HLBL CONTRIBUTION TO $a_\mu$: ENJL, CHIRAL QUARK MODELS AND CHIRAL LAGRANGIANS

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Old stuff: JB, E. Pallante, J. Prades


New stuff:

- JB, Mehran Zahiri Abyaneh, Johan Relefors
  HLbL pion loop contribution
Two main types of contributions

- HO HVP is like LO Had, can be derived from $e^+ e^- \rightarrow \text{hadrons}$. $a_{\mu}^{\text{HO HVP}} = -9.84(0.06) \times 10^{-10}$
- HLbL is the real problem: best estimate now: $a_{\mu}^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$
- Note that the sum is very small: but not an indication of the error
HLbL talks

- Melnikov
- Knecht
- Procura
- Vanderhaeghen
- Capiello
- this talk
- Greynat
- Nyffeler
- Several on the underlying form-factors
HLbL: the main object to calculate

- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
  - $p^4$, order 1: pion-loop
  - $p^8$, order $N_c$: quark-loop and heavier meson exchanges
  - $p^6$, order $N_c$: pion exchange

Does not fully solve the problem
only short-distance part of quark-loop is really $p^8$
but it’s a start
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
  - $p^4$, order 1: pion-loop
  - $p^8$, order $N_c$: quark-loop and heavier meson exchanges
  - $p^6$, order $N_c$: pion exchange

Implemented by two groups in the 1990s:

- Hayakawa, Kinoshita, Sanda: meson models, pion loop using hidden local symmetry, quark-loop with VMD, calculation in Minkowski space
- JB, Pallante, Prades: Try using as much as possible a consistent model-approach, ENJL, calculation in Euclidean space
General properties

\[ \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) = \]

\[ \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \]

Actually we really need
General properties

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3): \]

- In general 138 Lorentz structures (but only 28 contribute to \( g - 2 \))
- Using \( q_\rho \Pi^{\rho\nu\alpha\beta} = p_1\nu \Pi^{\rho\nu\alpha\beta} = p_2\alpha \Pi^{\rho\nu\alpha\beta} = p_3\beta \Pi^{\rho\nu\alpha\beta} = 0 \)
- 43 gauge invariant structures
- Bose symmetry relates some of them
- All depend on \( p_1^2, p_2^2 \) and \( q^2 \), but before derivative and \( p_3 \to 0 \) also \( p_2^2, p_1 \cdot p_2, p_1 \cdot p_3 \)
- Actually 2 less but singular basis Fischer et al.
- Compare HVP: one function, one variable
- General calculation from experiment: how difficult: Procura, Vanderhaeghen
- In four photon measurement: lepton contribution
General properties

\[ \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \] plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
- 5 remain: \( p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu \)
- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.

More recent: can do two more using Gegenbauer techniques Knecht-Nyffeler, Jegerlehner-Nyffeler, JB–Zahiri-Abyaneh–Relefors

- \( P_1^2, P_2^2 \) and \( Q^2 \) remain

- study \( a_{\mu}^{X} = \int dp_1 dp_2 a_{\mu}^{XLL} = \int dl_P dl_P dl_Q a_{\mu}^{XLLQ} \)
  \( l_P = \ln (P/\text{GeV}) \), to see where the contributions are

- Study the dependence on the cut-off for the photons
General properties

\[ \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \]

plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
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- study \( a_{\mu}^{X} = \int d l_{P_1} d l_{P_2} a_{\mu}^{XLL} = \int d l_{P_1} d l_{P_2} d l_{Q} a_{\mu}^{XLLQ} \)
- \( l_P = \ln \left( P/\text{GeV} \right) \), to see where the contributions are
- Study the dependence on the cut-off for the photons
ENJL: our main model

\[ \mathcal{L}_{\text{ENJL}} = \bar{q}^\alpha \left\{ i \gamma^\mu (\partial_\mu - i v_\mu - i a_\mu \gamma_5) - (\mathcal{M} + s - i p \gamma_5) \right\} q^\alpha \]
\[ + 2 g_S \left( \bar{q}_R^\alpha q_L^\beta \right) \left( \bar{q}_L^\beta q_R^\alpha \right) \]
\[ - g_V \left[ \left( \bar{q}_L^\alpha \gamma^\mu q_L^\beta \right) \left( \bar{q}_L^\beta \gamma_\mu q_L^\alpha \right) + \left( \bar{q}_R^\alpha \gamma^\mu q_R^\beta \right) \left( \bar{q}_R^\beta \gamma_\mu q_R^\alpha \right) \right] \]

