CHIRAL PERTURBATION THEORY IN NEW SURROUNDINGS

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Various ChPT: http://www.thep.lu.se/∼bijnens/chpt.html

Overview

Three new applications: i.e. Lund the last three years

- Hard Pion Chiral Perturbation Theory
  JB+ Alejandro Celis, arXiv:0906.0302 and JB + Ilaria Jemos, arXiv:1006.1197, arXiv:1011.6531
- Leading Logarithms to five loop order and large N (for $O(N)$)
  JB + Lisa Carloni, arXiv:0909.5086,arXiv:1008.3499
- Chiral Extrapolation Formulas for Technicolor and QCDlike theories
  i.e. equal mass ChPT for
  - $SU(n) \times SU(n)/SU(n)$
  - $SU(2n)/SO(2n)$
  - $SU(2n)/Sp(2n)$

  JB + Jie LU, arXiv:0910.5424 and arXiv:1102.0172

Hard pion ChPT?

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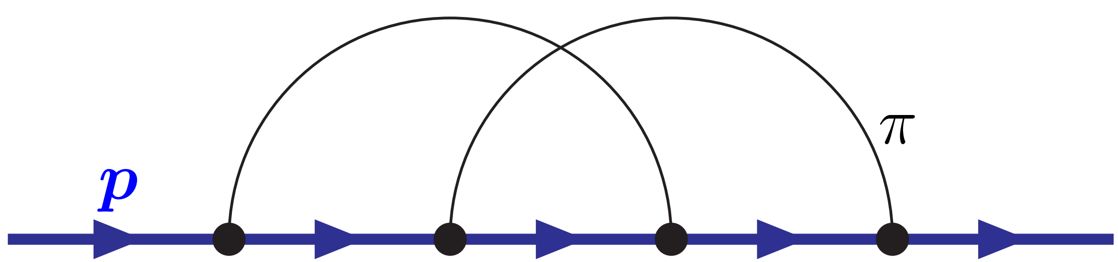
  - $p = M_B v + k$
  - Everything else soft
  - Works because baryon or b or c number conserved so the non soft line is continuous



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  - Decay constant works: takes away all heavy momentum
  - General idea: M_p dependence can always be reabsorbed in LECs, is analytic in the other parts k.

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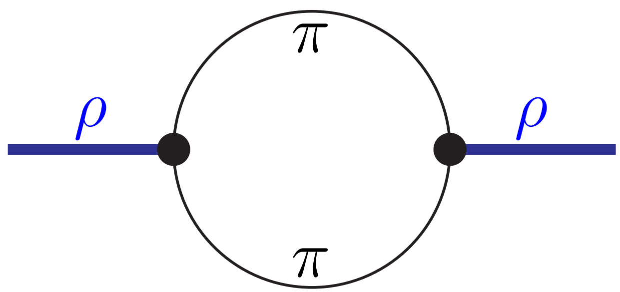

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  - (Vector) Meson: $p = M_V v + k$
  - Everyone else soft or $p = M_V v + k$
  - But (Heavy) (Vector) Meson ChPT decays strongly
    - First: keep diagrams where vectors always present
    - Applied to masses and decay constants
    - Decay constant works: takes away all heavy momentum
  - *It was argued that this could be done, the nonanalytic parts of diagrams with pions at large momenta are reproduced correctly* JB-Gosdzinsky-Talavera
  - Done both in relativistic and heavy meson formalism
  - General idea: M_V dependence can always be reabsorbed in LECs, is analytic in the other parts k.

Hard pion ChPT?

- Heavy Kaon ChPT:
  - $p = M_K v + k$
  - First: only keep diagrams where Kaon goes through
  - Applied to masses and πK scattering and decay constant Roessl,Allton et al.,...
  - Applied to $K_{\ell 3}$ at q^2_{max} Flynn-Sachrajda
  - Works like all the previous *heavy* ChPT

Hard pion ChPT?

- Heavy Kaon ChPT:
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  - Applied to masses and πK scattering and decay constant Roessl,Allton et al.,...
  - Applied to $K_{\ell 3}$ at q^2_{max} Flynn-Sachrajda
- Flynn-Sachrajda argued $K_{\ell 3}$ also for q^2 away from q^2_{max}.
- JB-Celis Argument generalizes to other processes with hard/fast pions and applied to $K \to \pi\pi$
- JB Jemos $B, D \to D, \pi, K, \eta$ vector formfactors and a two-loop check
- General idea: heavy/fast dependence can always be reabsorbed in LECs, is analytic in the other parts k.

Hard pion ChPT?

- nonanalyticities in the light masses come from soft lines
- soft pion couplings are constrained by current algebra

$$\lim_{q\to 0}\langle \pi^k(q)\alpha|O|\beta\rangle = -\frac{i}{F_\pi}\langle\alpha|\left[Q_5^k, O\right]|\beta\rangle\,,$$

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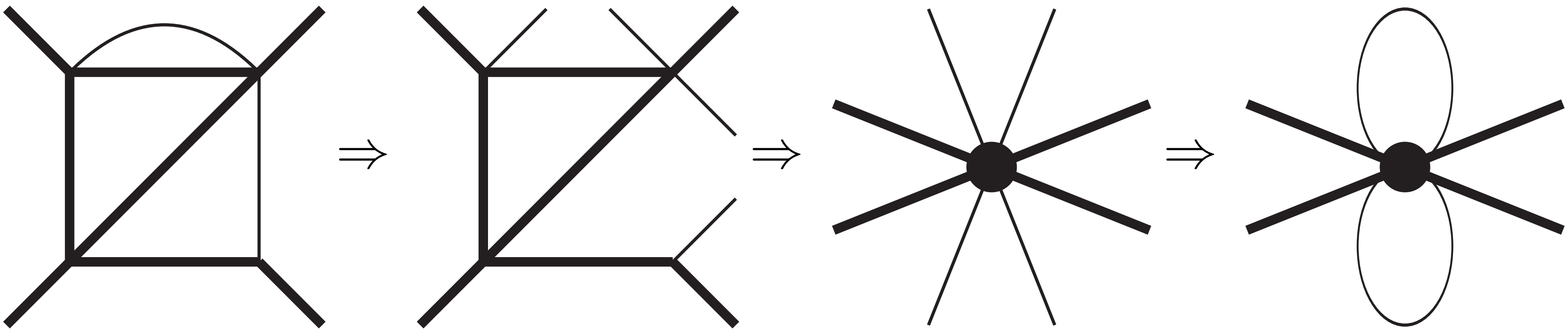
$$\lim_{q\to 0}\langle\pi^k(q)\alpha|O|\beta\rangle = -\frac{i}{F_\pi}\langle\alpha|\left[Q_5^k, O\right]|\beta\rangle\,,$$
- Nothing prevents hard pions to be in the states α or β
- So by heavily using current algebra I should be able to get the light quark mass nonanalytic dependence

Hard pion ChPT?

Field Theory: a process at given external momenta

- Take a diagram with a particular internal momentum configuration
- Identify the soft lines and cut them
- The result part is analytic in the soft stuff
- So should be describably by an effective Lagrangian with coupling constants dependent on the external given momenta
- If symmetries present, Lagrangian should respect them
- Lagrangian should be complete in *neighbourhood*
- Loop diagrams with this effective Lagrangian *should* reproduce the nonanalyticities in the light masses
  Crucial part of the argument

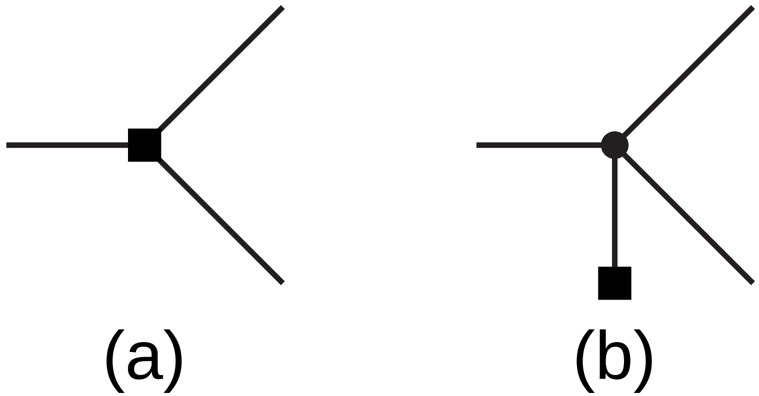
Hard pion ChPT?



