
- Meson-meson scattering: done to NNLO and some large $N_F$ relations at NNLO.
- Two-point functions and formfactors for precision observables: planned
Conclusions

Three new surroundings for ChPT:

- Hard Pion ChPT: a new application domain for EFT and first results
  - Many processes but limited domain
  - Power counting proof lacking so far, SCET?

- Leading Logarithms and large $N$: some progress in getting results at high loop orders, but hoped for patterns not seen (except large $N$ calculated)
  - Anybody recognize some funny functions?
  - Method applicable to many more cases

- Two-loop results for the equal mass case for different symmetry patterns. $SU(N) \times SU(N)/SU(N)$, $SU(2N)/SO(2N)$, $SU(2N)/Sp(2N)$