- \( \bar{q} \equiv (\bar{u}, \bar{d}, \bar{s}) \)
- \( v_\mu, a_\mu, s, p \): external vector, axial-vector, scalar and pseudoscalar matrix sources
- \( \mathcal{M} \) is the quark-mass matrix.
- \( g_V \equiv \frac{8\pi^2 G_V(\Lambda)}{N_c \Lambda^2} \), \( g_S \equiv \frac{4\pi^2 G_S(\Lambda)}{N_c \Lambda^2} \).
- \( G_V, G_S \) are dimensionless and valid up to \( \Lambda \)
- No confinement but has good pion, vector meson and OK axial vector-meson phenomenology
ENJL: our main model

- Gap equation: chiral symmetry spontaneously broken

\[ \rightarrow = \rightarrow + \circ \]

- Generates poles, i.e. mesons via bubble resummation
ENJL: our main model

- Can be thought of as a very simple rainbow and ladder approximation in the DSE equation with constant kernels for the one-gluon exchange
- Parameters fit via $F_\pi$, $L_i$, vector meson properties, etc.
- $G_S = 1.216$, $G_V = 1.263$, $\Lambda = 1.16$ GeV
- has $M_Q = 263$ MeV
- Has a number of decent matchings to short-distance, e.g. $\Pi_v - \Pi_A$ but fails in others.
- Generates always VMD in external legs (but with a twist)
- Hook together general processes by one-loop vertices and bubble-chain propagators
Separation of contributions

- Quark loop with external bubble-chains
- \( \approx \) Quark-loop with VMD

- Also internal bubble chain
- \( \approx \) meson exchange

Note that vertices have structure
- Off-shell effect in model included
"\( \pi^0 \) exchange

- "\( \pi^0 \)" = \( \frac{1}{(p^2 - m_{\pi}^2)} \)
- The blobs need to be modelled, and in e.g. ENJL contain corrections also to the \( \frac{1}{(p^2 - m_{\pi}^2)} \)
- Pointlike has a logarithmic divergence
- Numbers \( \pi^0 \), but also \( \eta, \eta' \)
### Introduction

#### General properties

**ENJL, CQM**

#### $\pi^0$-exchange

<table>
<thead>
<tr>
<th>Cutoff (GeV)</th>
<th>Point-like</th>
<th>ENJL–VMD</th>
<th>Pointlike VMD</th>
<th>Transverse VMD</th>
<th>CELLO-VMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.92(2)</td>
<td>3.29(2)</td>
<td>3.46(2)</td>
<td>3.60(3)</td>
<td>3.53(2)</td>
</tr>
<tr>
<td>0.7</td>
<td>7.68(4)</td>
<td>4.24(4)</td>
<td>4.49(3)</td>
<td>4.73(4)</td>
<td>4.57(4)</td>
</tr>
<tr>
<td>1.0</td>
<td>11.15(7)</td>
<td>4.90(5)</td>
<td>5.18(3)</td>
<td>5.61(6)</td>
<td>5.29(5)</td>
</tr>
<tr>
<td>2.0</td>
<td>21.3(2)</td>
<td>5.63(8)</td>
<td>5.62(5)</td>
<td>6.39(9)</td>
<td>5.89(8)</td>
</tr>
<tr>
<td>4.0</td>
<td>32.7(5)</td>
<td>6.22(17)</td>
<td>5.58(5)</td>
<td>6.59(16)</td>
<td>6.02(10)</td>
</tr>
</tbody>
</table>

BPP: All in reasonable agreement $a_\mu^{\pi^0} = 5.9 \times 10^{-10}$
\[ a_\mu^{\pi_0} = 5.9(0.9) \times 10^{-10} \]

BPP:

Nonlocal quark model:


DSE model:

Goecke, Fischer and Williams, Phys.Rev.D83 (2011)094006[1012.3886]

LMD+V:


Formfactor inspired by AdS/QCD:

Cappiello, Cata and D’Ambrosio, Phys.Rev.D83(2011)093006 [1009.1161]

Chiral Quark Model:


Constraint via magnetic susceptibility:


All in reasonable agreement
MV short-distance: $\pi^0$ exchange


- take $P_1^2 \approx P_2^2 \gg Q^2$: Leading term in OPE of two vector currents is proportional to axial current

$$\Pi^{\rho \nu \alpha \beta} \propto \frac{P_\rho}{P_1^2} \langle 0 | \mathcal{T} (J_{A\nu} J_{V\alpha} J_{V\beta}) | 0 \rangle$$

- $J_A$ comes from

- AVV triangle anomaly: extra info

- Implemented via setting one blob = 1

$$a_{A^0} = 7.7 \times 10^{-10}$$
\[ \pi^0 \text{ exchange} \]

- The pointlike vertex implements shortdistance part, not only \( \pi^0 \)-exchange
- Are these part of the quark-loop? See also in Dorokhov, Broniowski, Phys. Rev. D78(2008)07301
- BPP quarkloop + \( \pi^0 \)-exchange \( \approx \) MV \( \pi^0 \)-exchange
\( \pi^0 \) exchange

- \( a_\mu = \int dl_1 dl_2 a_{\mu}^{LL} \) with \( l_i = \log(P_i/\text{GeV}) \)

Which momentum regions do what: volume under the plot \( \propto a_\mu \)
Pseudoscalar exchange

- Point-like VMD: $\pi^0$, $\eta$ and $\eta'$ give 5.58, 1.38, 1.04.
- Models that include $U(1)_{A}$ breaking give similar ratios
- Pure large $N_c$ models use this ratio
- The MV argument should give some enhancement over the full VMD like models
- Total pseudo-scalar exchange is about
  $$a_{\mu}^{PS} = 8 - 10 \times 10^{-10}$$
- AdS/QCD estimate (includes excited pseudo-scalars)
  $$a_{\mu}^{PS} = 10.7 \times 10^{-10}$$
- Connected contribution only: you get a $\bar{u}u + \bar{d}d$ pseudoscalar, adds $25/9$ times the $\pi^0$ contribution
**Pure quark loop**

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^7$ Electron Loop</th>
<th>$a_\mu \times 10^9$ Muon Loop</th>
<th>$a_\mu \times 10^9$ Constituent Quark Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.41(8)</td>
<td>2.41(3)</td>
<td>0.395(4)</td>
</tr>
<tr>
<td>0.7</td>
<td>2.60(10)</td>
<td>3.09(7)</td>
<td>0.705(9)</td>
</tr>
<tr>
<td>1.0</td>
<td>2.59(7)</td>
<td>3.76(9)</td>
<td>1.10(2)</td>
</tr>
<tr>
<td>2.0</td>
<td>2.60(6)</td>
<td>4.54(9)</td>
<td>1.81(5)</td>
</tr>
<tr>
<td>4.0</td>
<td>2.75(9)</td>
<td>4.60(11)</td>
<td>2.27(7)</td>
</tr>
<tr>
<td>8.0</td>
<td>2.57(6)</td>
<td>4.84(13)</td>
<td>2.58(7)</td>
</tr>
<tr>
<td>Known Results</td>
<td>2.6252(4)</td>
<td>4.65</td>
<td>2.37(16)</td>
</tr>
</tbody>
</table>

- $M_Q : 300$ MeV
- now known fully analytically
- Us: $5+(3-1)$ integrals extra are Feynman parameters
- **Slow convergence:**
  - electron: all at 500 MeV
  - Muon: only half at 500 MeV, at 1 GeV still 20% missing
  - 300 MeV quark: at 2 GeV still 25% missing
Pure quark loop: momentum area

quark loop $m_Q = 0.3$ GeV

Most from $P_1 \approx P_2 \approx Q$, sizable large momentum part
## ENJL quark-loop

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ GeV</th>
<th>$a_\mu \times 10^{10}$ VMD</th>
<th>$a_\mu \times 10^{10}$ ENJL</th>
<th>$a_\mu \times 10^{10}$ masscut</th>
<th>$a_\mu \times 10^{10}$ sum ENJL+masscut</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>2.46</td>
<td>3.2</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>1.13</td>
<td>2.3</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>0.59</td>
<td>2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>0.13</td>
<td>1.9</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>0.03</td>
<td>2.0</td>
</tr>
<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>0.005</td>
<td>2.0</td>
</tr>
</tbody>
</table>