This procedure works at one loop level, matching at tree level, nonanalytic dependence at one loop:

- Toy models and vector meson ChPT JB, Gosdzinsky, Talavera
- Recent work on relativistic meson ChPT Gegelia, Scherer et al.
- Extra terms kept in $K \to 2\pi$ and semileptonic: a one-loop check
- Some two-loop checks

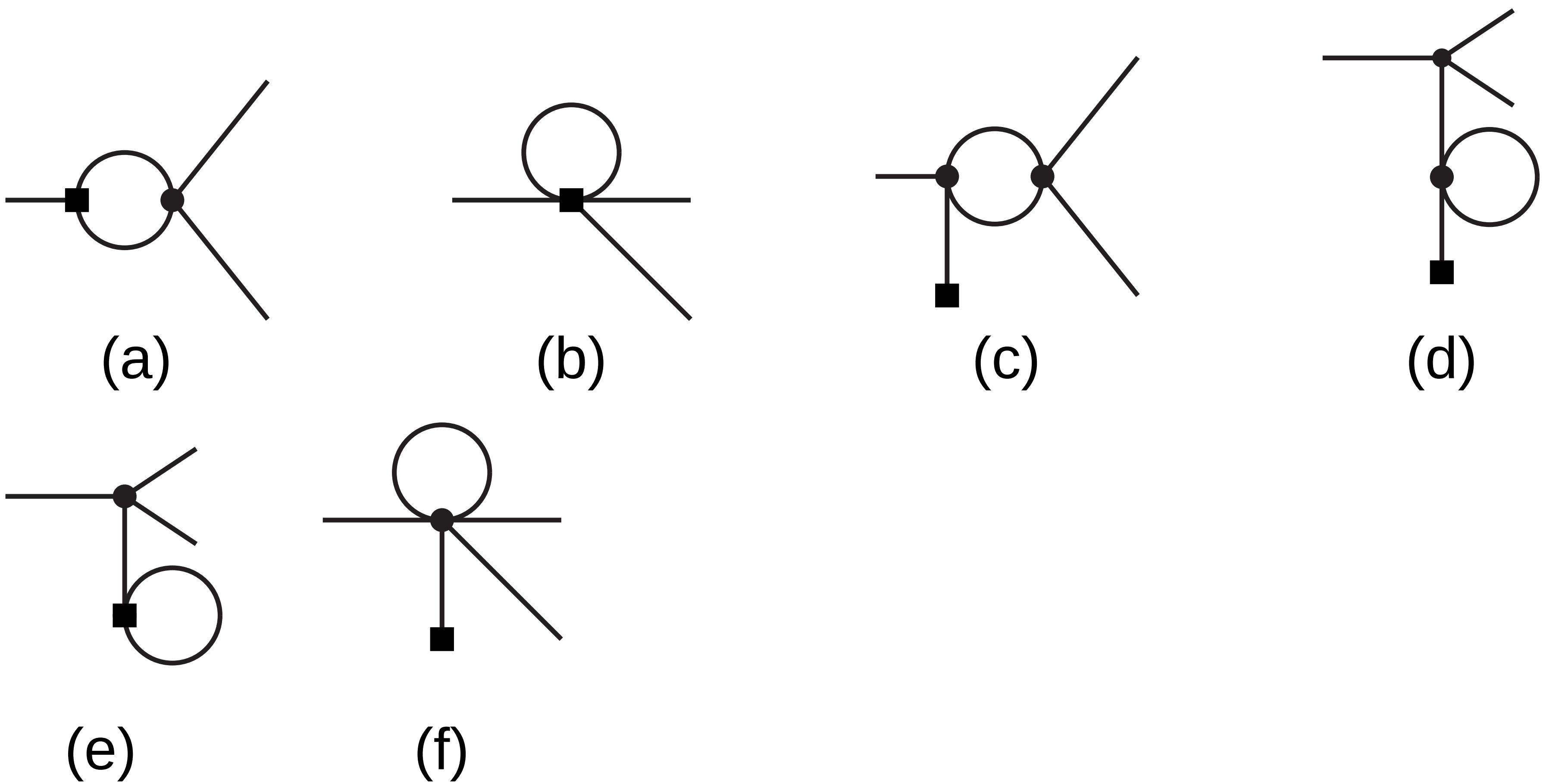
$K \to \pi\pi$: Tree level



$$A_0^{LO} = \frac{\sqrt{3}i}{2F^2}\left[-\frac{1}{2}E_1 + (E_2 - 4E_3)\,\overline{M}_K^2 + 2E_8\overline{M}_K^4 + A_1E_1\right]$$

$$A_2^{LO} = \sqrt{\frac{3}{2}}\frac{i}{F^2}\left[(-2D_1 + D_2)\,\overline{M}_K^2\right]$$

$K \to \pi\pi$: One loop



$K \to \pi\pi$: One-loop

$$A_0^{NLO} = A_0^{LO}\left(1+\frac{3}{8F^2}\overline{A}(M^2)\right)+\lambda_0 M^2+\mathcal{O}(M^4)\,,$$

$$A_2^{NLO} = A_2^{LO}\left(1+\frac{15}{8F^2}\overline{A}(M^2)\right)+\lambda_2 M^2+\mathcal{O}(M^4)\,.$$

$$\overline{A}(M^2) = -\frac{M^2}{16\pi^2}\log\frac{M^2}{\mu^2}$$

Hard Pion ChPT: A two-loop check

- Similar arguments to JB-Celis, Flynn-Sachrajda work for the pion vector and scalar formfactor JB-Jemos
- Therefore at any t the chiral log correction must go like the one-loop calculation.
- But note the one-loop log chiral log is with $t >> m_\pi^2$
- Predicts

$$F_V(t, M^2) = F_V(t, 0)\left(1 - \frac{M^2}{16\pi^2 F^2}\ln\frac{M^2}{\mu^2} + \mathcal{O}(M^2)\right)$$

$$F_S(t, M^2) = F_S(t, 0)\left(1 - \frac{5}{2}\frac{M^2}{16\pi^2 F^2}\ln\frac{M^2}{\mu^2} + \mathcal{O}(M^2)\right)$$

- Note that $F_{V,S}(t, 0)$ is now a coupling constant and can be complex

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- Note that $F_{V,S}(t,0)$ is now a coupling constant and can be complex
- Take the full two-loop ChPT calculation JB,Colangelo,Talavera and expand in $t >> m_\pi^2$.

A two-loop check

Full two-loop ChPT JB,Colangelo,Talavera, expand in $t >> m_\pi^2$:

$$F_V(t,M^2) = F_V(t,0)\left(1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\mu^2} + \mathcal{O}(M^2)\right)$$
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with

$$F_V(t,0) = 1 + \frac{t}{16\pi^2F^2}\left(\frac{5}{18} - 16\pi^2 l_6^r + \frac{i\pi}{6} - \frac{1}{6}\ln\frac{t}{\mu^2}\right)$$
$$F_S(t,0) = 1 + \frac{t}{16\pi^2F^2}\left(1 + 16\pi^2 l_4^r + i\pi - \ln\frac{t}{\mu^2}\right)$$

- The needed coupling constants are complex
- Both calculations have two-loop diagrams with overlapping divergences
- The chiral logs should be valid for any t where a pointlike interaction is a valid approximation

Electromagnetic formfactors

$$
\begin{aligned}
F_V^\pi(s) &= F_V^{\pi\chi}(s)\left(1+\frac{1}{F^2}\overline{A}(m_\pi^2)+\frac{1}{2F^2}\overline{A}(m_K^2)+\mathcal{O}(m_L^2)\right)\,,\\
F_V^K(s) &= F_V^{K\chi}(s)\left(1+\frac{1}{2F^2}\overline{A}(m_\pi^2)+\frac{1}{F^2}\overline{A}(m_K^2)+\mathcal{O}(m_L^2)\right)\,.
\end{aligned}
$$

$B, D \to \pi, K, \eta$

$$\left\langle P_f(p_f) \left| \overline{q}_i \gamma_\mu q_f \right| P_i(p_i) \right\rangle = (p_i + p_f)_\mu f_+(q^2) + (p_i - p_f)_\mu f_-(q^2)$$

$$f_{+B\to M}(t) = f^\chi_{+B\to M}(t) F_{B\to M}$$
$$f_{-B\to M}(t) = f^\chi_{-B\to M}(t) F_{B\to M}$$

- $F_{B\to M}$ always same for f_+, f_- and f_0
- This is not heavy quark symmetry: not valid at endpoint and valid also for $K \to \pi$.
- Not like Low's theorem, not only dependence on external legs
- Check: heavy meson and relativistic formalism
- Endpoint also done, η final state new

$B, D \to \pi, K, \eta$

$$F_{K\to\pi} = 1 + \frac{3}{8F^2}\overline{A}(m_\pi^2) \qquad (2-\text{flavour})$$

$$F_{B\to\pi} = 1 + \left(\frac{3}{8} + \frac{9}{8}g^2\right)\frac{\overline{A}(m_\pi^2)}{F^2} + \left(\frac{1}{4} + \frac{3}{4}g^2\right)\frac{\overline{A}(m_K^2)}{F^2} + \left(\frac{1}{24} + \frac{1}{8}g^2\right)\frac{\overline{A}(m_\eta^2)}{F^2},$$

$$F_{B\to K} = 1 + \frac{9}{8}g^2\frac{\overline{A}(m_\pi^2)}{F^2} + \left(\frac{1}{2} + \frac{3}{4}g^2\right)\frac{\overline{A}(m_K^2)}{F^2} + \left(\frac{1}{6} + \frac{1}{8}g^2\right)\frac{\overline{A}(m_\eta^2)}{F^2},$$

$$F_{B\to\eta} = 1 + \left(\frac{3}{8} + \frac{9}{8}g^2\right)\frac{\overline{A}(m_\pi^2)}{F^2} + \left(\frac{1}{4} + \frac{3}{4}g^2\right)\frac{\overline{A}(m_K^2)}{F^2} + \left(\frac{1}{24} + \frac{1}{8}g^2\right)\frac{\overline{A}(m_\eta^2)}{F^2},$$

