$\eta \rightarrow \pi \pi \pi$ at two-loop order in ChPT

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Various ChPT: http://www.thep.lu.se/~bijnens/chpt.html
Overview

- Introduction to Chiral Perturbation Theory (ChPT)
- $\eta \rightarrow 3\pi$: LO and NLO
- $\eta \rightarrow 3\pi$: dispersive: AL vs KWW vs previous talk
- $\eta \rightarrow 3\pi$: NNLO: the calculation and convergence
- $\eta \rightarrow 3\pi$: Comparison with experiment
- Conclusions

Note:

- Nonrelativistic EFT  Bissegger, Fuhrer, Gasser, Kubis, Rusetsky need dalitz plot as input.
- Perturbative dispersive: Zdrahal, Kampf, Knecht, Novotny good check, will be useful to combine ChPT and dispersive
Chiral Symmetry

QCD: 3 light quarks: equal mass: interchange: $U(3)_V$

But

$$\mathcal{L}_{QCD} = \sum_{q=u,d,s} \left[ i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q (\bar{q}_R q_L + \bar{q}_L q_R) \right]$$

So if $m_q = 0$ then $U(3)_L \times U(3)_R$.

Can also see that via $v < c, m_q \neq 0 \implies \quad v = c, m_q = 0 \implies$
Pseudoscalars are special

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But $\mathcal{L}_{QCD} = \sum_{q=u,d,s} [i\bar{q}_L \slashed{D} q_L + i\bar{q}_R \slashed{D} q_R - m_q (\bar{q}_R q_L + \bar{q}_L q_R)]$

So if $m_q = 0$ then $U(3)_L \times U(3)_R$.

- Hadrons do not come in parity doublets: symmetry must be broken
- A few very light hadrons: $\pi^0 \pi^+ \pi^-$ and also $K, \eta$
- Both can be understood from spontaneous Chiral Symmetry Breaking
Goldstone Modes

UNBROKEN: \( V(\phi) \)

Only massive modes around lowest energy state (=vacuum)

BROKEN: \( V(\phi) \)

Need to pick a vacuum
\( \langle \phi \rangle \neq 0 \): Breaks symmetry
No parity doublets
Massless mode along ridge

For QCD: \( \langle \phi \rangle \neq 0 \rightarrow \langle \bar{q}q \rangle \neq 0 \)

Explains why pions light

\[ U(3)_L \times U(3)_R \rightarrow U(3)_V \]
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$U(3)_L \times U(3)_R \rightarrow U(3)_V$

Explains why pions light but need NINE light particles

So WHY is the $\eta'$ NOT light?
Anomaly

\[ U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A \]

\[ SU(3)_L \times SU(3)_R \longrightarrow SU(3)_V \implies \pi, K, \eta \text{ light FINE} \]

\( U(1)_A \): Is NOT a good quantum symmetry

\[ \partial_\mu A^{0\mu} = 2\sqrt{N_f}\omega \]

\[ \omega = \frac{1}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} \text{tr} G_{\mu\nu} G_{\alpha\beta} \]

\( \omega \) is gluons: strongly interacting: \( \eta' \text{ heavy} \)
But

So quantum effects break $U(1)_A$

BUT $\omega$ is a total derivative $\Rightarrow$ How does it have an effect?
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So quantum effects break $U(1)_A$

BUT $\omega$ is a total derivative $\Rightarrow$ How does it have an effect?

’t Hooft:  
- winding number $\nu = \int d^4x \omega$
- instantons lead to an effect

Creates a new problem: $\mathcal{L}_{QCD} \longrightarrow \mathcal{L}_{QCD} - \theta \omega$

(Strong CP problem) BUT it solved the $\eta'$ problem

$\eta'$ has possibly large and very interesting nonperturbative effects and interaction with gluons as no other hadron

$m_s \neq \hat{m}$: This also affects $\eta$ physics
Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

$\eta \rightarrow \pi \pi \pi$ at two-loop order in ChPT
Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

Derivation from QCD:
Chiral Perturbation Theory

Degrees of freedom: Goldstone Bosons from Chiral Symmetry Spontaneous Breakdown (without $\eta'$)

Power counting: Dimensional counting in momenta/masses

Expected breakdown scale: Resonances, so $M_\rho$ or higher depending on the channel
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Power counting in momenta: Meson loops
Lagrangians

\[ U(\phi) = \exp(i\sqrt{2}\Phi / F_0) \] parametrizes Goldstone Bosons

\[ \Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ -\pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix} \].

