HADRONIC LIGHT-BY-LIGHT FOR THE MUON ANOMALY RENORMALIZATION GROUP

Johan Bijnens

Lund University

bijnens@thep.lu.se
http://www.thep.lu.se/∼bijnens

Various ChPT: http://www.thep.lu.se/∼bijnens/chpt.html

Overview

- Part I: Muon $g-2$
  - Overview
  - QED, Electroweak, Hadronic
  - Light-by-Light: the various contributions
  - Overall properties
  - The leading in N_c exchanges and quark-loop
  - π and K loop
  - Summary Light-by-light
- Part II: Renormalization group and leading logarithms
  - Leading logarithms: principle
  - $O(N)$ model: mass
  - Large N
  - Anomaly

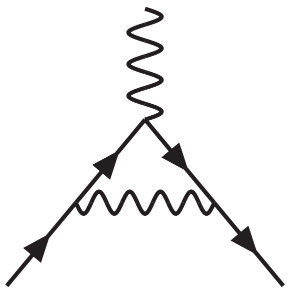
Muon $g-2$: overview

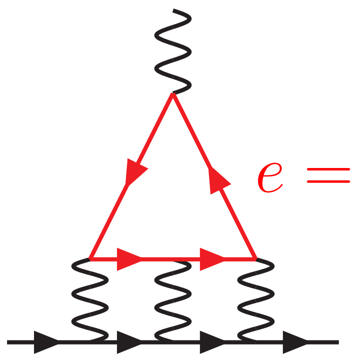
- in terms of the anomaly $a_\mu = (g-2)/2$
- Data dominated by BNL E821 (statistics)(systematic)
  $a_{\mu^+}^{\rm exp} = 11659204(6)(5) \times 10^{-10}$
  $a_{\mu^-}^{\rm exp} = 11659215(8)(3) \times 10^{-10}$
  $a_{\mu}^{\rm exp} = 11659208.9(5.4)(3.3) \times 10^{-10}$
- Theory is off somewhat (electroweak)(LO had)(HO had)
  $a_{\mu}^{\rm SM} = 11659180.2(2)(4.2)(2.6) \times 10^{-10}$
- $\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.7(6.3)(4.9) \times 10^{-10}$ (PDG)
- E821 goes to Fermilab, expect factor of four in precision
- Many BSM models CAN predict a value in this range (often a lot more or a lot less)

Muon $g-2$: QED

$$a_\mu^{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857410(27)\left(\frac{\alpha}{\pi}\right)^2 + 24.05050964(43)\left(\frac{\alpha}{\pi}\right)^3 + 130.8055(80)\left(\frac{\alpha}{\pi}\right)^4 + 663(20)\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

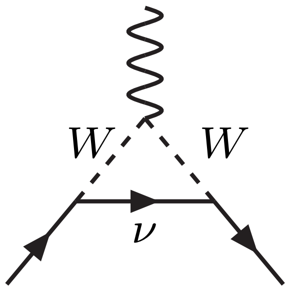
- First three loops known analytically
- four-loops fully done numerically
- Five loops estimate
- Kinoshita, Laporta, Remiddi, Schwinger,...
- α fixed from the electron $g-2$: $\alpha = 1/137.035999084(51)$
- $a_\mu^{\text{QED}} = 11658471.809(0.015) \times 10^{-10}$