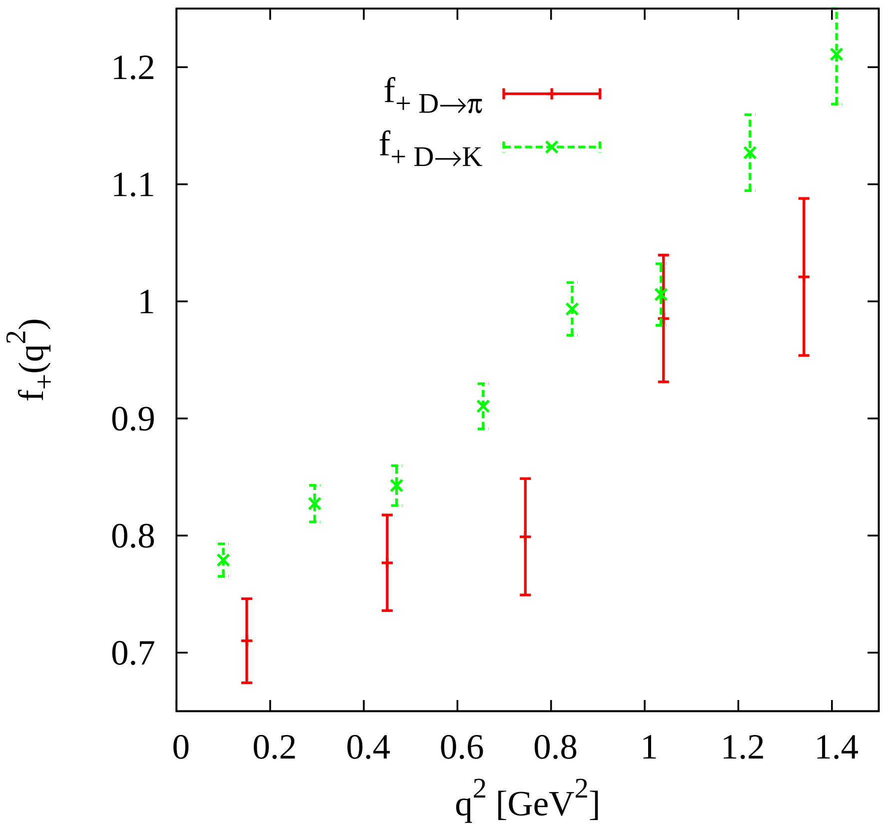
$$F_{B_s\to K} = 1 + \frac{3}{8}\frac{\overline{A}(m_\pi^2)}{F^2} + \left(\frac{1}{4} + \frac{3}{2}g^2\right)\frac{\overline{A}(m_K^2)}{F^2} + \left(\frac{1}{24} + \frac{1}{2}g^2\right)\frac{\overline{A}(m_\eta^2)}{F^2},$$

$$F_{B_s\to\eta} = 1 + \left(\frac{1}{2} + \frac{3}{2}g^2\right)\frac{\overline{A}(m_K^2)}{F^2} + \left(\frac{1}{6} + \frac{1}{2}g^2\right)\frac{\overline{A}(m_\eta^2)}{F^2}.$$

$F_{B_s\to\pi}$ vanishes at this order due to the possible flavour quantum numbers.

Experimental check

CLEO data on $f_+(q^2)|V_{cq}|$ for $D \to \pi$ and $D \to K$ with $|V_{cd}| = 0.2253$, $|V_{cs}| = 0.9743$



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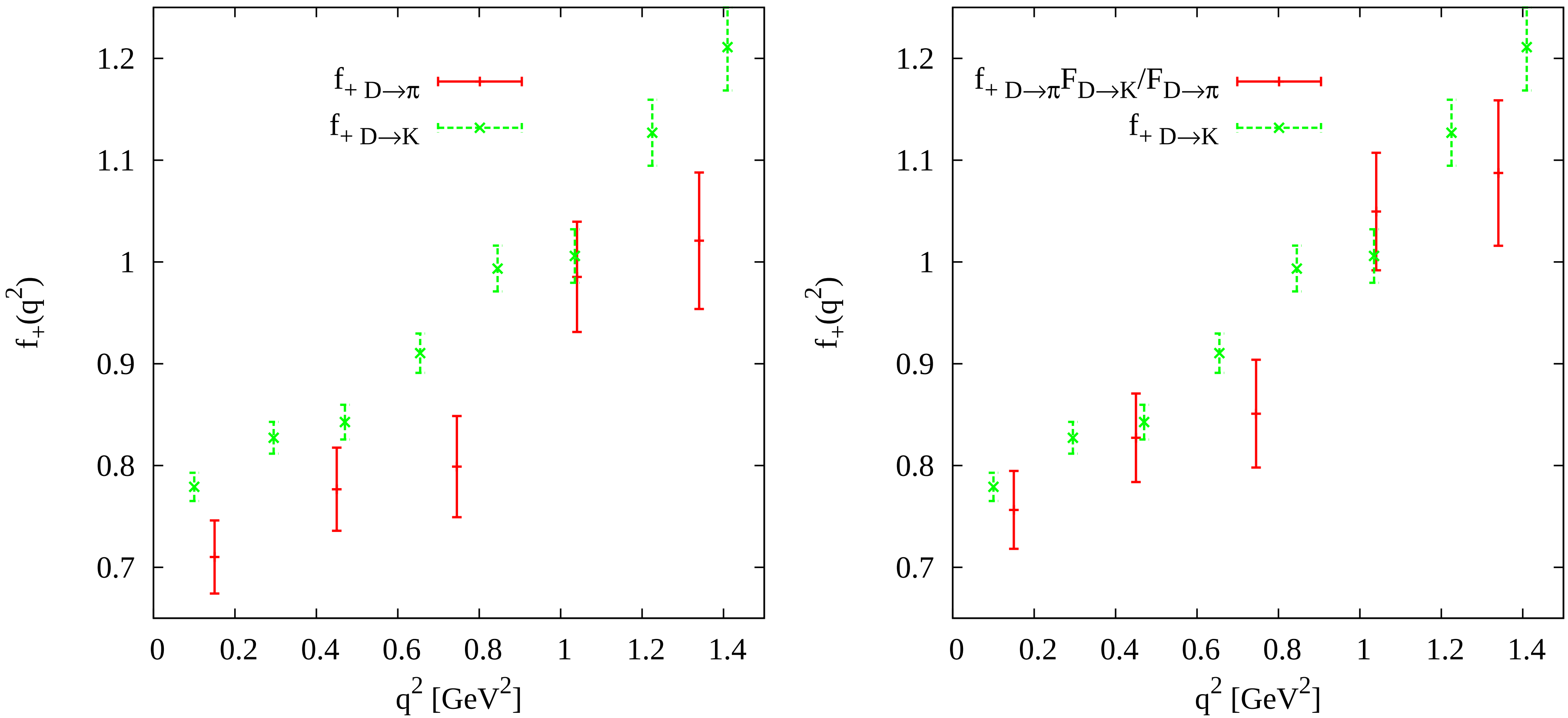

$$f_{+D\to\pi} = f_{+D\to K} F_{D\to\pi}/F_{D\to K}$$

Hard Pion ChPT: summary

Why is this useful:

- Lattice works actually around the strange quark mass
- need only extrapolate in m_u and m_d for $K \to \pi$ and $K \to \pi\pi$
- Applicable in momentum regimes where usual ChPT might not work
- Three flavour case useful for B, D decays

Leading Logarithms

- Take a quantity with a single scale: $F(M)$
- The dependence on the scale in field theory is typically logarithmic
- $L = \log(\mu/M)$
- $F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + F_3^3 L^3 + \cdots$
- Leading Logarithms: The terms $F_m^m L^m$

The F_m^m can be more easily calculated than the full result

- $\mu\,(dF/d\mu) \equiv 0$
- Ultraviolet divergences in Quantum Field Theory are always local

Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$
- f_i^j are pure numbers
- $\mu \dfrac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$

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- f_i^j are pure numbers
- $\mu \dfrac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$
- $\mu \dfrac{dF}{d\mu} = 0 \Longrightarrow \boxed{\beta_0 = -f_1^1 = f_2^2 = -f_3^3 = \cdots}$

Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu\frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–'t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the α the same in all orders
- LL one-loop β_0
- NLL two-loop β_1, f_0^1

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- In effective field theories: different Lagrangian at each order
- The recursive argument does not work

Weinberg's argument

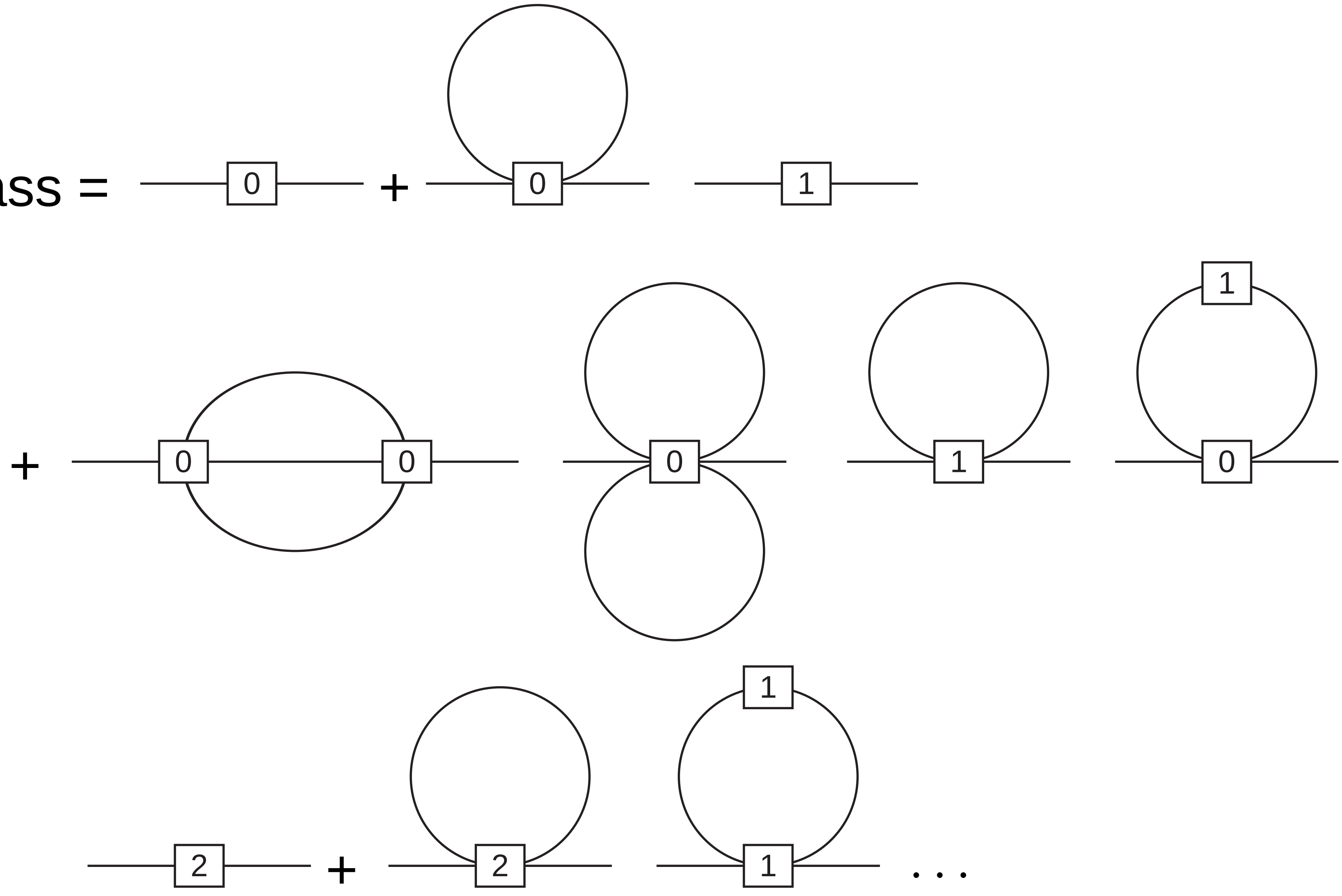
- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop
- Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- General at 2 loop: JB, Colangelo, Ecker, hep-ph/9808421
- Proof at all orders using β-functions
  Büchler, Colangelo, hep-ph/0309049
- Proof with diagrams: JB, Carloni, arXiv:0909.5086