LO Lagrangian: \[ \mathcal{L}_2 = \frac{F_0^2}{4}\left\{ \langle D_\mu U^\dagger D^\mu U \rangle + \langle \chi^\dagger U + \chi U^\dagger \rangle \right\} , \]

\[ D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu , \]

left and right external currents: \[ r(l)_\mu = v_\mu + (-) a_\mu \]

Scalar and pseudoscalar external densities: \[ \chi = 2B_0(s + ip) \]

quark masses via scalar density: \[ s = \mathcal{M} + \cdots \]

\[ \langle A \rangle = Tr_F(A) \]
Lagrangians

\[ \mathcal{L}_4 = L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle + L_3 \langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \rangle + L_4 \langle D^\mu U^\dagger D_\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle + L_5 \langle D^\mu U^\dagger D_\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2
\]

\[ + L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle - iL_9 \langle F^{R}_{\mu\nu} D_\mu U D^\nu U^\dagger + F^{L}_{\mu\nu} D_\mu U^\dagger D^\nu U \rangle + L_{10} \langle U^\dagger F^{R}_{\mu\nu} U F^{L}_{\mu\nu} \rangle + H_1 \langle F^{R}_{\mu\nu} F^{R}_{\mu\nu} + F^{L}_{\mu\nu} F^{L}_{\mu\nu} \rangle + H_2 \langle \chi^\dagger \chi \rangle \]

\[ L_i : \text{Low-energy-constants (LECs)} \]
\[ H_i : \text{Values depend on definition of currents/densities} \]

These absorb the divergences of loop diagrams: \( L_i \rightarrow L_i^r \)

Renormalization: order by order in the powercounting
Lagrangians

Lagrangian Structure:

<table>
<thead>
<tr>
<th>p^2</th>
<th>2 flavour</th>
<th>3 flavour</th>
<th>3+3 PQChPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>p^2</td>
<td>F, B</td>
<td>F_0, B_0</td>
<td>F_0, B_0</td>
</tr>
<tr>
<td>p^4</td>
<td>l^r_i, h^r_i</td>
<td>L^r_i, H^r_i</td>
<td>\hat{L}^r_i, \hat{H}^r_i</td>
</tr>
<tr>
<td>p^6</td>
<td>c^r_i</td>
<td>C^r_i</td>
<td>K^r_i</td>
</tr>
</tbody>
</table>

\[ p^2 : \text{Weinberg 1966} \]
\[ p^4 : \text{Gasser, Leutwyler 84,85} \]
\[ p^6 : \text{JB, Colangelo, Ecker 99,00} \]

\[ \Rightarrow \text{replica method} \rightarrow \text{PQ obtained from } N_F \text{ flavour} \]
\[ \Rightarrow \text{All infinities known} \]
\[ \Rightarrow \text{3 flavour special case of 3+3 PQ: } \hat{L}^r_i, K^r_i \rightarrow L^r_i, C^r_i \]
\[ \Rightarrow 53 \rightarrow 52 \text{ arXiv:0705.0576 [hep-ph]} \]
Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms

\[
m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\left(\frac{2B\hat{m}}{\mu^2}\right) + 2l_3^r(\mu)\right] + \cdots
\]

\[
M^2 = 2B\hat{m}
\]

\[B \neq B_0, \, F \neq F_0 \text{ (two versus three-flavour)}\]
LECs and $\mu$

$$l_3^r(\mu)$$

$$\bar{l}_i = \frac{32\pi^2}{\gamma_i} l_i^r(\mu) - \log \frac{M_\pi^2}{\mu^2}.$$  
Independent of the scale $\mu$.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

$\mu$:

- $m_\pi, m_K$: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$ large $N_c$ arguments????
- compromise: $\mu = m_\rho = 0.77$ GeV
Expand in what quantities?