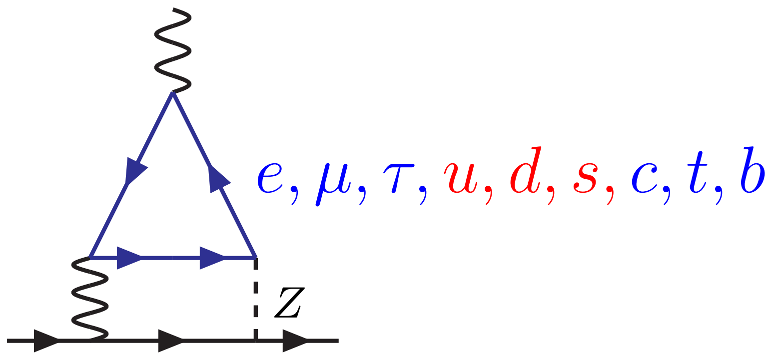




$e = 20.95, \mu = 0.37, \tau = 0.002$

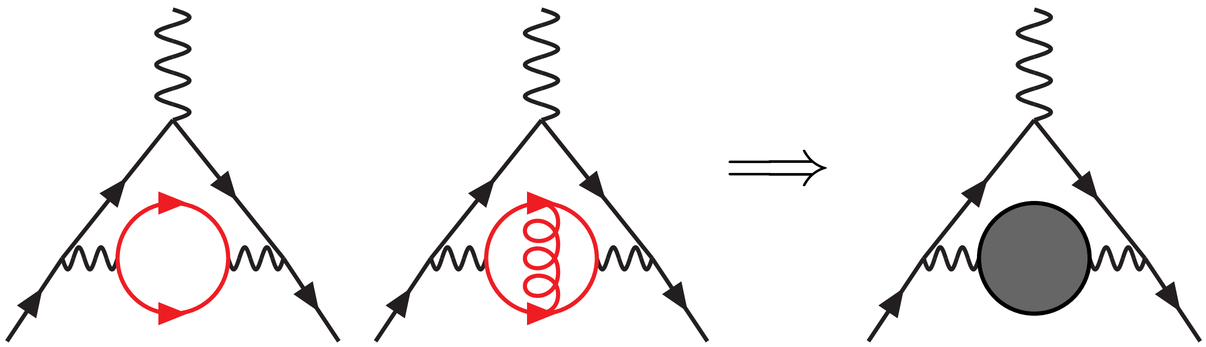
Muon $g-2$: Electroweak





- $a_\mu^{\mathrm{EW}}[1\text{−loop}] = 19.48 \times 10^{-10}$
- $a_\mu^{\mathrm{EW}}[2\text{−loop}] = -4.07(0.10)(0.18) \times 10^{-10}$
- $a_\mu^{\mathrm{EW}} = 15.4(0.1)(0.2) \times 10^{-10}$ (triangle)(Higgs mass)

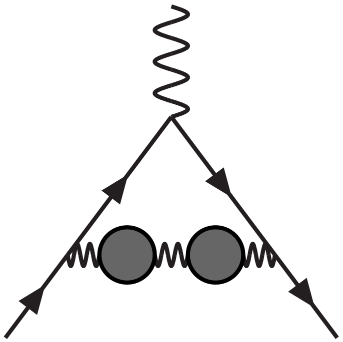
Muon $g-2$: LO hadronic



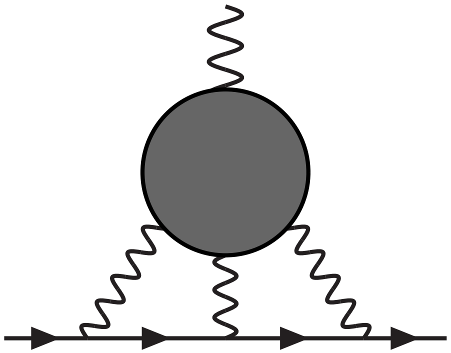
- $a_\mu^{\mathrm{LOhad}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds \frac{K(s)}{s} R^{(0)}(s)$
- $R^{(0)}(s)$ bare cross-section ratio $\dfrac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$
- Bare, many different evaluations,...
- $a_\mu^{\mathrm{LOHad}} = 692.3(4.2)(0.3) \times 10^{-10}$ (exp)(pert. QCD)