Weinberg's argument

- μ: dimensional regularization scale
- $d = 4 - w$
- loop-expansion $\equiv \hbar$-expansion
- $\mathcal{L}^{\text{bare}} = \sum_{n\geq 0} \hbar^n \mu^{-nw} \mathcal{L}^{(n)} =$
  $\sum_{n\geq 0} \hbar^n \mu^{-nw} \sum_i \left(\sum_{k=0,n} \frac{c_{ki}^{(n)}}{w^k} \right) \mathcal{O}_i^{(n)}$
- L_l^n l-loop contribution at order $\hbar^n$
- Expand in divergences from the loops (not from the counterterms) $L_l^n = \sum_{k=0,l} \frac{1}{w^k} L_{kl}^n$
- $\mathcal{L}^{(n)} =$ $\boxed{\text{n}}$

Weinberg's argument

Mass =



Weinberg's argument

- $\hbar^0$: L^0_0
- $\hbar^1$: $\frac{1}{w}\left(\mu^{-w}L^1_{00}(\{c\}^1_1) + L^1_{11}\right) + \mu^{-w}L^1_{00}(\{c\}^1_0) + L^1_{10}$

Weinberg's argument

- $\hbar^0$: L_0^0
- $\hbar^1$: $\frac{1}{w}\left(\mu^{-w}L_{00}^1(\{c\}_1^1) + L_{11}^1\right) + \mu^{-w}L_{00}^1(\{c\}_0^1) + L_{10}^1$
  - Expand $\mu^{-w} = 1 - w\log\mu + \frac{1}{2}w^2\log^2\mu + \cdots$
  - $1/w$ must cancel: $L_{00}^1(\{c\}_1^1) + L_{11}^1 = 0$
    this determines the c_{1i}^1
  - Explicit $\log\mu$: $-\log\mu\, L_{00}^1(\{c\}_0^1) \equiv \log\mu\, L_{11}^1$

All orders

- $\hbar^n$:
$$\frac{1}{w^n}\Big(\mu^{-nw}L^n_{00}(\{c\}^n_n)+\mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1})+\cdots$$
$$+\mu^{-w}L^n_{n-1\;n-1}(\{c\}^1_1)+L^n_{nn}\Big)+\frac{1}{w^{n-1}}\cdots$$

All orders

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$$+\mu^{-w}L^n_{n-1\;n-1}(\{c\}^1_1)+L^n_{nn}\Big)+\frac{1}{w^{n-1}}\cdots$$
- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\;n-i}(\{c\}^i_i)=0 \qquad j=0,..,n-1.$$

All orders

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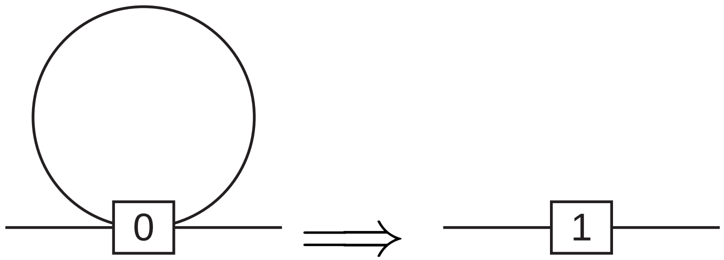
$$\frac{1}{w^n}\left(\mu^{-nw}L^n_{00}(\{c\}^n_n) + \mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1}) + \cdots\right.$$
$$\left. + \mu^{-w}L^n_{n-1\ n-1}(\{c\}^1_1) + L^n_{nn}\right) + \frac{1}{w^{n-1}}\cdots$$
- $1/w^n, \log\mu/w^{n-1}, \ldots, \log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\ n-i}(\{c\}^i_i) = 0 \qquad j = 0,..,n-1.$$
- Solution: $L^n_{n-i\ n-i}(\{c\}^i_i) = (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} L^n_{nn}$
- explicit leading $\log\mu$ dependence and divergence
$$\log^n\mu\frac{(-1)^{n-1}}{n}L^n_{11}(\{c\}^{n-1}_{n-1}) \qquad L^n_{00}(\{c\}^n_n) = -\frac{1}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$

Mass to $\hbar^2$

- $\hbar^1$: 

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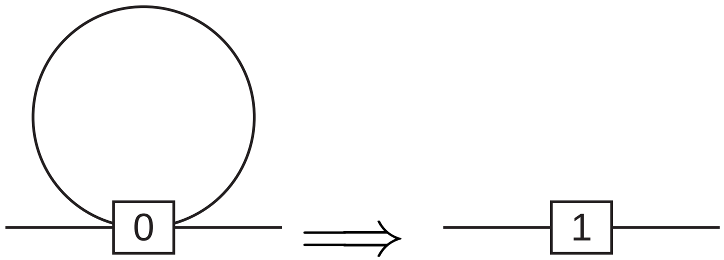

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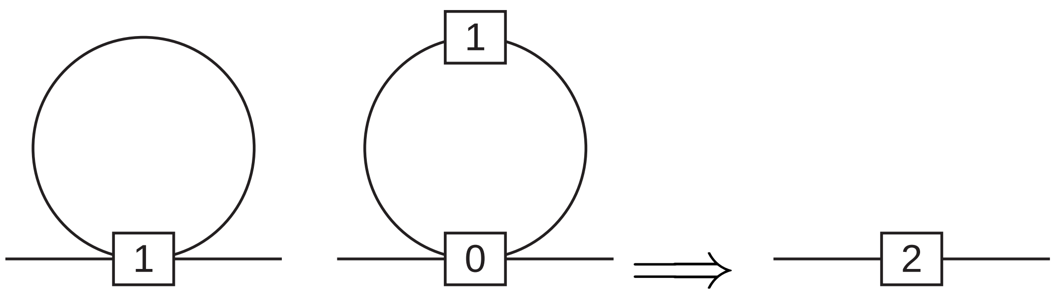