- **Very stable**
- ENJL cuts off slower than pure VMD
- masscut: $M_Q = \Lambda$ to have short-distance and no problem with momentum regions
- Quite stable in region 1-4 GeV
ENJL: scalar

\[ \Pi^{\rho \nu \alpha \beta} = \prod_{ab}^{VVS} (p_1, r) g_s (1 + g_s \Pi^S (r)) \prod_{cd}^{SVV} (p_2, p_3) \nu^{abcd \rho \nu \alpha \beta} + \text{permutations} \]

\[ g_s (1 + g_s \Pi^S) = \frac{g_A (r^2) (2M_Q)^2}{2f^2 (r^2)} \frac{1}{M_S^2 (r^2) - r^2} \]

\[ \nu^{abcd \rho \nu \alpha \beta} : \text{ENJL VMD legs} \]

In ENJL only scalar + quark-loop properly chiral invariant
ENJL: scalar/QL

<table>
<thead>
<tr>
<th>Cut-off Λ GeV</th>
<th>$a_\mu \times 10^{10}$ Quark-loop VMD</th>
<th>$a_\mu \times 10^{10}$ Quark-loop ENJL</th>
<th>$a_\mu \times 10^{10}$ Scalar Exchange</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>−0.22</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>−0.46</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>−0.60</td>
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<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
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<td>4.0</td>
<td>0.98</td>
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<td>−0.68</td>
</tr>
<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>−0.68</td>
</tr>
</tbody>
</table>

- ENJL only scalar+quark-loop properly chiral invariant
- Note: ENJL+scalar (BPP) ≈ Quark-loop VMD (HKS)
- $M_S \approx 620$ MeV certainly an overestimate for real scalars
- If scalar is $\sigma$: related to pion loop part?
- quark-loop: $a_\mu^{ql} \approx 1 \times 10^{-10}$
DSE model: $a_{\mu}^{ql} = 10.7(0.2) \times 10^{-10}$ T. Goecke, C. S. Fischer and R. Williams, arXiv:1210.1759

- Not a full calculation (yet) but includes an estimate of some of the missing parts
- A lot larger than bare quark loop with constituent mass
- DSE model (Maris-Roberts) does reproduce a lot of low-energy phenomenology. My guess was: numbers similar to ENJL.

Can one find something in between full DSE and ENJL that is easier to handle?

- Nonlocal chiral quark model or nonlocal NJL (but no vector vertex, i.e. no rho) A. E. Dorokhov, A. E. Radzhabov and A. S. Zhevlakov, arXiv:1502.04487 [hep-ph].
  $a_{\mu}^{ql} = 11.0(0.9) \times 10^{-10}$
Other quark loop

- de Rafael-Greynat 1210.3029 $(7.6 - 8.9) \times 10^{-10}$
- Boughezal-Melnikov 1104.4510 $(11.8 - 14.8) \times 10^{-10}$
- Masjuan-Vanderhaeghen 1212.0357 $(7.6 - 12.5) \times 10^{-10}$

Various interpretations: the full calculation or not

All (even DSE) have in common that a low quark mass is used for a large part of the integration range, not shielded by formfactors
Axial-vector exchange exchange

<table>
<thead>
<tr>
<th>Cut-off Λ (GeV)</th>
<th>( a_\mu \times 10^{10} ) from Axial-Vector Exchange ( \mathcal{O}(N_c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.05(0.01)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.07(0.01)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.13(0.01)</td>
</tr>
<tr>
<td>2.0</td>
<td>0.24(0.02)</td>
</tr>
<tr>
<td>4.0</td>
<td>0.59(0.07)</td>
</tr>
</tbody>
</table>

There is some pseudo-scalar exchange piece here as well, off-shell not quite clear what is what.