Muon $g - 2$: HO hadronic

- Two main types of contributions



HO HVP



HLbL

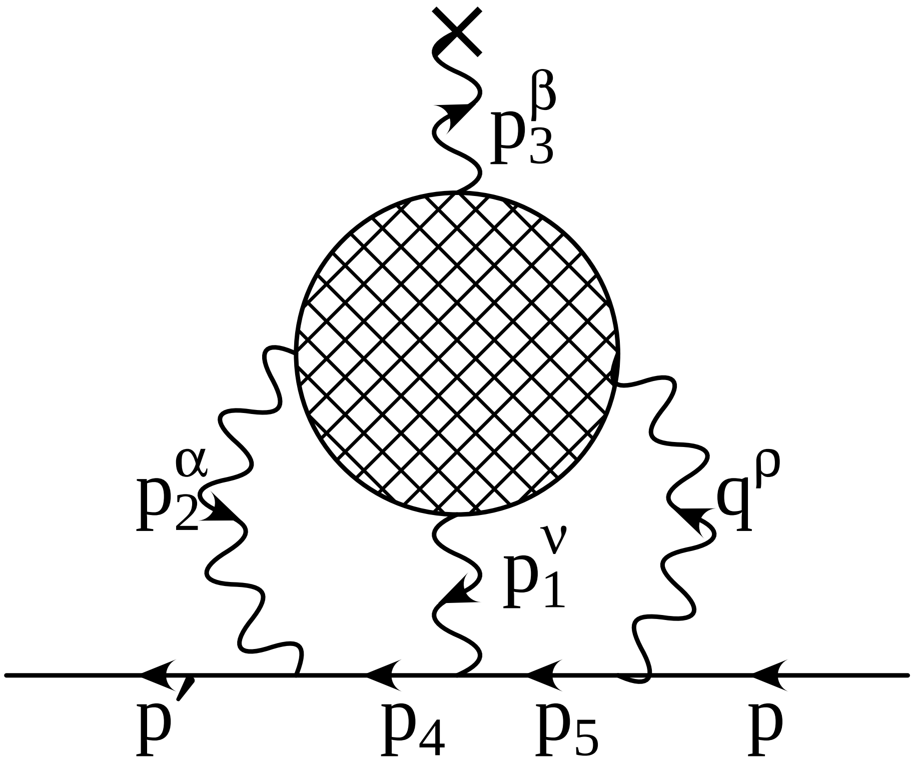
- HO HVP is like LO Had but a more complicated function
  $K(s)\ a_\mu^{\text{HO HVP}} = -9.84(0.06) \times 10^{-10}$
- HLbL is the real problem: best estimate now:
  $a_\mu^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$

Summary of Muon $g-2$ contributions

	$10^{10} a_\mu$	
exp	11 659 208.9	6.3
theory	11 659 180.2	5.0
QED	11 658 471.8	0.0
EW	15.4	0.2
LO Had	692.3	4.2
HO HVP	-9.8	0.1
HLbL	10.5	2.6
difference	28.7	8.1

- Error on LO had
  all e^+e^- based OK
  τ based 2 σ
- Error on HLbL
- Errors added quadratically
- 3.5 σ
- Difference:
  4% of LO Had

Our object



- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks

A separation proposal: a start

E. de Rafael, “Hadronic contributions to the muon g-2 and low-energy QCD,” Phys. Lett. **B322** (1994) 239-246. [hep-ph/9311316].

- Use ChPT p counting and large N_c
- p^4, order 1: pion-loop
- p^8, order N_c: quark-loop and heavier meson exchanges
- p^6, order N_c: pion exchange

Does not fully solve the problem
only short-distance quark-loop is really p^8
but it’s a start

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- Hayakawa, Kinoshita, Sanda: meson models, pion loop using hidden local symmetry, quark-loop with VMD, calculation in Minkowski space
- JB, Pallante, Prades: Try using as much as possible a consistent model-approach, calculation in Euclidean space

Papers: BPP and HKS

- JB, E. Pallante and J. Prades
  - “Comment on the pion pole part of the light-by-light contribution to the muon g-2,” Nucl. Phys. B **626** (2002) 410 [arXiv:hep-ph/0112255].
  - “Analysis of the Hadronic Light-by-Light Contributions to the Muon $g-2$,” Nucl. Phys. B **474** (1996) 379 [arXiv:hep-ph/9511388].
  - “Hadronic light by light contributions to the muon g-2 in the large N(c) limit,” Phys. Rev. Lett. **75** (1995) 1447 [Erratum-ibid. **75** (1995) 3781] [arXiv:hep-ph/9505251].
- Hayakawa, Kinoshita, (Sanda)
  - “Pseudoscalar pole terms in the hadronic light by light scattering contribution to muon g - 2,” Phys. Rev. **D57** (1998) 465-477. [hep-ph/9708227], Erratum-ibid.D66 (2002) 019902[hep-ph/0112102].
  - “Hadronic light by light scattering contribution to muon g-2,” Phys. Rev. **D54** (1996) 3137-3153. [hep-ph/9601310].
  - “Hadronic light by light scattering effect on muon g-2,” Phys. Rev. Lett. **75** (1995) 790-793. [hep-ph/9503463].