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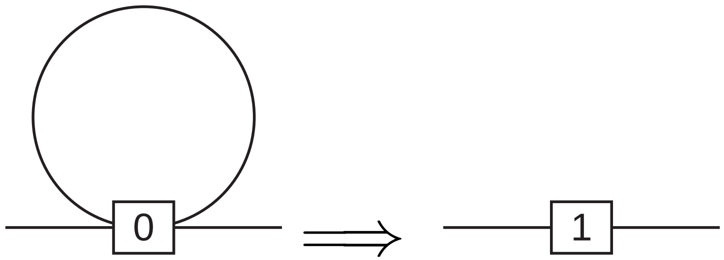
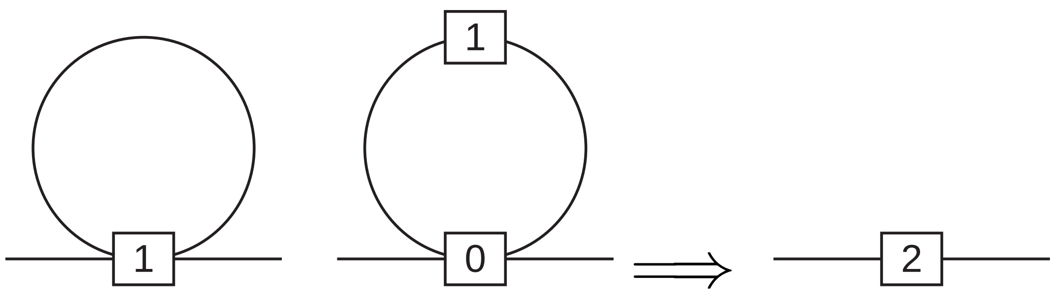
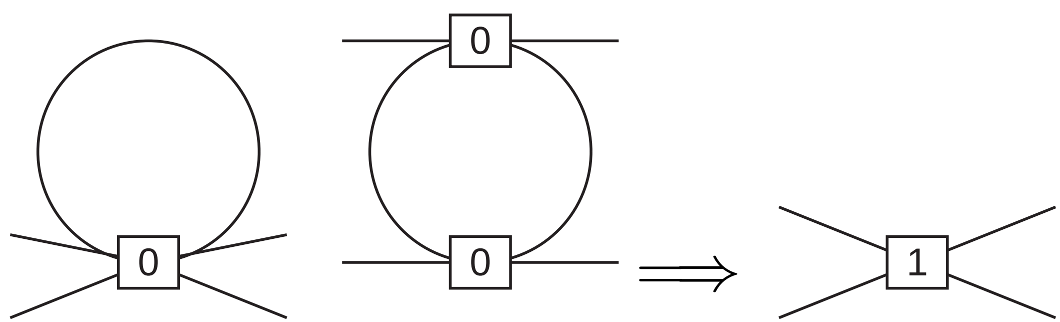
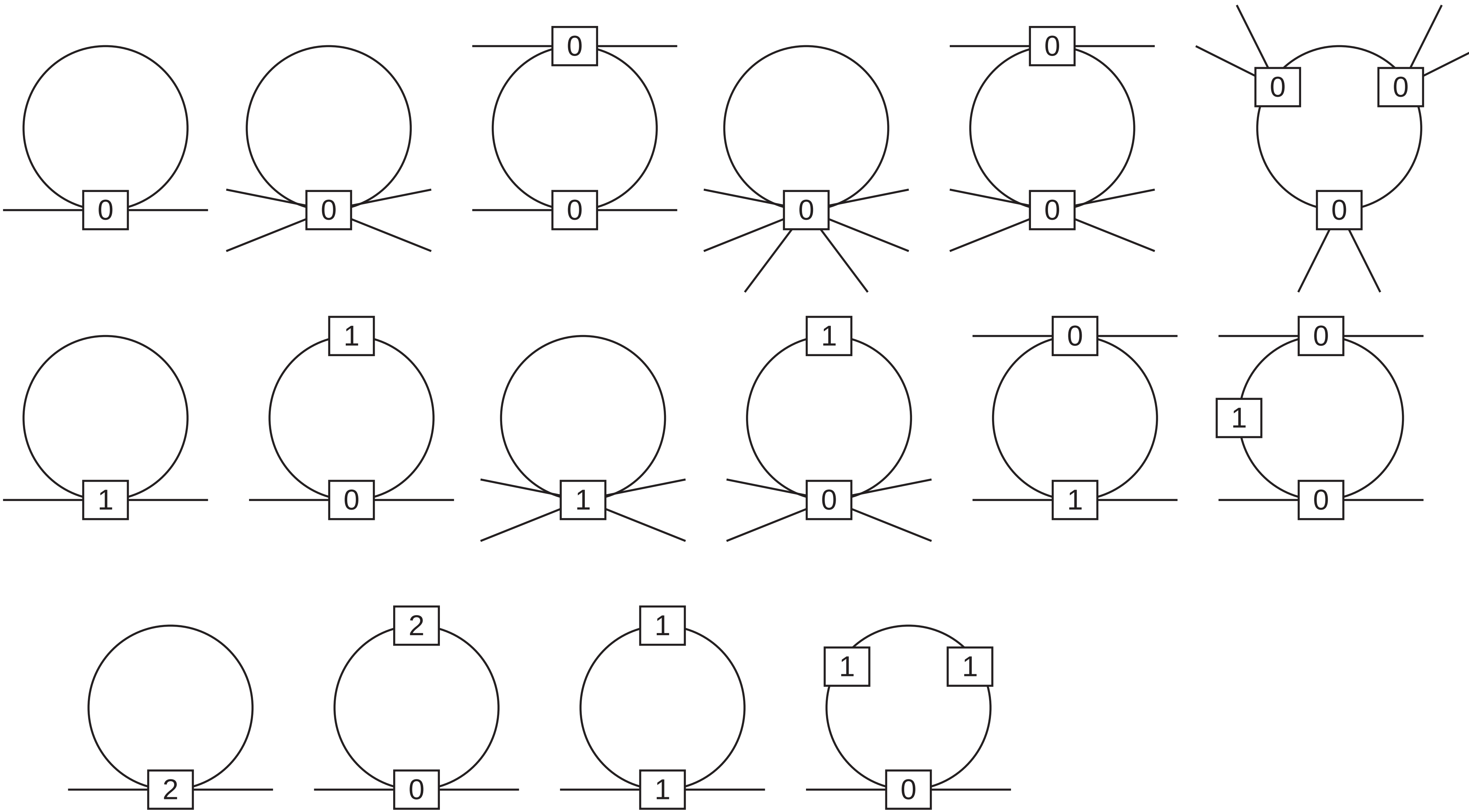
- $\hbar^1$: 

- $\hbar^2$: 

- but also needs $\hbar^1$: 

Mass to order $\hbar^3$



Mass+decay to $\hbar^5$

- $\hbar^1$: 18 + 27
- $\hbar^2$: 26 + 45
- $\hbar^3$: 33 + 51
- $\hbar^4$: 26 + 33
- $\hbar^5$: 13 + 13

- Calculate the divergence
- rewrite it in terms of a local Lagrangian
- Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
- We keep all terms to have all 1PI (one particle irreducible) diagrams finite

Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu\Phi^T\partial^\mu\Phi + F^2\chi^T\Phi\,.$
- Φ is a real $N+1$ vector; $\Phi \to O\Phi$; $\Phi^T\Phi = 1$.
- Vacuum expectation value $\langle\Phi^T\rangle = (1\ 0\ldots 0)$
- Explicit symmetry breaking: $\chi^T = \left(M^2\ 0\ldots 0\right)$
- Both spontaneous and explicit symmetry breaking
- N-vector ϕ
- N (pseudo-)Nambu-Goldstone Bosons
- $N = 3$ is two-flavour Chiral Perturbation Theory

Massive $O(N)$ sigma model: Φ vs ϕ

- $$\Phi_1 = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}$$ Gasser, Leutwyler

- $$\Phi_2 = \frac{1}{\sqrt{1+\frac{\phi^T\phi}{F^2}}} \begin{pmatrix} 1 \\ \frac{\phi}{F} \end{pmatrix}$$ similar to Weinberg

  $$\Phi_3 = \begin{pmatrix} 1-\frac{1}{2}\frac{\phi^T\phi}{F^2} \\ \sqrt{1-\frac{1}{4}\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{F} \end{pmatrix}$$ only mass term

- $$\Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$ CCWZ

Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use the four different parametrizations
- compare with known results:

$$M^2_{phys} = M^2 \left(1 - \frac{1}{2} L_M + \frac{17}{8} L_M^2 + \cdots \right) \,,$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{\mathcal{M}^2}$$

  Usual choice $\mathcal{M} = M$.
- large N (but known results only for massless case)
  Coleman, Jackiw, Politzer 1974
- large N massive later found partly in appendix of Kivel, Polyakov, Vladimirov on distribution functions.

Results

$$M^2_{\rm phys} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

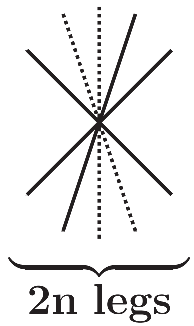
i	a_i, $N=3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$

$F_{\rm phys}, \langle \bar q_i q_i \rangle$ as well done

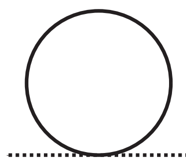
Anyone recognize any funny functions?

Large N

Power counting: pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N$, $M^2 \sim 1$

- 

  $\Leftrightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$

  $\Leftrightarrow N$

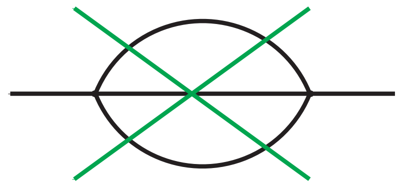
- 1PI diagrams:

$$\left.\begin{array}{l} N_L = N_I - \sum_n N_{2n} + 1 \\ 2N_I + N_E = \sum_n 2nN_{2n} \end{array}\right\} \Rightarrow N_L = \sum_n (n-1)N_{2n} - \frac{1}{2}N_E + 1$$

- diagram suppression factor: $\frac{N^{N_L}}{N^{N_E/2-1}}$

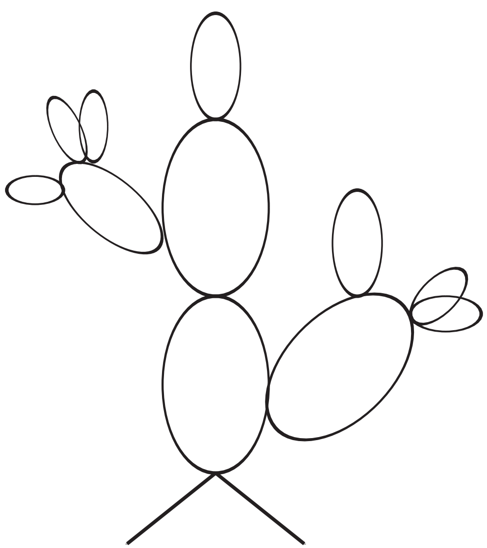
Large N

- diagrams with shared lines are suppressed



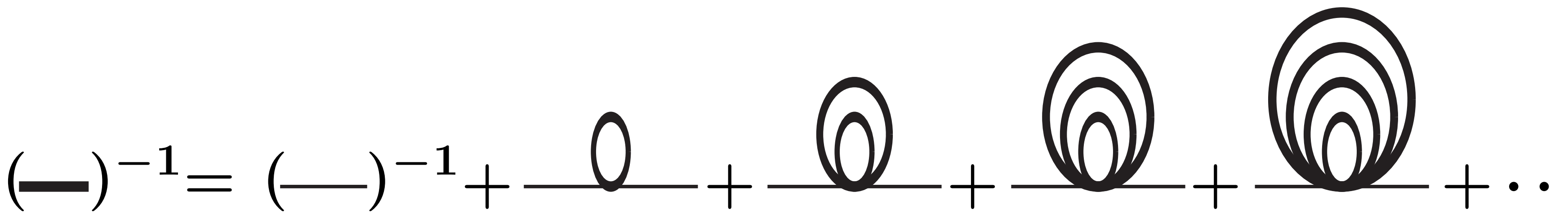
each new loop needs also a new flavour loop