- \( a_{axial}^\mu = 0.6 \times 10^{-10} \)
- MV: short distance enhancement + mixing (both enhance about the same)
  \( a_{axial}^\mu = 2.2 \times 10^{-10} \)
A bare $\pi$-loop (sQED) give about $-4 \cdot 10^{-10}$

The $\pi\pi\gamma^*$ vertex is always done using VMD

$\pi\pi\gamma^*\gamma^*$ vertex two choices:
- Hidden local symmetry model: only one $\gamma$ has VMD
- Full VMD
- Both are chirally symmetric
- The HLS model used has problems with $\pi^+ - \pi^0$ mass difference (due to not having an $a_1$)

Final numbers quite different: $-0.45$ and $-1.9 \times 10^{-10}$

For BPP stopped at 1 GeV but within 10% of higher $\Lambda$
\[ \pi \text{ loop: Bare vs VMD} \]

- plotted \( a_{LLQ}^\mu \) for \( P_1 = P_2 \)
- \( a_\mu = \int dP_1 dP_2 dQ \ a_{LLQ}^\mu \)
- \( l_Q = \log(Q/1 \text{ GeV}) \)
\section*{Introduction}

General properties

ENJL

$\pi^0$-exchange

Quark-loop

Scalar

$a_1$-exchange

$\pi$-loop

Summary

Future

\section*{HLbL for $a_\mu$:
ENJL, CQM
and $\chi C$}

Johan Bijnens

$\pi$ loop: VMD vs HLS

Usual HLS, $a = 2$
HLS with $a = 1$, satisfies more short-distance constraints
\( \pi \) loop

- \( \pi \pi \gamma^* \gamma^* \) for \( q_1^2 = q_2^2 \) has a short-distance constraint from the OPE as well.
- HLS does not satisfy it
- full VMD does: so probably better estimate
  - Ramsey-Musolf suggested to do pure ChPT for the \( \pi \) loop
  - Polarizability \((L_9 + L_{10})\) up to 10\%, charge radius 30\% at low energies, more at higher
  - Both HLS and VMD have charge radius effect but not polarizability
\( \pi \) loop

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- Both HLS and VMD have charge radius effect but not polarizability
\[ \pi \] loop: \( L_9, L_{10} \)

- ChPT for muon \( g - 2 \) at order \( p^6 \) is not powercounting finite so no prediction for \( a_\mu \) exists.
- But can be used to study the low momentum end of the integral over \( P_1, P_2, Q \)
- The four-photon amplitude is finite still at two-loop order (counterterms start at order \( p^8 \))
- Add \( L_9 \) and \( L_{10} \) vertices to the bare pion loop
- JB-Relefors-Zahiri-Abyaneh
low scale, charge radius effect well reproduced
\( \pi \) loop: VMD vs \( L_9 \) and \( L_{10} \)

- \( L_9 + L_{10} \neq 0 \) gives an enhancement of 10-15%
- To do it fully need to get a model: include \( a_1 \)
Include $a_1$

- $L_9 + L_{10}$ effect is from

- But to get gauge invariance correctly need
Include $a_1$

- Consistency problem: full $a_1$-loop?
- Treat $a_1$ and $\rho$ classical and $\pi$ quantum: there must be a $\pi$ that closes the loop
  - Argument: integrate out $\rho$ and $a_1$ classically, then do pion loops with the resulting Lagrangian
- To avoid problems: representation without $a_1-\pi$ mixing
- Check for curiosity what happens if we add $a_1$-loop
Include $a_1$

- Use antisymmetric vector representation for $a_1$ and $\rho$
- Fields $A_{\mu\nu}$, $V_{\mu\nu}$ (nonets)
- Kinetic terms: $-\frac{1}{2} \langle \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{1}{2} V_{\mu\nu} V^{\mu\nu} \rangle$
  $-\frac{1}{2} \langle \nabla^\lambda A_{\lambda\mu} \nabla_\nu A^{\nu\mu} - \frac{1}{2} A_{\mu\nu} A^{\mu\nu} \rangle$
- Terms that give contributions to the $L_i^r$:
  $\frac{F_V}{2\sqrt{2}} \langle f_{+\mu\nu} V^{\mu\nu} \rangle + \frac{iG_V}{\sqrt{2}} \langle V^{\mu\nu} u_\mu u_\nu \rangle + \frac{F_A}{2\sqrt{2}} \langle f_{-\mu\nu} A^{\mu\nu} \rangle$
- $L_9 = \frac{F_V G_V}{2M_V^2}$, $L_{10} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}$
- Weinberg sum rules: (Chiral limit)
  $F_V^2 = F_A^2 + F_{\pi}^2$
  $F_V^2 M_V^2 = F_A^2 M_A^2$
- VMD for $\pi\pi\gamma$:
  $F_V G_V = F_{\pi}^2$
$V_{\mu\nu}$ only