Differences

- HK(S)
  - Purely hadronic exchanges
  - quark-loop with hadronic VMD
  - Studied dependence of everything on m_V
- BPP
  - Use the ENJL as an overall model to have a similar uncertainty on all low-energy parts
  - repair some of the worst short-comings
  - Add the short-distance quark-loop
  - Study of cut-off dependence

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- Sign mistake
  - HKS: Euclidean versus Minkowski $\varepsilon^{\mu\nu\alpha\beta}$
  - BPP: notes all correct sign, program had wrong sign, probably minus sign from fermion loop not removed

The overall

$$a_\mu^{\mathrm{HLbL}} = \frac{-1}{48 m_\mu} \mathrm{tr}[(\not p + m_\mu) M^{\lambda\beta}(0)\,(\not p + m_\mu)[\gamma_\lambda, \gamma_\beta]]\,.$$

$$M^{\lambda\beta}(p_3) = |e|^6 \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \frac{1}{q^2\, p_1^2\, p_2^2 (p_4^2 - m_\mu^2)\,(p_5^2 - m_\mu^2)}$$
$$\times \left[\frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}}\right] \gamma_\alpha (\not p_4 + m_\mu) \gamma_\nu (\not p_5 + m_\mu) \gamma_\rho\,.$$

- We used: $\Pi^{\rho\nu\alpha\lambda}(p_1, p_2, p_3) = -p_{3\beta} \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}}\,.$
- Can calculate at $p_3 = 0$ but must take derivative
- derivative makes in quark-loop each permutation finite
- Four point function of $V_i^\mu(x) \equiv \sum_i Q_i\ [\bar q_i(x) \gamma^\mu q_i(x)]$

$$\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \equiv$$
$$i^3 \int d^4x \int d^4y \int d^4z\, e^{i(p_1 \cdot x + p_2 \cdot y + p_3 \cdot z)} \langle 0 | T \left(V_a^\rho(0) V_b^\nu(x) V_c^\alpha(y) V_d^\beta(z) \right) |0\rangle$$

General properties

$\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)$:

- In general 138 Lorentz structures (but only 32 contribute to $g-2$)
- Using $q_\rho \Pi^{\rho\nu\alpha\beta} = p_{1\nu}\Pi^{\rho\nu\alpha\beta} = p_{2\alpha}\Pi^{\rho\nu\alpha\beta} = p_{3\beta}\Pi^{\rho\nu\alpha\beta} = 0$
  43 gauge invariant structures
- Bose symmetry relates some of them
- All depend on p_1^2, p_2^2 and q^2, but before derivative and $p_3 \to 0$ there are more
- Compare HVP: one function, one variable
- General calculation from experiment difficult to see how

General properties

$\int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4}$ plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
- 5 remain: $p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu$
- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.
- More recent: can do two more using Gegenbauer techniques Knecht-Nyffeler, Jegerlehner-Nyffeler,JB–Zahiri-Abyaneh
- P_1^2, P_2^2 and Q^2 remain

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- P_1^2, P_2^2 and Q^2 remain
- study $a_\mu^{\mathrm{X}} = \int dl_{P_1} dl_{P_2} a_\mu^{\mathrm{XLL}} = \int dl_{P_1} dl_{P_2} dl_Q a_\mu^{\mathrm{XLLQ}}$
  $l_P = \ln\left(P/GeV\right)$, to see where the contributions are