- in the large N limit only “*cactus*” diagrams survive:



large N: propagator

Generate recursively via a Gap equation



$\Rightarrow$ resume the series and look for the pole

$$M^2 = M^2_{\text{phys}}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

$$\overline{A}(m^2) = \frac{m^2}{16\pi^2}\log\frac{\mu^2}{m^2}.$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N

$$F_{\text{phys}} = F\sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

$$\langle \bar{q}q \rangle_{\text{phys}} = \langle \bar{q}q \rangle_0 \sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

Comments:

- These are the full* leading N results, not just leading log
- But depends on the choice of N-dependence of higher order coefficients
- Assumes higher LECs zero ($< N^{n+1}$ for $\hbar^n$)
- Large N as in $O(N)$ not large N_c

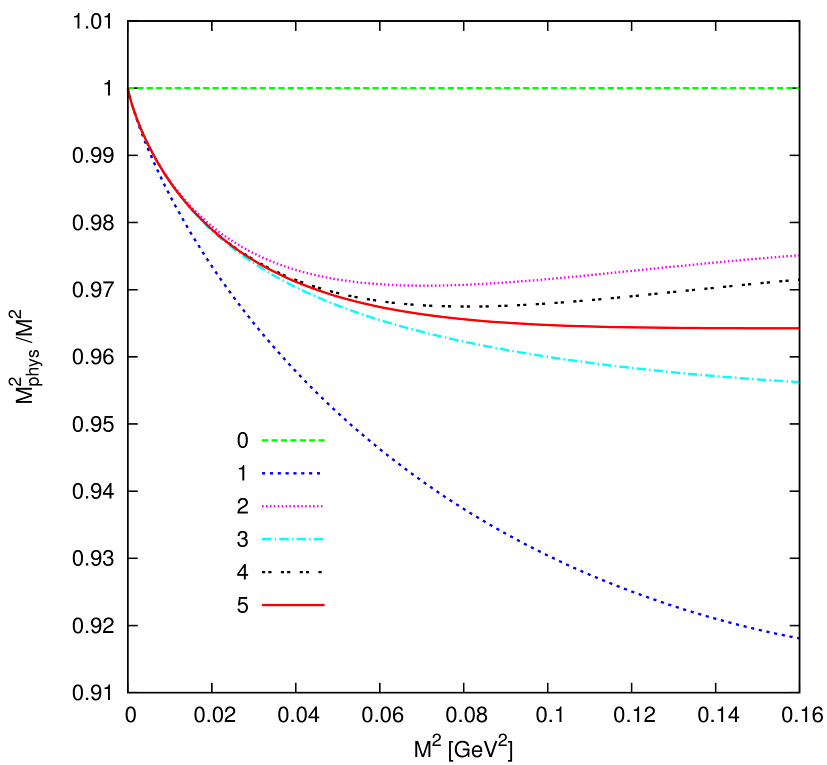
Large N: Checking expansions

$$M^2 = M^2_{\rm phys}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\rm phys})}$$

much smaller expansion coefficients than the table, try

$$M^2 = M^2_{\rm phys}(1+d_1 L_{M_{\rm phys}} + d_2 L^2_{M_{\rm phys}} + d_3 L^3_{M_{\rm phys}} + \ldots)$$

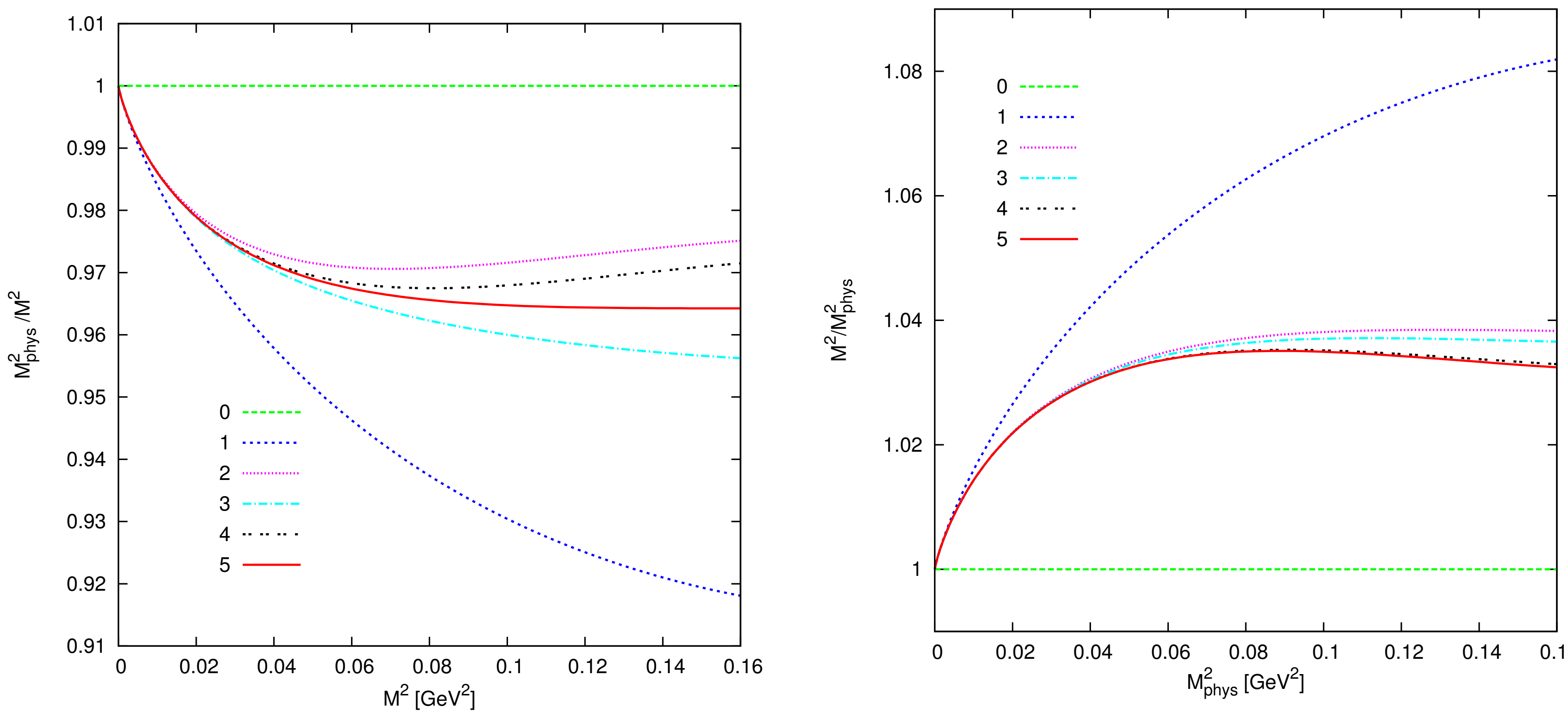
Numerical results



Left: $\frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

F = 90 MeV, $\mu = 0.77$ GeV

Numerical results



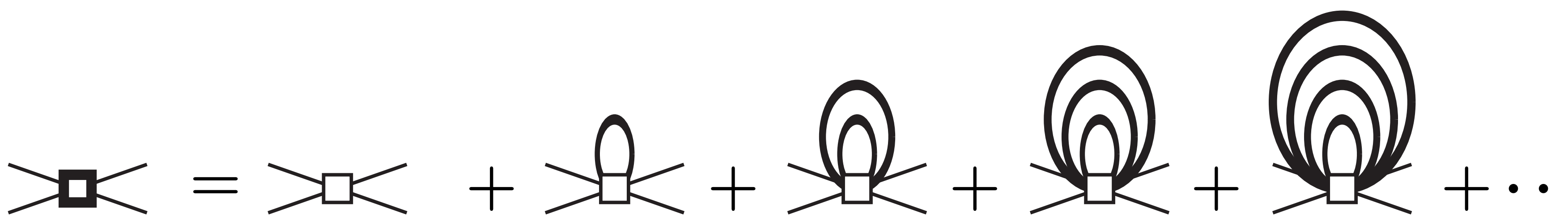
Left: $\frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

Right: $\frac{M^2}{M^2_{\text{phys}}} = 1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + \cdots$

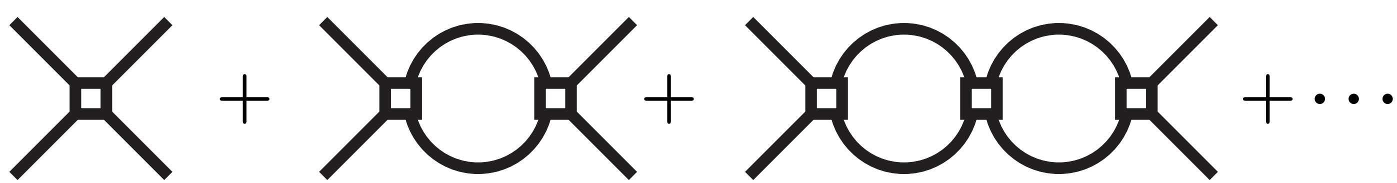
F = 90 MeV, $\mu = 0.77$ GeV

Large N: $\pi\pi$-scattering

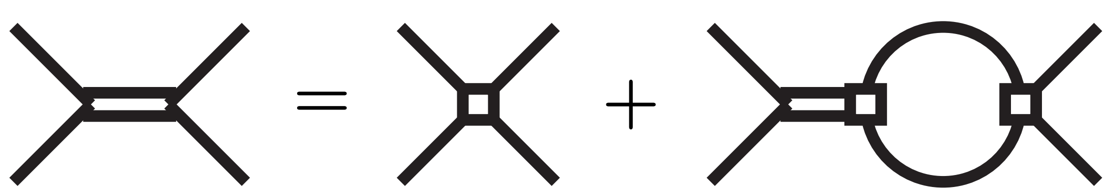
- Cactus diagrams for $A(s,t,u)$
- Branch with no momentum: resummed by
- Branch starting at vertex: resum by