- $\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)$ is not finite
  (but was also not finite for HLS)

- But $\left. \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \right|_{p_3=0}$ also not finite
  (but was finite for HLS)

- Derivative one finite for $G_V = F_V/2$

- Surprise: $g - 2$ identical to HLS with $a = \frac{F_V^2}{F_\pi^2}$

- Yes I know, different representations are identical BUT they do differ in higher order terms and even in what is higher order

- Same comments as for HLS numerics
\[ V_{\mu \nu} \text{ only} \]

- \( \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) \) is not finite
  (but was also not finite for HLS)
- But \( \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \) also not finite
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- Yes I know, different representations are identical BUT they do differ in higher order terms and even in what is higher order
- Same comments as for HLS numerics
**V_{\mu\nu} and A_{\mu\nu}**

- Add $a_1$
- Calculate a lot

$$\frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0}$$

finite for:

- $G_V = F_V = 0$ and $F_A^2 = -2F_\pi^2$
- If adding full $a_1$-loop $G_V = F_V = 0$ and $F_A^2 = -F_\pi^2$
Start by adding $\rho a_1 \pi$ vertices

\[
\lambda_1 \langle [V_{\mu\nu}, A_{\mu\nu}] \chi_- \rangle + \lambda_2 \langle [V_{\mu\nu}, A_{\nu\alpha}] h_{\mu}^{\nu} \rangle \\
+ \lambda_3 \langle i [\nabla^\mu V_{\mu\nu}, A_{\nu\alpha}] u_\alpha \rangle + \lambda_4 \langle i [\nabla_\alpha V_{\mu\nu}, A_{\alpha\nu}] u^\mu \rangle \\
+ \lambda_5 \langle i [\nabla^\alpha V_{\mu\nu}, A_{\mu\nu}] u_\alpha \rangle + \lambda_6 \langle i [V_{\mu\nu}, A_{\mu\nu}] f^-_{\alpha} \rangle \\
+ \lambda_7 \langle iV_{\mu\nu} A^{\mu\rho} A^\nu_{\rho} \rangle
\]

All lowest dimensional vertices of their respective type

Not all independent, there are three relations

Follow from the constraints on $V_{\mu\nu}$ and $A_{\mu\nu}$ (thanks to Stefan Leupold)
\( V_{\mu\nu} \) and \( A_{\mu\nu} \): big disappointment

- Work a whole lot
  \[
  \delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \left|_{\delta p_3\lambda} \right. \bigg|_{p_3=0}
  \]
  not obviously finite

- Work a lot more

- Prove that
  \[
  \delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \left|_{\delta p_3\lambda} \right. \bigg|_{p_3=0}
  \]
  finite, only same solutions as before

- Try the combination that show up in \( g - 2 \) only

- Work a lot

- Again, only same solutions as before

- Small loophole left: after the integration for \( g - 2 \) could be finite but many funny functions of \( m_\pi, m_\mu, M_V \) and \( M_A \) show up.
a₁-loop: cases with good $L_9$ and $L_{10}$

- Add $F_V$, $G_V$ and $F_A$
- Fix values by Weinberg sum rules and VMD in $\gamma^* \pi \pi$
- no $a_1$-loop
**a₁-loop: cases with good L₉ and L₁₀**

- Add $F_V$, $G_V$ and $F_A$
- Fix values by Weinberg sum rules and VMD in $\gamma^* \pi \pi$
- With $a_1$-loop (is different plot!!)
**a_1-loop: cases with good L_9 and L_{10}**

- Add $a_1$ with $F_A^2 = +F_\pi^2$
- Add the full VMD as done earlier for the bare pion loop
Integration results

\[ a_1 F_A^2 = -2F^2 \]

\[ a_1 F_A^2 = -1 \ a_1\text{-loop} \]

HLS

HLS \( a=1 \)

VMD

\[ a_1 \text{ VMD} \]

\[ a_1 \text{ Weinberg} \]

\[ P_1, P_2, Q \leq \Lambda \]
Integration results with $a_1$

- Problem: get high energy behaviour good enough
- But all models with reasonable $L_9$ and $L_{10}$ fall way inside the error quoted earlier ($-1.9 \pm 1.3 \times 10^{-10}$)
- Tentative conclusion: Use hadrons only below about 1 GeV: $a_\mu^{\pi-\text{loop}} = (-2.0 \pm 0.5) \times 10^{-10}$
- Note that Engel and Ramsey-Musolf, arXiv:1309.2225 is a bit more pessimistic quoting numbers from $(-1.1 \text{ to } -7.1) \times 10^{-10}$
### Summary: ENJL vc PdRV

