ISOSPIN BREAKING AT ORDER $p^6$

in $K\ell_3$ DECAYS

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Various ChPT: http://www.thep.lu.se/~bijnens/chpt.html
Overview

- Why $K_{\ell 3}$ (and similar things)?
- List of papers or earlier results
- Formfactor relations
- Some aspects of the calculation
- Numerical results
  - $f_+(0)$
  - $f_+(t), r_{0-}(t)$
  - $f_0(t), r_{0-}^0(t)$
  - $\Delta CT$
Why look at $K_{\ell 3}$?

\[
\Gamma \left( K^i \rightarrow \pi^j \ell^+ \nu_\ell \right) = \\
C_{ij}^2 \frac{G_F^2 S_{EW} m_K^5}{192 \pi^3} \left| V_{us} f_{+}^{K^i \pi^j}(0) \right|^2 \mathcal{I}_{\ell}^{ij} \left( 1 + 2 \Delta_{EM}^{ij} \right).
\]

$\mathcal{I}_{\ell}^{ij}$: depends on measured formfactors
In principle well known

Similar formulas exist for $K \rightarrow \pi \nu \bar{\nu}$
\[ \langle \pi^0 (p') | \bar{s} \gamma_\mu u(0) | K^+(p) \rangle = \frac{1}{\sqrt{2}} \left[ (p' + p)_\mu f_+^{K^0 \pi^0}(t) + (p - p')_\mu f_-^{K^0 \pi^0}(t) \right] \]

\[ \langle \pi^- (p') | \bar{s} \gamma_\mu u(0) | K^0(p) \rangle = \left[ (p' + p)_\mu f_+^{K^0 \pi^-}(t) + (p - p')_\mu f_-^{K^0 \pi^-}(t) \right] \]

\[ \langle \pi^+ (p') | \bar{s} \gamma_\mu d(0) | K^+(p) \rangle = \left[ (p' + p)_\mu f_+^{K^+ \pi^+}(t) + (p - p')_\mu f_-^{K^+ \pi^+}(t) \right] \]

\[ \langle \pi^0 (p') | \bar{s} \gamma_\mu d(0) | K^0(p) \rangle = \frac{-1}{\sqrt{2}} \left[ (p' + p)_\mu f_+^{K^0 \pi^0}(t) + (p - p')_\mu f_-^{K^0 \pi^0}(t) \right] \]

8 form-factors, \( t = (p' - p)^2 \)

\[ f_+^{K^i \pi^j}(0) = 1 \text{ in the } SU(3) \text{ limit } m_u = m_d = m_s \]

Isospin limit: \( f_\pm = f_\pm^{K^0} = f_\pm^{K^+ \pi^0} = f_\pm^{K^0 \pi^-} = f_\pm^{K^+ \pi^+} = f_\pm^{K^0 \pi^0} \)
$K_{\ell 3}$ Definitions

Scalar formfactor: 
\[ f_0(t) = f_+(t) + \frac{t}{m_{K^i}^2 - m_{\pi^j}^2} f_-(t) \]

\[ f_{l}^{K^i \pi^j} = LO + NLO + NNLO + \cdots \]

\[ \delta = m_u - m_d, \quad e^2 \text{ and } p^2 \text{ expansion} \]

\[ f_{l=+,0} = 1 + \delta + \delta p^2 + e^2 p^2 + \delta p^4 + \cdots \]
Definitions

Scalar formfactor:

\[ f_0(t) = f_+(t) + \frac{t}{m_{K^i}^2 - m_{\pi^j}^2} f_-(t) \]

\[ f^{K^i\pi^j}_l = LO + NLO + NNLO + \cdots \]

\[ \delta = m_u - m_d, \ e^2 \ and \ p^2 \ expansion \]

\[ f_{l=+,0} = 1 + \delta + \delta p^2 + e^2 p^2 + \delta p^4 + \cdots \]

LO is also tree diagrams from \( p^2 \)-Lagrangian: \( p^2 \)

NLO is also tree diagrams from \( p^4 \)-Lagrangian: \( p^4 \)

NNLO is also tree diagrams from \( p^6 \)-Lagrangian: \( p^6 \)
$K_{\ell 3}$: (subset of) papers


- V. Cirigliano et al., hep-ph/0503108, JHEP 0504 (2005) 006 $C_i p^4$
$K_{\ell 3}$: (subset of) papers