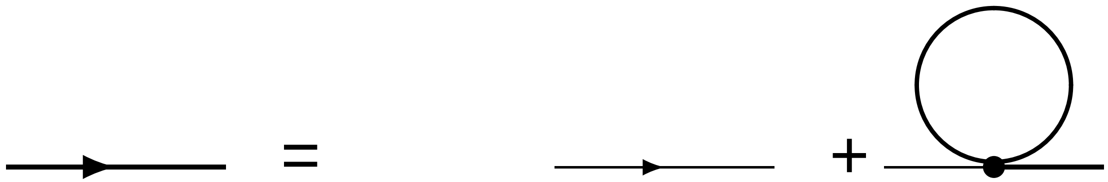
ENJL: our main model

$$\mathcal{L}_{\mathrm{ENJL}} = \overline{q}^\alpha \left\{ i\gamma^\mu \left(\partial_\mu - i v_\mu - i a_\mu \gamma_5\right) - \left(\mathcal{M} + s - i p \gamma_5\right)\right\} q^\alpha + 2\, g_S \left(\overline{q}^\alpha_R q^\beta_L\right)\left(\overline{q}^\beta_L q^\alpha_R\right)$$
$$- g_V \left[\left(\overline{q}^\alpha_L \gamma^\mu q^\beta_L\right)\left(\overline{q}^\beta_L \gamma_\mu q^\alpha_L\right) + \left(\overline{q}^\alpha_R \gamma^\mu q^\beta_R\right)\left(\overline{q}^\beta_R \gamma_\mu q^\alpha_R\right)\right]$$

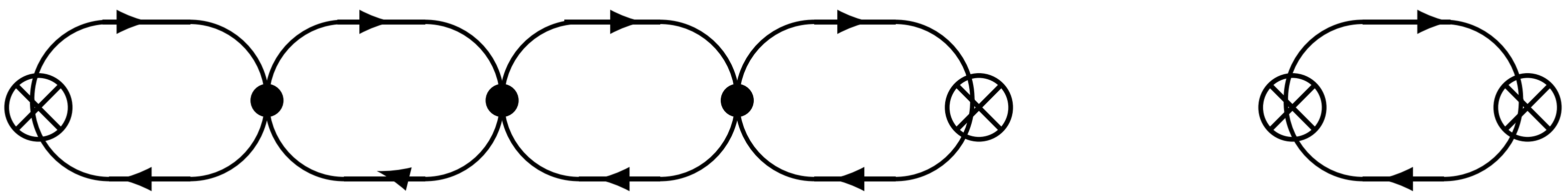
- $\overline{q} \equiv \left(\overline{u}, \overline{d}, \overline{s}\right)$
- v_μ, a_μ, s, p: external vector, axial-vector, scalar and pseudoscalar matrix sources
- $\mathcal{M}$ is the quark-mass matrix.
- $g_V \equiv \dfrac{8\pi^2 G_V(\Lambda)}{N_c \Lambda^2}$, $g_S \equiv \dfrac{4\pi^2 G_S(\Lambda)}{N_c \Lambda^2}$.
- G_V, G_S are dimensionless and valid up to Λ
- No confinement but has good pion, vector meson and OK axial vector-meson phenomenology

ENJL: our main model

- (this) ENJL JB, Bruno, de Rafael, Nucl. Phys. B390 (1993) 501 [hep-ph/9206236]; JB, Phys. Rep. 265 (1996) 369 [hep-ph/9502335] (review)
- Gap equation: chiral symmetry spontaneously broken



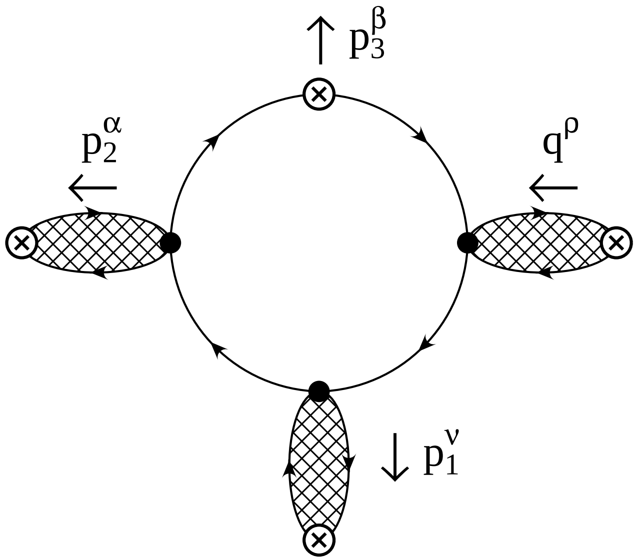
- Generates poles, i.e. mesons via bubble resummation