<table>
<thead>
<tr>
<th></th>
<th>BPP</th>
<th>PdRV arXiv:0901.0306</th>
</tr>
</thead>
<tbody>
<tr>
<td>quark-loop</td>
<td>$(2.1 \pm 0.3) \cdot 10^{-10}$</td>
<td>$(11.4 \pm 1.3) \cdot 10^{-10}$</td>
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<tr>
<td>pseudo-scalar</td>
<td>$(8.5 \pm 1.3) \cdot 10^{-10}$</td>
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<tr>
<td>axial-vector scalar</td>
<td>$(0.25 \pm 0.1) \cdot 10^{-10}$</td>
<td>$(1.5 \pm 1.0) \cdot 10^{-10}$</td>
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<tr>
<td>scalar</td>
<td>$(-0.68 \pm 0.2) \cdot 10^{-10}$</td>
<td>$(-0.7 \pm 0.7) \cdot 10^{-10}$</td>
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<tr>
<td>$\pi K$-loop</td>
<td>$(-1.9 \pm 1.3) \cdot 10^{-10}$</td>
<td>$(-1.9 \pm 1.9) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>errors sum</td>
<td>linearly</td>
<td>quadratically</td>
</tr>
<tr>
<td></td>
<td>$(8.3 \pm 3.2) \cdot 10^{-10}$</td>
<td>$(10.5 \pm 2.6) \cdot 10^{-10}$</td>
</tr>
</tbody>
</table>
What can we do more?

The ENJL model can certainly be improved:

- Chiral nonlocal quark-model (like nonlocal ENJL): so far no rho in the model
- DSE: $\pi^0$-exchange similar to everyone else, quark-loop very different, looking forward to final results

More resonances models should be tried, AdS/QCD is one approach, $R_{\chi T}$ (Valencia et al.) possible,…

Note short-distance matching must be done in many channels, there are theorems JB,Gamiz,Lipartia,Prades that with only a few resonances this requires compromises

$\pi$-loop: HLS smaller than double VMD (understood) models with $\rho$ and $a_1$: difficulties with infinities
What can we do more?

- **Constraints from experiment:**
  

  Studying three formfactors $P\gamma^*\gamma^*$ in $P \to \ell^+\ell^-\ell'^+\ell'^-$, $e^+e^- \to e^+e^- P$ exact tree level and for $g - 2$ (but beware sign):
  
  - **Conclusion:** possible but VERY difficult
  - Two $\gamma^*$ off-shell not so important for our choice of form-factor
  - See also the other talks here

- All information on hadrons and 1-2-3-4 off-shell photons is welcome: constrain the models

- More short-distance constraints: MV, Nyffeler integrate with all contributions, not just $\pi^0$-exchange

- **Need a new overall evaluation with consistent approach.**

- **Lattice:** Lehner

- **Dispersion theory:** Procura, Vanderhaeghen
#### Summary of Muon $g-2$ contributions

<table>
<thead>
<tr>
<th></th>
<th>$10^{10} a_{\mu}$</th>
</tr>
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<tbody>
<tr>
<td>exp</td>
<td>11 659 209.1</td>
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<td>theory</td>
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<tr>
<td>QED</td>
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<tr>
<td>EW</td>
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<tr>
<td>LO Had</td>
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<td>HO HVP</td>
<td>$-9.8$</td>
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<tr>
<td>HLbL</td>
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<tr>
<td>difference</td>
<td>28.8</td>
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</tbody>
</table>

- Error on LO had
- Error on HLbL
- Errors added quadratically
- 3.6 $\sigma$
- Difference: $4\%$ of LO Had
- $270\%$ of HLbL
- $1\%$ of leptonic LbL

**Generic SUSY**: $12.3 \times 10^{-10} \left( \frac{100 \text{ GeV}}{M_{\text{SUSY}}} \right)^2 \tan \beta$

$M_{\text{SUSY}} \approx 66 \text{ GeV} \sqrt{\tan \beta}$