- Expansion is in momenta and masses
- But is not unique: relations between masses (Gell-Mann–Okubo) exists
- Express orders in terms of physical masses and quantities ($F_\pi, F_K$)?
- Express orders in terms of lowest order masses?
- E.g. $s + t + u = 2m_\pi^2 + 2m_K^2$ in $\pi K$ scattering
- Relative sizes of order $p^2, p^2, p^4, \ldots$ can vary considerably
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- Relative sizes of order $p^2$, $p^2$, $p^4$, . . . can vary considerably
- I prefer physical masses
- Thresholds correct
- Chiral logs are from physical particles propagating
LECs

Some combinations of order $p^6$ LECs are known as well: curvature of the scalar and vector formfactor, two more combinations from $\pi\pi$ scattering (implicit in $b_5$ and $b_6$)

General observation:

- Obtainable from kinematical dependences: known
- Only via quark-mass dependence: poorly known
Most analysis use: $C^r_i$ from (single) resonance approximation

Motivated by large $N_c$: large effort goes in this

\[
\mathcal{L}_V = -\frac{1}{4} \langle V_{\mu\nu} V^{\mu\nu} \rangle + \frac{1}{2} m_V^2 \langle V_\mu V^\mu \rangle - \frac{f_V}{2\sqrt{2}} \langle V_{\mu\nu} f^{\mu\nu}_+ \rangle - \frac{ig_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + f_X \langle V_\mu [u^\mu, \chi^-] \rangle
\]

\[
\mathcal{L}_A = -\frac{1}{4} \langle A_{\mu\nu} A^{\mu\nu} \rangle + \frac{1}{2} m_A^2 \langle A_\mu A^\mu \rangle - \frac{f_A}{2\sqrt{2}} \langle A_{\mu\nu} f_{-\mu\nu} \rangle
\]

\[
\mathcal{L}_S = \frac{1}{2} \langle \nabla^\mu S \nabla_\mu S - M_S^2 S^2 \rangle + c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi^+ \rangle
\]

\[
\mathcal{L}_{\eta'} = \frac{1}{2} \partial_\mu P_1 \partial^\mu P_1 - \frac{1}{2} M_{\eta'}^2 P_1^2 + i \tilde{d}_m P_1 \langle \chi^- \rangle
\]

\[f_V = 0.20, \quad f_X = -0.025, \quad g_V = 0.09, \quad c_m = 42 \text{ MeV}, \quad c_d = 32 \text{ MeV}, \quad \tilde{d}_m = 20 \text{ MeV},\]

\[m_V = m_\rho = 0.77 \text{ GeV}, \quad m_A = m_{a_1} = 1.23 \text{ GeV}, \quad m_S = 0.98 \text{ GeV}, \quad m_{P_1} = 0.958 \text{ GeV}\]

\[f_V, g_V, f_X, f_A: \text{ experiment}\]

\[c_m \text{ and } c_d \text{ from resonance saturation at } O(p^4)\]
Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: $C_{r_i}^{\tau}$ in the masses/decay constants and how these effects correlate into the rest
- No $\mu$ dependence: obviously only estimate

What we do/did about it:

- Vary resonance estimate by factor of two
- Vary the scale $\mu$ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult
\[ \eta \to 3\pi \]


\[ s = (p_{\pi^+} + p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2 \]
\[ t = (p_{\pi^-} + p_{\pi^0})^2 = (p_\eta - p_{\pi^+})^2 \]
\[ u = (p_{\pi^+} + p_{\pi^0})^2 = (p_\eta - p_{\pi^-})^2 \]

\[ s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0 . \]

\[ \langle \pi^0\pi^+\pi^-|\eta \rangle = i (2\pi)^4 \delta^4 (p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}) A(s, t, u) . \]

\[ \langle \pi^0\pi^0\pi^0|\eta \rangle = i (2\pi)^4 \delta^4 (p_\eta - p_1 - p_2 - p_3) \overline{A}(s_1, s_2, s_3) \]

\[ \overline{A}(s_1, s_2, s_3) = A(s_1, s_2, s_3) + A(s_2, s_3, s_1) + A(s_3, s_1, s_2) , \]
η → 3π: Lowest order (LO)