- F. Mescia and C. Smith, arXiv:0705.2025 $f_+(t) \delta p^2, e^2 p^2$
- J. Bijnens and K. Ghorbani, arXiv:0711.0148 $f_{+,0}(t) \delta p^4$
MS (and us) noticed:

\[ f_\ell K^+\pi^0(t) - f_\ell K^0\pi^-(t) - f_\ell K^+\pi^0(t) + f_\ell K^0\pi^0(t) = 0 + \mathcal{O}(\delta^2), \]

\[ r(t) \equiv \frac{f_\ell K^+\pi^0(t) f_\ell K^0\pi^0(t)}{f_\ell K^0\pi^-(t) f_\ell K^+\pi^+(t)} = 1 + \mathcal{O}(\delta^2) \]

GL noticed:

\[ r_{0-}(t) = \frac{f^+_K\pi^0(t)}{f^+_K\pi^-(t)} = \text{constant} \]
Formfactor relations

\[ r(t) = 1 + \delta^2 \] is always true, also for \( r^-(t) \) and \( r^0(t) \)

Vector operator is \( I = 1/2 \)
\[(m_u - m_d)(\bar{u}u - \bar{d}d) \] is \( I = 1 \)

\[
\begin{align*}
  f_{K^+\pi^0}(t) &= f_A^{\ell}(t) + \delta f_B^{\ell}(t) + O(\delta^2), \\
  f_{K^0\pi^-}(t) &= f_A^{\ell}(t) - \delta f_D^{\ell}(t) + O(\delta^2), \\
  f_{K^+\pi^+}(t) &= f_A^{\ell}(t) + \delta f_D^{\ell}(t) + O(\delta^2), \\
  f_{K^0\pi^0}(t) &= f_A^{\ell}(t) - \delta f_B^{\ell}(t) + O(\delta^2),
\end{align*}
\]

Photon exchange has \( I = 0, 1, 2 \), \( I = 2 \) part breaks relation

Relation also true for scalar currents \( \bar{s}u, \bar{s}d \)
Formfactor relations

\[ r_{0-}(t) \text{ constant not true at } \delta p^4 \text{ (MS: } C_i, \text{ BG also the rest)} \]

\[ r_{0-}^0(t), r_{0-}^-(t) \text{ not true at } \delta p^2 \]
New for isospin breaking

Take LSZ into account properly

Amoros, JB, Talavera, 2001

Matrix element:

\[ A_{i_1\ldots i_n} = \left( \frac{(-i)^n}{\sqrt{Z_{i_1}\ldots Z_{i_n}}} \right) \prod_{i=1}^{n} \lim_{k_i^2 \to m_i^2} \left( k_i^2 - m_i^2 \right) G_{i_1\ldots i_n}(k_1, \ldots, k_n) \]
Deal with the complications of mixing via matrices/determinant

\[ A_3 = G_3^{(2)} + \left\{ G_3^{(4)} - \frac{1}{2} Z_{33}^{(4)} G_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m^2} G_8^{(2)} \right\} \]

\[ + \left[ G_3^{(6)} - \frac{1}{2} Z_{33}^{(6)} G_3^{(2)} - \frac{1}{2} Z_{33}^{(4)} G_3^{(4)} + \frac{3}{8} \left( Z_{33}^{(4)} \right)^2 G_3^{(2)} \right] \]

\[ + \frac{Z_{33}^{(4)} \Pi_{38}^{(4)}}{\Delta m^2} G_3^{(2)} - \frac{1}{2} \left( \frac{\Pi_{38}^{(4)}}{\Delta m^2} \right)^2 G_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m^2} G_8^{(4)} \]

\[ - \frac{\Pi_{38}^{(6)}}{\Delta m^2} G_8^{(2)} + \frac{\Pi_{38}^{(4)} \Pi_{88}^{(4)}}{\Delta m^2} G_8^{(2)} + \frac{1}{2} Z_{33}^{(4)} \frac{\Pi_{38}^{(4)}}{\Delta m^2} G_8^{(2)} \]
Diagrams

\[ \bullet : p^2 \text{ vertex} \]
\[ \times : p^4 \text{ vertex} \]
\[ \otimes : p^6 \text{ vertex} \]
\[ f_+(0) : \text{NO isospin violation} \]

<table>
<thead>
<tr>
<th></th>
<th>(f_{+}K^+\pi^0)</th>
<th>(f_{+}K^0\pi^-)</th>
<th>(f_{+}K^+\pi^+)</th>
<th>(f_{+}K^0\pi^0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order (p^2)</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>order (p^4)</td>
<td>−0.02276</td>
<td>−0.02266</td>
<td>−0.02226</td>
<td>−0.02316</td>
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<tr>
<td>order (p^6)</td>
<td>0.01423</td>
<td>0.01462</td>
<td>0.01406</td>
<td>0.01480</td>
</tr>
<tr>
<td>(p^6) 2-loop</td>
<td>0.01104</td>
<td>0.01130</td>
<td>0.01090</td>
<td>0.01145</td>
</tr>
<tr>
<td>(p^6) (L^r_i)</td>
<td>0.00320</td>
<td>0.00332</td>
<td>0.00316</td>
<td>0.00336</td>
</tr>
<tr>
<td>sum</td>
<td>0.99156</td>
<td>0.99196</td>
<td>0.99180</td>
<td>0.99164</td>
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</tbody>
</table>