- The full result is then



- Can be summarized by a recursive equation



Large N: $\pi\pi$ scattering

$$y = \frac{N}{F^2}\overline{A}(M^2_{\rm phys})$$

$$A(s,t,u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{N}{2}\left(\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}\right)\overline{B}(M^2_{\rm phys}, M^2_{\rm phys}, s)}$$

or

$$A(s,t,u) = \frac{\frac{s-M^2_{\rm phys}}{F^2_{\rm phys}}}{1 - \frac{N}{2}\frac{s-M^2_{\rm phys}}{F^2_{\rm phys}}\overline{B}(M^2_{\rm phys}, M^2_{\rm phys}, s)}$$

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results

Other results

- Bissegger, Fuhrer, hep-ph/0612096 Dispersive methods, massless Π_S to five loops
- Kivel, Polyakov, Vladimirov, 0809.3236, 0904.3008, 1004.2197
  - In the massless case tadpoles vanish
  - hence the number of external legs needed does not grow
  - All 4-meson vertices via Legendre polynomials
  - can do divergence of all one-loop diagrams analytically
  - algebraic (but quadratic) recursion relations
  - massless $\pi\pi$, F_V and F_S to arbitrarily high order
  - large N agrees with Coleman, Wess, Zumino
  - large N is not a good approximation

Other results

- JB,Carloni, arXiv:1008.3499
  - massive case: $\pi\pi$, F_V and F_S to 4-loop order
  - large N for these cases also for massive $O(N)$.
  - done using bubble resummations or recursion eqation which can be solved analytically

Conclusions Leading Logs

- Several quantities in massive $O(N)$ LL known to high loop order
- Large N in massive $O(N)$ model solved
- Had hoped: recognize the series also for general N
- Limited essentially by CPU time and size of intermediate files
- Some first studies on convergence etc.
- $\pi\pi$, F_V and F_S to four-loop order
- The technique can be generalized to other models/theories
  - $SU(N) \times SU(N)/SU(N)$
  - One nucleon sector

QCDlike and/or technicolor theories

A typical gaugegroup and N_F fermions:

- QCD or complex: $q^T = (q_1\ q_2 \ldots\ q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$
    $q_L \to g_L q_L$ and $g_R \to g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$
  - Conserved $H = SU(N_F)$ $g_L = g_R$ $\Sigma_{ij} \to \Sigma_{ij}$
  - q in complex prepresentation of gauge group

QCDlike and/or technicolor theories

A typical gaugegroup and N_F fermions:

- Real (e.g. adjoint):
  - $\tilde{q}_{Ri} \equiv C\overline{q}_{Li}^T$ is in the same gauge group representation as q_{Ri}
  - $\hat{q}^T = (q_{R1} \ \dots \ q_{RN_F} \ \tilde{q}_{R1} \ \dots \ \tilde{q}_{RN_F})$
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - Vacuum condensate $\langle \overline{q}_j q_i \rangle$ is really $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto J_{Sij}$

    $$J_S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
  - Conserved symmetry part has $g J_S g^T = J_S$
  - $H = SO(2N_F)$
  - some Goldstone bosons have baryonnumber

QCDlike and/or technicolor theories

A typical gaugegroup and N_F fermions:

- Pseudoreal (e.g. two-colours):
  - $\tilde{q}_{R\alpha i} \equiv \epsilon_{\alpha\beta} C \bar{q}^T_{L\beta i}$ is in the same gauge group representation as $q_{R\alpha i}$
  - $\hat{q}^T = (q_{R1} \ \ldots \ q_{RN_F} \ \tilde{q}_{R1} \ \ldots \ \tilde{q}_{RN_F})$
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - Vacuum condensate $\langle \overline{q}_j q_i \rangle$ is really

    $$\epsilon_{\alpha\beta} \langle (\hat{q}_{\alpha j})^T C \hat{q}_{\beta i} \rangle \propto J_{Aij} \ J_A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$
  - Conserved symmetry part has $g J_A g^T = J_A$
  - $H = Sp(2N_F)$
  - some Goldstone bosons have baryonnumber

Lagrangians

In arXiv:0910.5424 we showed that there is a very similar way of phrasing the two theories using $u = \exp\left(\frac{i}{\sqrt{2}F}\phi^a X^a\right)$

But the matrices X^a are:

- Complex or $SU(N) \times SU(N)/SU(N)$: all $SU(N)$ generators
- Real or $SU(2N)/SO(2N)$: $SU(2N)$ generator with $X^a J_S = J_S X^{aT}$
- Pseudoreal or $SU(2N)/Sp(2N)$: $SU(2N)$ generator with $X^a J_A = J_A X^{aT}$
- Note that the latter are not the usual ways of parametrizing $SO(2N)$ and $Sp(2N)$ matrices

Divergences etc

Calculating for equal mass case goes through using:

$$\begin{aligned}
\text{QCD}: \quad & \langle X^a A X^a B\rangle = \langle A\rangle\langle B\rangle - \frac{1}{N_F}\langle AB\rangle\,, \\
& \langle X^a A\rangle\langle X^a B\rangle = \langle AB\rangle - \frac{1}{N_F}\langle A\rangle\langle B\rangle\,. \\
\text{Adjoint}: \quad & \langle X^a A X^a B\rangle = \frac{1}{2}\langle A\rangle\langle B\rangle + \frac{1}{2}\left\langle A J_S B^T J_S\right\rangle - \frac{1}{2N_F}\langle AB\rangle\,, \\
& \langle X^a A\rangle\langle X^a B\rangle = \frac{1}{2}\langle AB\rangle + \frac{1}{2}\left\langle A J_S B^T J_S\right\rangle - \frac{1}{2N_F}\langle A\rangle\langle B\rangle\,. \\
2-\text{colour}: \quad & \langle X^a A X^a B\rangle = \frac{1}{2}\langle A\rangle\langle B\rangle + \frac{1}{2}\left\langle A J_A B^T J_A\right\rangle - \frac{1}{2N_F}\langle AB\rangle\,, \\
& \langle X^a A\rangle\langle X^a B\rangle = \frac{1}{2}\langle AB\rangle - \frac{1}{2}\left\langle A J_A B^T J_A\right\rangle - \frac{1}{2N_F}\langle A\rangle\langle B\rangle
\end{aligned}$$

So can do the calculations for all cases

Vacuum expectation value

All cases: $\langle \overline{q}q\rangle_{\text{LO}} \equiv \sum_{i=1,N_F}\langle \overline{q}_{Ri}q_{Li} + \overline{q}_{Li}q_{Ri}\rangle_{\text{LO}} = -N_F B_0 F^2$

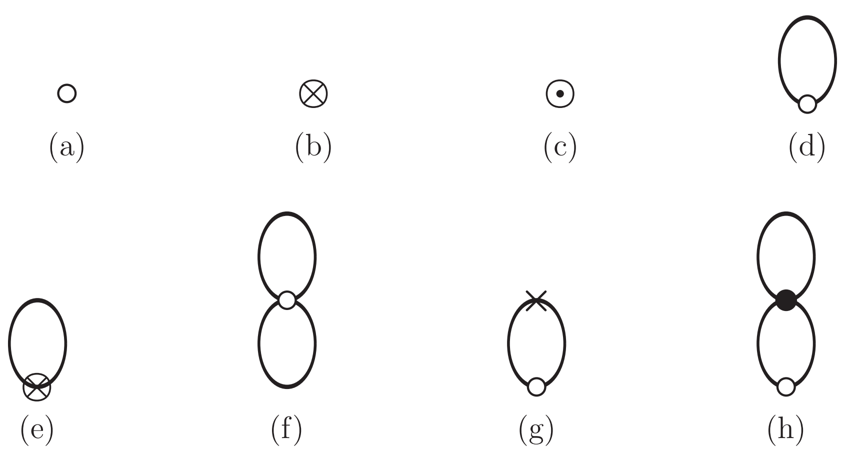
$M^2 = 2B_0\hat{m}$ and $\overline{A}(M^2) = -\frac{M^2}{16\pi^2}\log\frac{M^2}{\mu^2}$.