Pions are in $I = 1$ state $\implies A \sim (m_u - m_d)$ or $\alpha_{em}$

- $\alpha_{em}$ effect is small (but large via $m_{\pi^+} - m_{\pi^0}$)
- $\eta \rightarrow \pi^+ \pi^- \pi^0 \gamma$ needs to be included directly
- Ditsche-Kubis-Meißner, Baur-Kambor-Wyler
\[ \eta \rightarrow 3\pi: \text{ Lowest order (LO)} \]

Pions are in \( I = 1 \) state \( \implies A \sim (m_u - m_d) \text{ or } \alpha_{em} \)

ChPT: Cronin 67: 
\[
A(s, t, u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F^2_\pi} \left\{ 1 + \frac{3(s - s_0)}{m^2_\eta - m^2_\pi} \right\}
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with $Q^2 \equiv \frac{m_\pi^2 - \hat{m}^2}{m_d^2 - m_u^2}$ or $R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$ \quad $\hat{m} = \frac{1}{2}(m_u + m_d)$

$A(s, t, u) = \frac{1}{Q^2} \frac{m_K^2}{m_\pi^2} (m_\pi^2 - m_K^2) \frac{M(s, t, u)}{3\sqrt{3}F^2_\pi}$,

$A(s, t, u) = \frac{\sqrt{3}}{4R} M(s, t, u)$
\( \eta \rightarrow 3\pi: \) Lowest order (LO)

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\( \hat{m} = \frac{1}{2}(m_u + m_d) \)

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\]

\[
A(s, t, u) = \frac{\sqrt{3}}{4R} M(s, t, u)
\]

LO: 

\[
M(s, t, u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2} \quad M(s, t, u) = \frac{1}{F_\pi^2}\left( \frac{4}{3}m_\pi^2 - s \right)
\]
\[ \eta \to 3\pi \text{ beyond } p^4: p^2 \text{ and } p^4 \]

\[ \Gamma (\eta \to 3\pi) \propto |A|^2 \propto Q^{-4} \]

allows a PRECISE measurement

\[ Q \approx 24 \]

gives lowest order \( \Gamma (\eta \to \pi^+ \pi^- \pi^0) \approx 66 \text{ eV} \).
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Other Source from \[ m_{K^+}^2 - m_{K^0}^2 \sim Q^{-2} \implies Q = 20.0 \pm 1.5 \]

Lowest order prediction \[ \Gamma (\eta \rightarrow \pi^+\pi^-\pi^0) \approx 140 \text{ eV} . \]
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At order \( p^4 \) Gasser-Leutwyler 1985:

\[
\frac{\int dLIPS |A_2 + A_4|^2}{\int dLIPS |A_2|^2} = 2.4 ,
\]

\( (LIPS=\text{Lorentz invariant phase-space}) \)

**Major source:** large \( S \)-wave final state rescattering

**Experiment:** \( 295 \pm 16 \text{ eV} \) (PDG 2008)
\[ \eta \rightarrow 3\pi \text{ beyond } p^4: \text{ Dispersive} \]

Try to resum the \( S \)-wave rescattering:

Anisovich-Leutwyler (AL), Kambor, Wiesendanger, Wyler (KWW)

Different method but similar approximations
Here: simplified version of AL

Up to \( p^8 \): No absorptive parts from \( \ell \geq 2 \)

\[ \Rightarrow M(s, t, u) = M_0(s) + (s - u)M_1(t) + (s - t)M_1(t) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s) \]

\( M_I \): “roughly” contributions with isospin 0, 1, 2
$\eta \rightarrow 3\pi$ beyond $p^4$: Dispersive

3 body dispersive: difficult: keep only 2 body cuts

start from $\pi\eta \rightarrow \pi\pi$ ($m_{\eta}^2 < 3m_{\pi}^2$) standard dispersive analysis

analytically continue to physical $m_{\eta}^2$.

\[
M_I(s) = \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \frac{\text{Im} M_I(s')}{s' - s - i\varepsilon}
\]

\[
\text{Im} M_I(s')\longrightarrow \text{disc} M_I(s) = \frac{1}{2i} (M_I(s + i\varepsilon) - M_I(s - i\varepsilon))
\]

\[
M_0(s) = a_0 + b_0 s + c_0 s^2 + \frac{s^3}{\pi} \int \frac{ds'}{s'^3} \frac{\text{disc} M_0(s')}{s' - s - i\varepsilon},
\]

\[
M_1(s) = a_1 + b_1 s + s^2 \pi \int \frac{ds'}{s'^2} \frac{\text{disc} M_1(s')}{s' - s - i\varepsilon},
\]

\[
M_2(s) = a_2 + b_2 s + c_2 s^2 + \frac{s^3}{\pi} \int \frac{ds'}{s'^3} \frac{\text{disc} M_2(s')}{s' - s - i\varepsilon}.
\]
\[ \eta \to 3\pi \text{ beyond } p^4 \]