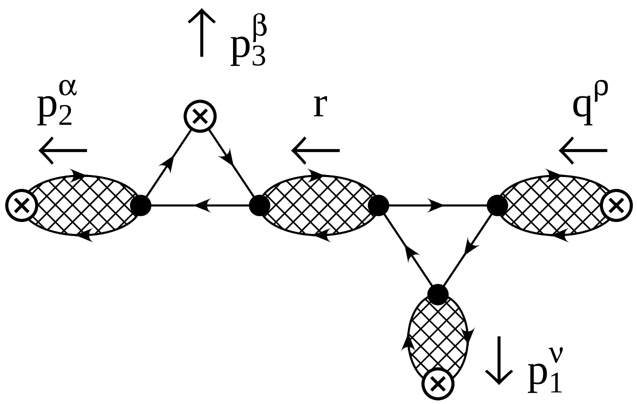
ENJL: our main model

- Can be thought of as a very simple rainbow and ladder approximation in the DSE equation with constant kernels for the one-gluon exchange
- Parameters fit via F_π, L_i^r, vector meson properties,. . .
- $G_S = 1.216$, $G_V = 1.263$, $\Lambda = 1.16$ GeV
- has $M_Q = 263$ MeV
- Has a number of decent matchings to short-distance, e.g. $\Pi_V - \Pi_A$ but fails in others.
- Generates always VMD in external legs (but with a twist)
- Hook together general processes by one-loop vertices and bubble-chain propagators

Separation of contributions



(a)

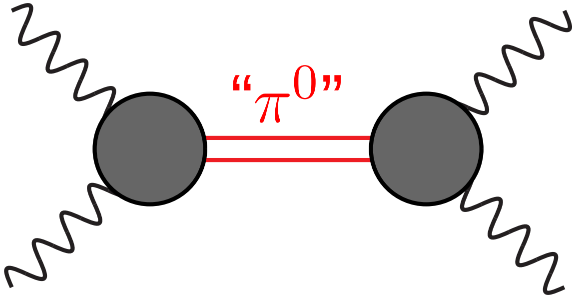


(b)

- Quark loop with external bubble-chains
- $\approx$ Quark-loop with VMD

- Also internal bubble chain
- $\approx$ meson exchange
- Note that vertices have structure
- Off-shell effect in model included

π^0 exchange



- "π^0"$= 1/(p^2 - m_\pi^2)$
- The blobs need to be modelled, and in e.g. ENJL contain corrections also to the $1/(p^2 - m_\pi^2)$
- Pointlike has a logarithmic divergence

π^0 exchange

Cut-off μ (GeV)	$a_\mu \times 10^{10}$ Point-like	$a_\mu \times 10^{10}$ ENJL–VMD	$a_\mu \times 10^{10}$ Point-Like-VMD	$a_\mu \times 10^{10}$ Transverse-VMD	$a_\mu \times 10^{10}$ Transverse-VMD
0.5	4.92(2)	3.29(2)	3.46(2)	3.60(3)	3.53(2)
0.7	7.68(4)	4.24(4)	4.49(3)	4.73(4)	4.57(4)
1.0	11.15(7)	4.90(5)	5.18(3)	5.61(6)	5.29(5)
2.0	21.3(2)	5.63(8)	5.62(5)	6.39(9)	5.89(8)
4.0	32.7(5)	6.22(17)	5.58(5)	6.59(16)	6.02(10)

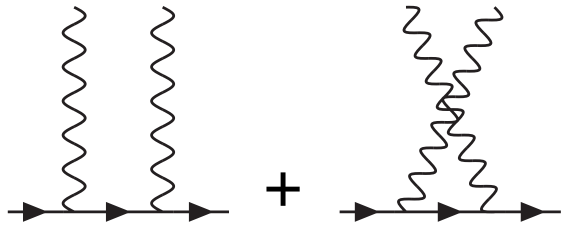
BPP: All in reasonable agreement $a_\mu^{\pi^0} = 5.9 \times 10^{-10}$

π^0 exchange

- BPP $a_\mu^{\pi^0} = 5.9 \times 10^{-10}$
- Nonlocal quark model: $a_\mu^{\pi^0} = 6.27 \times 10^{-10}$ A. E. Dorokhov, W. Broniowski, Phys. Rev. **D78** (2008) 073011. [arXiv:0805.0760 [hep-ph]]
- DSE model: $a_\mu^{\pi^0} = 5.75 \times 10^{-10}$ T. Goecke, C. S. Fischer and R. Williams, Phys. Rev. D **83** (2011) 094006 [arXiv:1012.3886 [hep-ph]]
- LMD+V: $a_\mu^{\pi^0} = (5.8 - 6.3) \times 10^{-10}$ M. Knecht, A. Nyffeler, Phys. Rev. **D65**(2002)073034, [hep-ph/0111058]
- Formfactor inspired by AdS/QCD: $a_\mu^{\pi^0} = 6.54 \cdot 10^{-10}$ L. Cappiello, O. Cata and G. D'Ambrosio, Phys. Rev. D **83** (2011) 093006 [arXiv:1009.1161 [hep-ph]]