JB-Talavera, JB-Ghorbani

The \(C^r_i\) contribution: nothing new here and not included
\[ f_+(0): \frac{m_u}{m_d} = 0.585 (\text{NLO}) \]

\( \frac{m_u}{m_d} \) NLO, Dashen’s theorem

<table>
<thead>
<tr>
<th></th>
<th>( f_{+}^{K^+\pi^0} )</th>
<th>( f_{+}^{K^0\pi^-} )</th>
<th>( f_{+}^{K^+\pi^+} )</th>
<th>( f_{+}^{K^0\pi^0} )</th>
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<tbody>
<tr>
<td>order ( p^2 )</td>
<td>1.01702</td>
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<td>order ( p^4 )</td>
<td>-0.01931</td>
<td>-0.02282</td>
<td>-0.02202</td>
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<tr>
<td>order ( p^6 )</td>
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<td>0.01467</td>
<td>0.01395</td>
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<tr>
<td>( p^6 ) 2-loop</td>
<td>0.00435</td>
<td>0.01142</td>
<td>0.01084</td>
<td>0.01815</td>
</tr>
<tr>
<td>( p^6 ) ( L_i^r )</td>
<td>0.00551</td>
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<td>0.00311</td>
<td>0.00104</td>
</tr>
<tr>
<td>sum</td>
<td>1.00757</td>
<td>0.99186</td>
<td>0.99193</td>
<td>0.97532</td>
</tr>
</tbody>
</table>

\( \sin \epsilon = 0.00986 \) JB-Ghorbani

The \( C_i^r \) contribution: not included
$f_+(0): m_u/m_d = 0.45(\text{NNLO})$

NNLO and violations of Dashen's theorem

<table>
<thead>
<tr>
<th>Order</th>
<th>$f_+^{K^+\pi^0}$</th>
<th>$f_+^{K^0\pi^-}$</th>
<th>$f_+^{K^+\pi^+}$</th>
<th>$f_+^{K^0\pi^0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^2$</td>
<td>1.02465</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.97514</td>
</tr>
<tr>
<td>$p^4$</td>
<td>-0.01775</td>
<td>-0.02292</td>
<td>-0.02197</td>
<td>-0.02838</td>
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<tr>
<td>$p^6$</td>
<td>0.00809</td>
<td>0.01470</td>
<td>0.01391</td>
<td>0.02095</td>
</tr>
<tr>
<td>$p^6$ 2-loop</td>
<td>0.00159</td>
<td>0.01145</td>
<td>0.01081</td>
<td>0.02092</td>
</tr>
<tr>
<td>$p^6$ $L^r_i$</td>
<td>0.00650</td>
<td>0.00325</td>
<td>0.00309</td>
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<tr>
<td>Sum</td>
<td>1.01499</td>
<td>0.99177</td>
<td>0.99194</td>
<td>0.96772</td>
</tr>
</tbody>
</table>

$\sin \epsilon = 0.0143$ JB-Ghorbani

The $C^r_i$ contribution: not included

$C^r_i$ from $\eta'$: $f_+^{K^+\pi^0}(0) \big|_{P_1} = 0.00065$ $f_+^{K^0\pi^0}(0) \big|_{P_1} = -0.00065$
\[ f_+(0) \]

\[ r_{0-}(0) = 1.02465 + 0.00587 - 0.00711 = 1.02341 , \]

Palutan: KAON07

\[ r_{0-exp} = 1 + \Delta_{SU(2)} = 1.0284 \pm 0.0040 . \]

As we see, we obtain a reasonable agreement.
For $K^0 \rightarrow \pi^- \ell^+ \nu$ numerically very little change,
\( t\)-dependence: \( p^6 \)

For \( K^0 \rightarrow \pi^- \ell^+ \nu \) numerically very little change,
**t-dependence:** $r_{0-}(t)$
For $K^0 \rightarrow \pi^- \ell^+ \nu$ numerically very little change,
$t$-dependence: $r_{0-}(t)$
\( f_0(t) \) Isospin limit