- Technical complications in solving
- Only 4 relevant constants:
  \[ M(s, t, u) = a + bs + cs^2 - d(s^2 + tu) \]

\[ M_0(s) + \frac{4}{3} M_2(s) s M_1(s) + M_2(s) + s^2 \frac{4L_3 - 1/(64\pi^2)}{F_{\pi}^2(m_\eta^2 - m_\pi^2)} \]

Converge better

\[ c = c_0 + \frac{4}{3} c_2 = \frac{1}{\pi} \int \frac{ds'}{s'^3} \left\{ \text{disc} M_0(s') + \frac{4}{3} \text{disc} M_2(s') \right\} , \]
\[ d = -\frac{4L_3 - 1/(64\pi^2)}{F_{\pi}^2(m_\eta^2 - m_\pi^2)} + \frac{1}{\pi} \int \frac{ds'}{s'^3} \left\{ s' \text{disc} M_1(s') + \text{disc} M_2(s') \right\} \]

Fix \( a, b \) by matching to tree level or \( p^4 \) amplitude
\[ \eta \rightarrow 3\pi \text{ beyond } \rho^4 \]

Along \( s = u \) KWW

Along \( s = u \) AL

Adler zero
Two Loop Calculation: why

- In $K_{\ell 4}$ dispersive gave about half of $p^6$ in amplitude
- Same order in ChPT as masses for consistency check on $m_u/m_d$
- Check size of 3 pion dispersive part
- At order $p^4$ unitarity about half of correction
- Technology exists:
  - Dealing with the mixing $\pi^0$-$\eta$: Amorós, JB, Dhonte, Talavera 01
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  - Two-loops: Amorós, JB, Dhonte, Talavera, ...
  - Dealing with the mixing $\pi^0-\eta$:
    - Amorós, JB, Dhonte, Talavera 01
  - Dealing with the mixing $\pi^0-\eta$: extended to $\eta \to 3\pi$
Include mixing, renormalize, pull out factor $\sqrt{3} / 4R$, ...  
Two independent calculations (comparison major amount of work)
$x$ variation: vertical

$y$ variation: parallel to $t = u$
$\eta \to 3\pi$: $M(s, t = u)$

Along $t = u$

Along $t = u$ parts
\( \eta \to 3\pi: M(s, t = u) \)

Along \( t = u \)

\( L_i = C_i = 0 \)

Along \( t = u \): \( \mu \) dependence

I.e. where \( C_i^r(\mu) \) estimated
η → 3π: \( M(s = u, t) \)

Shape agrees with AL

Correction larger: 20-30% in amplitude

Along \( s = u \)
Adler zero along $s = u$

Does move around

Along $s = u$
Dalitz plot

\[ x = \sqrt{3} \frac{T_+ - T_-}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta} (u - t) \]

\[ y = \frac{3T_0}{Q_\eta} - 1 = \frac{3 ((m_\eta - m_{\pi^0})^2 - s)}{2m_\eta Q_\eta} - 1 \overset{\text{iso}}{=} \frac{3}{2m_\eta Q_\eta} (s_0 - s) \]

\[ Q_\eta = m_\eta - 2m_{\pi^+} - m_{\pi^0} \]

\( T^i \) is the kinetic energy of pion \( \pi^i \)

\[ z = \frac{2}{3} \sum_{i=1,3} \left( \frac{3E_i - m_\eta}{m_\eta - 3m_{\pi^0}} \right)^2 \]

\( E_i \) is the energy of pion \( \pi^i \)

\[ |M|^2 = A_0^2 \left( 1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \cdots \right) \]

\[ |\overline{M}|^2 = \overline{A}_0^2 \left( 1 + 2\alpha z + \cdots \right) \]
## Experiment: charged