$\langle \overline{q}q\rangle = \langle \overline{q}q\rangle_{\text{LO}} + \langle \overline{q}q\rangle_{\text{NLO}} + \langle \overline{q}q\rangle_{\text{NNLO}}$.

$$\langle \overline{q}q\rangle_{\text{NLO}} = \langle \overline{q}q\rangle_{\text{LO}}\left(a_V\frac{\overline{A}(M^2)}{F^2} + b_V\frac{M^2}{F^2}\right),$$

$$\langle \overline{q}q\rangle_{\text{NNLO}} = \langle \overline{q}q\rangle_{\text{LO}}\left(c_V\frac{\overline{A}(M^2)^2}{F^4} + \frac{M^2\overline{A}(M^2)}{F^4}\left(d_V + \frac{e_V}{16\pi^2}\right) + \frac{M^4}{F^4}\left(f_V + \frac{g_V}{16\pi^2}\right)\right).$$

Diagrams:



Vacuum expectation value

	QCD	
a_V	$n - \frac{1}{n}$	
b_V	$16nL_6^r + 8L_8^r + 4H_2^r$	
c_V	$\frac{3}{2}\left(-1 + \frac{1}{n^2}\right)$	
d_V	$-24\left(n^2-1\right)\left(L_A + \frac{1}{n}L_B\right)$	$L_A = L_4^r - 2L_6^r$
e_V	$1 - \frac{1}{n^2}$	$L_B = L_5^r - 2L_8^r$
f_V	$48\left(K_{25}^r + nK_{26}^r + n^2K_{27}^r\right)$	
g_V	$8\left(n^2-1\right)\left(L_A + \frac{1}{n}L_B\right)$	
	Adjoint	**2-colour**
a_V	$n + \frac{1}{2} - \frac{1}{2n}$	$n - \frac{1}{2} - \frac{1}{2n}$
b_V	$32nL_6^r + 8L_8^r + 4H_2^r$	$32nL_6^r + 8L_8^r + 4H_2^r$
c_V	$\frac{3}{8}\left(-1 + \frac{1}{n^2} - \frac{2}{n} + 2n\right)$	$\frac{3}{8}\left(-1 + \frac{1}{n^2} + \frac{2}{n} - 2n\right)$
d_V	$-12\left(2n^2+n-1\right)\left(2L_A + \frac{1}{n}L_B\right)$	$-12\left(2n^2-n-1\right)\left(2L_A + \frac{1}{n}L_B\right)$
e_V	$\frac{1}{4}\left(1 - \frac{1}{n^2} + \frac{2}{n} - 2n\right)$	$\frac{1}{4}\left(1 - \frac{1}{n^2} - \frac{2}{n} + 2n\right)$
f_V	r_{VA}^r	r_{VT}^r
g_V	$4\left(2n^2+n-1\right)\left(2L_A + \frac{1}{n}L_B\right)$	$4\left(2n^2-n-1\right)\left(2L_A + \frac{1}{n}L_B\right)$

$\phi\phi \to \phi\phi$

- $\pi\pi$ scattering
  - Amplitude in terms of $A(s,t,u)$

    $M_{\pi\pi}(s,t,u) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t)\,.$
  - Three intermediate states $I = 0,1,2$
- Our three cases
  - Two amplitudes needed $B(s,t,u)$ and $C(s,t,u)$

$$\begin{aligned} M(s,t,u) &= \left[\langle X^aX^bX^cX^d\rangle + \langle X^aX^dX^cX^b\rangle\right]B(s,t,u) \\ &+ \left[\langle X^aX^cX^dX^b\rangle + \langle X^aX^bX^dX^c\rangle\right]B(t,u,s) \\ &+ \left[\langle X^aX^dX^bX^c\rangle + \langle X^aX^cX^bX^d\rangle\right]B(u,s,t) \\ &+\delta^{ab}\delta^{cd}C(s,t,u) + \delta^{ac}\delta^{bd}C(t,u,s) + \delta^{ad}\delta^{bc}C(u,s,t)\,. \end{aligned}$$

$$B(s,t,u) = B(u,t,s) \qquad C(s,t,u) = C(s,u,t)\,.$$

  - 7, 6 and 6 possible intermediate states

$\phi\phi \to \phi\phi$

- calculate all the diagrams
- Do all integrals, renormalize,...
- Construct states for all the presentations and their projection operators
- Get the amplitudes for all intermediate states
- Get all scattering lengths
- All formulas similar length to $\pi\pi$ cases but there are so many of them
- arXiv:1102.0172:
  - Very long appendix part
  - References for the Young diagrams, tensor algebra we did ourselves but probably exists (e.g. Cvitanovic group theory book)

$\phi\phi \to \phi\phi$

Some curious large $N_F = n$ relations

Leading in n:

$$
\begin{aligned}
a_0^I|_{\rm complex} &= a_0^I|_{\rm real} = a_0^I|_{\rm pseudoreal} =_{LO} \frac{x_2}{\pi}\,\frac{n}{8}\,,\\
a_0^S|_{\rm complex} &= a_0^S|_{\rm real} = a_0^A|_{\rm pseudoreal} =_{LO} \frac{x_2}{\pi}\,\frac{n}{16}\,,\\
a_1^A|_{\rm complex} &= a_1^A|_{\rm real} = a_1^S|_{\rm pseudoreal} =_{LO} \frac{x_2}{\pi}\,\frac{n}{48}\,,
\end{aligned}
$$

Subleading:

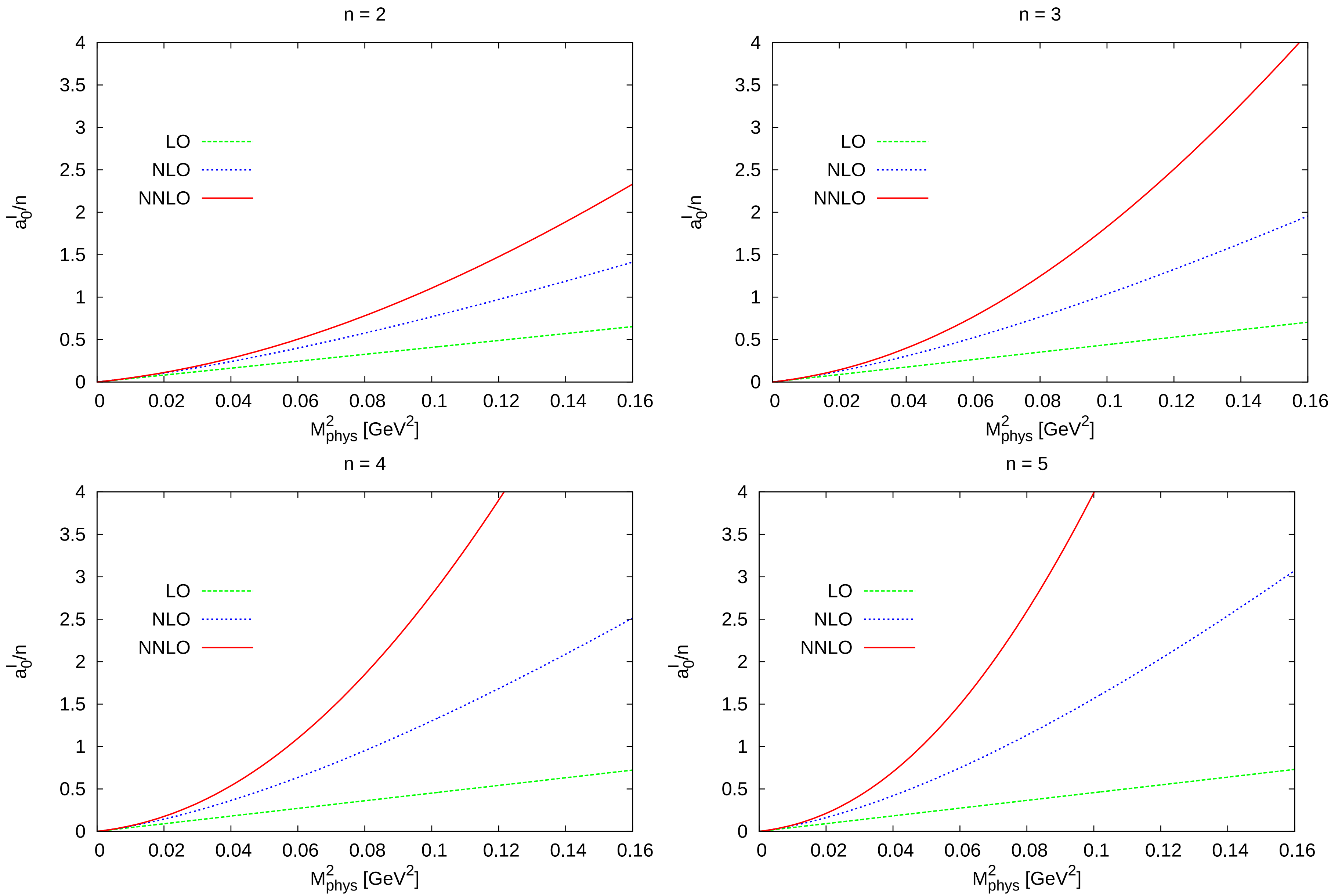
$$
\begin{aligned}
a_0^{SS}|_{\rm complex} &= a_0^{FS}|_{\rm real} = 2a_0^{MS}|_{\rm pseudoreal} =_{LO} \frac{x_2}{\pi}\,\frac{-1}{16}\,,\\
a_0^{AA}|_{\rm complex} &= 2a_0^{MS}|_{\rm real} = a_0^{FA}|_{\rm pseudoreal} =_{LO} \frac{x_2}{\pi}\,\frac{1}{16}\,.
\end{aligned}
$$

Subsubleading:

$$
a_1^{SA}|_{\rm complex} = a_1^{AS}|_{\rm complex} = 2a_1^{MA}|_{\rm real} = 2a_1^{MA}|_{\rm pseudoreal} =_{LO} 0\,.
$$

At NNLO here violated by an $L_4^r L_6^r$ term

$\phi\phi \to \phi\phi$: a_0^I/n



Other results: fully to NNLO

- M^2_{phys}
- F_{phys}
- Meson-meson scattering
- Equal mass case: allows to get fully analytical result just as for 2-flavour ChPT
- Note: large N_F here not cactus but planar diagrams (in flavour lines)

QCDlike: conclusions

- Different symmetry patterns can appear for different gaugegroups and fermion representations
- Nonperturbative: lattice needs extrapolation formulae
- Masses, decay constant and VEV: done to NNLO
- Meson-meson scattering: done to NNLO and some large N_F relations at NNLO.
- Two-pointfunctions and formfactors for precision observables: planned

Conclusions

Three new surroundings for ChPT:

- Hard Pion ChPT: a new application domain for EFT and first results
  - Many processes but limited domain
  - power counting proof lacking so far, SCET?
- Leading Logarithms and large N: some progress in getting results at high loop orders, but hoped for patterns not seen (except large N calculated)
  - Anybody recognize some funny functions?
  - Method applicable to many more cases
- Two-loop results for the equal mass case for different symmetry patterns. $SU(N) \times SU(N)/SU(N)$, $SU(2N)/SO(2N)$, $SU(2N)/Sp(2N)$