Old Main Result: JB-Talavera

\[
f_0(t) = 1 - \frac{8}{F_\pi^4} \left( C_{12}^r + C_{34}^r \right) (m_K^2 - m_\pi^2)^2 \\
+8 \frac{t}{F_\pi^4} \left( 2C_{12}^r + C_{34}^r \right) (m_K^2 + m_\pi^2) + \frac{t}{m_K^2 - m_\pi^2} (F_K/F_\pi - 1) \\
- \frac{8}{F_\pi^4} t^2 C_{12}^r + \overline{\Delta}(t) + \Delta(0).
\]

\( \overline{\Delta}(t) \) and \( \Delta(0) \) contain NO \( C_i^r \) and only depend on the \( L_i^r \) at order \( p^6 \) \( \implies \)

All needed parameters can be determined experimentally

Numerically definitely OK, relation, still checking
Callan-Treiman point

Callan-Treiman: \[ f_0 \left( m_K^2 - m_{\pi}^2 \right) = \frac{F_K}{F_\pi} + \mathcal{O}(m_u, m_d) . \]

\[ \Delta_{CT} \equiv f_0 \left( m_K^2 - m_\pi^2 \right) - \frac{F_K}{F_\pi} = -3.5 \times 10^{-3} . (\text{GL, iso}) \]

\[ \Delta_{CT} = -6.2 \times 10^{-3} . \text{ NNLO using BT formulas} \]

With \( \delta p^4 \) Calculated with \( F_K / F_\pi = 1.22 \)

\[ \Delta_{CT}^{K^+\pi^0} = 15.1 \times 10^{-3} , \]
\[ \Delta_{CT}^{K^0\pi^-} = -5.6 \times 10^{-3} , \]
\[ \Delta_{CT}^{K^+\pi^+} = -9.4 \times 10^{-3} , \]
\[ \Delta_{CT}^{K^0\pi^0} = -26.4 \times 10^{-3} . \]
Most analysis use: $C^r_i$ from (single) resonance approximation

Motivated by large $N_c$: large effort goes in this

Ananthanarayan, JB, Cirigliano, Donoghue, Ecker, Gamiz, Golterman, Kaiser, Knecht, Peris, Pich, Prades, Portoles, de Rafael, ...
\[ \mathcal{L}_V = -\frac{1}{4} \langle V_{\mu\nu} V_{\mu\nu} \rangle + \frac{1}{2} m_V^2 \langle V_\mu V_\mu \rangle - \frac{f_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle \\
- \frac{ig_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + f_\chi \langle V_\mu [u^\mu, \chi^-] \rangle \]

\[ \mathcal{L}_A = -\frac{1}{4} \langle A_{\mu\nu} A^{\mu\nu} \rangle + \frac{1}{2} m_A^2 \langle A_\mu A^\mu \rangle - \frac{f_A}{2\sqrt{2}} \langle A_{\mu\nu} f_+^{\mu\nu} \rangle \]

\[ \mathcal{L}_S = \frac{1}{2} \langle \nabla_\mu S \nabla_\mu S - M_S^2 S^2 \rangle + c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi^- \rangle \]

\[ \mathcal{L}_{\eta'} = \frac{1}{2} \partial_\mu P_1 \partial^\mu P_1 - \frac{1}{2} M_{\eta'}^2 P_1^2 + i\bar{d}_m P_1 \langle \chi^- \rangle. \]

\[ f_V = 0.20, \quad f_\chi = -0.025, \quad g_V = 0.09, \quad c_m = 42 \text{ MeV}, \quad c_d = 32 \text{ MeV}, \quad \bar{d}_m = 20 \text{ MeV}, \]

\[ m_V = m_\rho = 0.77 \text{ GeV}, \quad m_A = m_{a_1} = 1.23 \text{ GeV}, \quad m_S = 0.98 \text{ GeV}, \quad m_{P_1} = 0.958 \text{ GeV} \]

\[ f_V, g_V, f_\chi, f_A : \text{ experiment} \]

\[ c_m \text{ and } c_d \text{ from resonance saturation at } \mathcal{O}(p^4) \]
Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: $C_i^r$ in the masses/decay constants and how these effects correlate into the rest
- No $\mu$ dependence: obviously only estimate
Problems:

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What we did about it:

- Vary resonance estimate by factor of two
- Vary the scale $\mu$ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult
Conclusions

- Have calculated all $K \to \pi$ transitions to NNLO, i.e. $\delta p^4$
- NNLO contributions lower the isospin breaking compared to NLO
- But $m_u/m_d$ also smaller at NNLO (But see $\eta \to 3\pi$)
- Slight increase when all put together but fits experimental ratios well.
- Determination of $C^r_i$ still needed:
- Analytical playing with the amplitude: started
- Future: checking whether we have more nontrivial relations between different observables for the $C^r_i$ contribution.
- New fit of the $L^r_i$ with better $C^r_i$ treatment.