<table>
<thead>
<tr>
<th>Exp.</th>
<th>a</th>
<th>b</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>KLOE</td>
<td>$-1.090 \pm 0.005^{+0.008}_{-0.019}$</td>
<td>$0.124 \pm 0.006 \pm 0.010$</td>
<td>$0.057 \pm 0.006^{+0.007}_{-0.016}$</td>
</tr>
<tr>
<td>Crystal Barrel</td>
<td>$-1.22 \pm 0.07$</td>
<td>$0.22 \pm 0.11$</td>
<td>$0.06 \pm 0.04$</td>
</tr>
<tr>
<td>Layter et al.</td>
<td>$-1.08 \pm 0.014$</td>
<td>$0.034 \pm 0.027$</td>
<td>$0.046 \pm 0.031$</td>
</tr>
<tr>
<td>Gormley et al.</td>
<td>$-1.17 \pm 0.02$</td>
<td>$0.21 \pm 0.03$</td>
<td>$0.06 \pm 0.04$</td>
</tr>
</tbody>
</table>

KLOE has: $f = 0.14 \pm 0.01 \pm 0.02$.

Crystal Barrel: $d$ input, but $a$ and $b$ insensitive to $d$
### Theory: charged

<table>
<thead>
<tr>
<th></th>
<th>$A_0^2$</th>
<th>a</th>
<th>b</th>
<th>d</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LO</strong></td>
<td>120</td>
<td>−1.039</td>
<td>0.270</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td><strong>NLO</strong></td>
<td>314</td>
<td>−1.371</td>
<td>0.452</td>
<td>0.053</td>
<td>0.027</td>
</tr>
<tr>
<td><strong>NLO ($L^r_i = 0$)</strong></td>
<td>235</td>
<td>−1.263</td>
<td>0.407</td>
<td>0.050</td>
<td>0.015</td>
</tr>
<tr>
<td><strong>NNLO</strong></td>
<td>538</td>
<td>−1.271</td>
<td>0.394</td>
<td>0.055</td>
<td>0.025</td>
</tr>
<tr>
<td><strong>NNLOp ($y$ from $T^0$)</strong></td>
<td>574</td>
<td>−1.229</td>
<td>0.366</td>
<td>0.052</td>
<td>0.023</td>
</tr>
<tr>
<td><strong>NNLOq (incl $(x, y)^4$)</strong></td>
<td>535</td>
<td>−1.257</td>
<td>0.397</td>
<td>0.076</td>
<td>0.004</td>
</tr>
<tr>
<td><strong>NNLO ($\mu = 0.6$ GeV)</strong></td>
<td>543</td>
<td>−1.300</td>
<td>0.415</td>
<td>0.055</td>
<td>0.024</td>
</tr>
<tr>
<td><strong>NNLO ($\mu = 0.9$ GeV)</strong></td>
<td>548</td>
<td>−1.241</td>
<td>0.374</td>
<td>0.054</td>
<td>0.025</td>
</tr>
<tr>
<td><strong>NNLO ($C^r_i = 0$)</strong></td>
<td>465</td>
<td>−1.297</td>
<td>0.404</td>
<td>0.058</td>
<td>0.032</td>
</tr>
<tr>
<td><strong>NNLO ($L^r_i = C^r_i = 0$)</strong></td>
<td>251</td>
<td>−1.241</td>
<td>0.424</td>
<td>0.050</td>
<td>0.007</td>
</tr>
<tr>
<td>dispersive (KWW)</td>
<td>—</td>
<td>−1.33</td>
<td>0.26</td>
<td>0.10</td>
<td>—</td>
</tr>
<tr>
<td>tree dispersive</td>
<td>—</td>
<td>−1.10</td>
<td>0.33</td>
<td>0.001</td>
<td>—</td>
</tr>
<tr>
<td>absolute dispersive</td>
<td>—</td>
<td>−1.21</td>
<td>0.33</td>
<td>0.04</td>
<td>—</td>
</tr>
<tr>
<td>error</td>
<td>18</td>
<td>0.075</td>
<td>0.102</td>
<td>0.057</td>
<td>0.160</td>
</tr>
</tbody>
</table>

**NLO to NNLO:**

Little change

**Error on**

$$|M(s, t, u)|^2:$$

$$|M^{(6)} M(s, t, u)|$$
### Experiment: neutral

<table>
<thead>
<tr>
<th>Exp.</th>
<th>$\alpha$</th>
<th>$\bar{A}_0^2$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crystal Ball (MAMI C)</td>
<td>$-0.032 \pm 0.003$</td>
<td>1090</td>
<td>0.000</td>
</tr>
<tr>
<td>Crystal Ball (MAMI B)</td>
<td>$-0.032 \pm 0.002 \pm 0.002$</td>
<td>2810</td>
<td>0.013</td>
</tr>
<tr>
<td>WASA/COSY</td>
<td>$-0.027 \pm 0.008 \pm 0.005$</td>
<td>2100</td>
<td>0.016</td>
</tr>
<tr>
<td>KLOE 2007</td>
<td>$-0.027 \pm 0.004 \pm 0.006$</td>
<td>4790</td>
<td>0.013</td>
</tr>
<tr>
<td>KLOE (prel)</td>
<td>$-0.014 \pm 0.005 \pm 0.004$</td>
<td>4790</td>
<td>0.014</td>
</tr>
<tr>
<td>Crystal Ball (BNL)</td>
<td>$-0.031 \pm 0.004$</td>
<td>4140</td>
<td>0.011</td>
</tr>
<tr>
<td>WASA/CELSIUS</td>
<td>$-0.026 \pm 0.010 \pm 0.010$</td>
<td>2220</td>
<td>0.016</td>
</tr>
<tr>
<td>Crystal Barrel</td>
<td>$-0.052 \pm 0.017 \pm 0.010$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>GAMS2000</td>
<td>$-0.022 \pm 0.023$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>SND</td>
<td>$-0.010 \pm 0.021 \pm 0.010$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>NLO ($L_i^r = 0$)</td>
<td></td>
<td>2100</td>
<td>0.016</td>
</tr>
<tr>
<td>NNLO</td>
<td></td>
<td>4790</td>
<td>0.013</td>
</tr>
<tr>
<td>NNLO ($C_i^r = 0$)</td>
<td></td>
<td>4140</td>
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</tr>
<tr>
<td>NNLO ($L_i^r = C_i^r = 0$)</td>
<td></td>
<td>2220</td>
<td>0.016</td>
</tr>
<tr>
<td>dispersive (KWW)</td>
<td></td>
<td>—</td>
<td>$(0.007-0.014)$</td>
</tr>
<tr>
<td>tree dispersive</td>
<td></td>
<td>—</td>
<td>$-0.0065$</td>
</tr>
<tr>
<td>absolute dispersive</td>
<td></td>
<td>—</td>
<td>$-0.007$</td>
</tr>
<tr>
<td>Borasoy</td>
<td></td>
<td>—</td>
<td>$-0.031$</td>
</tr>
<tr>
<td>error</td>
<td></td>
<td>160</td>
<td>0.032</td>
</tr>
</tbody>
</table>

Note: NNLO ChPT gets $a_0^0$ in $\pi\pi$ correct.
\( \alpha \) is difficult

Expand amplitudes and isospin:

\[
M(s, t, u) = A \left( 1 + \tilde{a}(s - s_0) + \tilde{b}(s - s_0)^2 + \tilde{d}(u - t)^2 + \cdots \right)
\]

\[
\overline{M}(s, t, u) = A \left( 3 + (\tilde{b} + 3\tilde{d}) \left( (s - s_0)^2 + (t - s_0)^2 + (u - s_0)^2 \right) \right) + \cdots
\]

Gives relations \((R_\eta = (2m_\eta Q_\eta)/3)\)

\[
a = -2R_\eta \text{Re}(\tilde{a}), \quad b = R^2_\eta \left( |\tilde{a}|^2 + 2\text{Re}(\tilde{b}) \right), \quad d = 6R^2_\eta \text{Re}(\tilde{d}) \cdot
\]

\[
\alpha = \frac{1}{2}R^2_\eta \text{Re} \left( \tilde{b} + 3\tilde{d} \right) = \frac{1}{4} \left( d + b - R^2_\eta |\tilde{a}|^2 \right) \leq \frac{1}{4} \left( d + b - \frac{1}{4}a^2 \right)
\]

equality if \(\text{Im}(\tilde{a}) = 0\)

Large cancellation in \(\alpha\), overestimate of \(b\) likely the problem
\( r \) and decay rates

\[
\sin \epsilon = \frac{\sqrt{3}}{4R} + \mathcal{O}(\epsilon^2)
\]

\[
\Gamma(\eta \to \pi^+\pi^-\pi^0) = \sin^2 \epsilon \cdot 0.572 \text{ MeV} \quad \text{LO,}
\]
\[
\sin^2 \epsilon \cdot 1.59 \text{ MeV} \quad \text{NLO},
\]
\[
\sin^2 \epsilon \cdot 2.68 \text{ MeV} \quad \text{NNLO},
\]
\[
\sin^2 \epsilon \cdot 2.33 \text{ MeV} \quad \text{NNLO } C_i^r = 0,
\]

\[
\Gamma(\eta \to \pi^0\pi^0\pi^0) = \sin^2 \epsilon \cdot 0.884 \text{ MeV} \quad \text{LO,}
\]
\[
\sin^2 \epsilon \cdot 2.31 \text{ MeV} \quad \text{NLO},
\]
\[
\sin^2 \epsilon \cdot 3.94 \text{ MeV} \quad \text{NNLO,}
\]
\[
\sin^2 \epsilon \cdot 3.40 \text{ MeV} \quad \text{NNLO } C_i^r = 0.
\]
$r$ and decay rates

\[ r \equiv \frac{\Gamma(\eta \to \pi^0\pi^0\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)} \]

\[ r_{\text{LO}} = 1.54 \]

\[ r_{\text{NLO}} = 1.46 \]

\[ r_{\text{NNLO}} = 1.47 \]

\[ r_{\text{NNLO}\ C_i^r=0} = 1.46 \]

PDG 2008

\[ r = 1.48 \pm 0.05 \quad \text{our average} . \]

\[ r = 1.432 \pm 0.026 \quad \text{our fit} , \]

Good agreement
### $R$ and $Q$

<table>
<thead>
<tr>
<th></th>
<th>LO</th>
<th>NLO</th>
<th>NNLO</th>
<th>NNLO ($C_i^T = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ ($\eta$)</td>
<td>19.1</td>
<td>31.8</td>
<td>42.2</td>
<td>38.7</td>
</tr>
<tr>
<td>$R$ (Dashen)</td>
<td>44</td>
<td>44</td>
<td>37</td>
<td>—</td>
</tr>
<tr>
<td>$R$ (Dashen-violation)</td>
<td>36</td>
<td>37</td>
<td>32</td>
<td>—</td>
</tr>
<tr>
<td>$Q$ ($\eta$)</td>
<td>15.6</td>
<td>20.1</td>
<td>23.2</td>
<td>22.2</td>
</tr>
<tr>
<td>$Q$ (Dashen)</td>
<td>24</td>
<td>24</td>
<td>22</td>
<td>—</td>
</tr>
<tr>
<td>$Q$ (Dashen-violation)</td>
<td>22</td>
<td>22</td>
<td>20</td>
<td>—</td>
</tr>
</tbody>
</table>

**LO from**

$$R = \frac{m_{K^0}^2 + m_{K^+}^2 - 2m_{\pi^0}^2}{2 \left( m_{K^0}^2 - m_{K^+}^2 \right)} \quad \text{(QCD part only)}$$

**NLO and NNLO from masses:** Amorós, JB, Talavera 2001

$$Q^2 = \frac{m_s + \hat{m}}{2\hat{m}} R = 12.7 R \quad \text{($m_s/\hat{m} = 24.4$)}$$
Conclusions

- High quality distributions are a must: strong constraints on theory
- $\eta \rightarrow 3\pi$: needed for $m_u - m_d$
- NNLO ChPT: fails to reproduce distributions
- NNLO ChPT: reproduces $\pi\pi$ scattering lengths
Conclusions

- High quality distributions are a must: strong constraints on theory
- $\eta \to 3\pi$: needed for $m_u - m_d$
- NNLO ChPT: fails to reproduce distributions
- NNLO ChPT: reproduces $\pi\pi$ scattering lengths

In progress, planned, wished:
- new fit of the $L_i^r$: JB, Jemos
- combining dispersive and NNLO ChPT