MV short-distance: π^0 exchange

- K. Melnikov, A. Vainshtein, Phys. Rev. **D70** (2004) 113006. [hep-ph/0312226]
- take $p_1^2 \approx p_2^2 \gg q^2$: Leading term in OPE of two vector currents is proportional to axial current
- These come from



- Are these part of the quark-loop? See also in Dorokhov,Broniowski, phys.Rev. D78(2008)07301
- Implemented via setting one blob = 1
- $a_\mu^{\pi^0} = 7.7 \times 10^{-10}$

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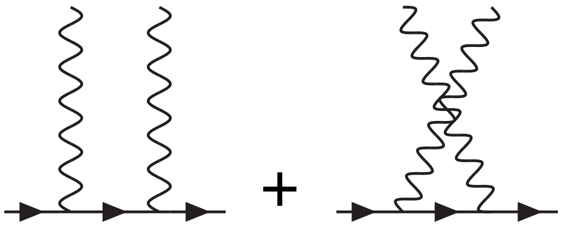

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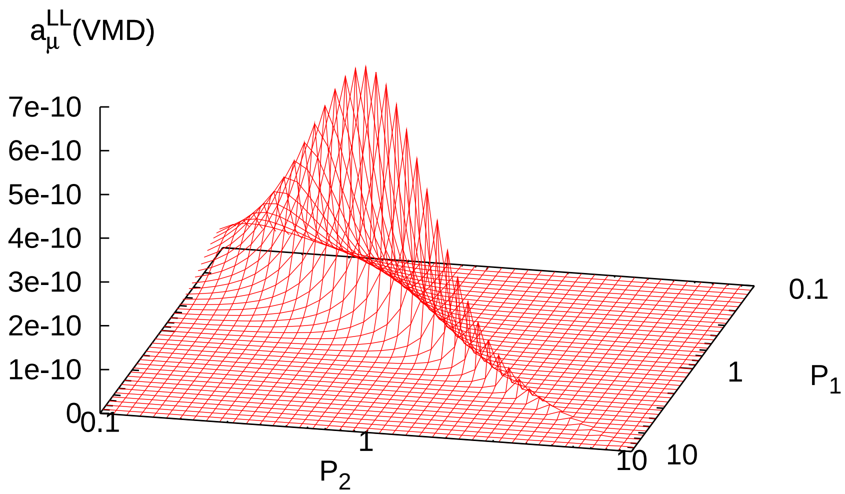
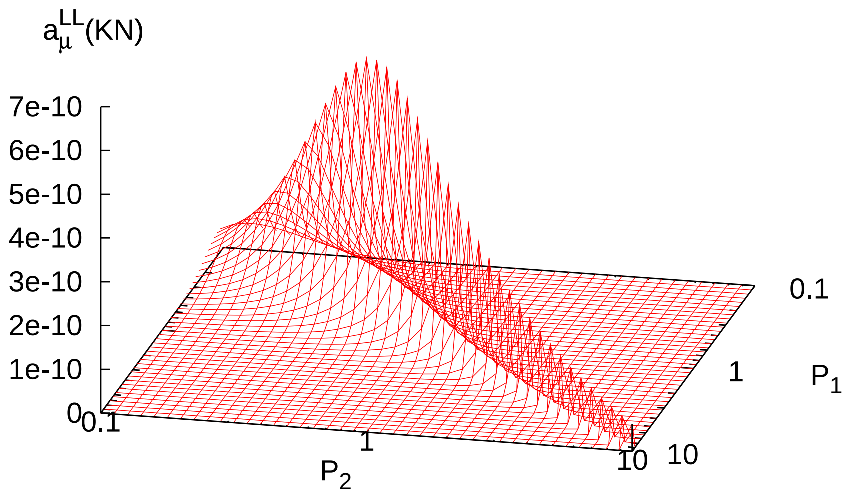
- A. Nyffeler: constraint via magnetic susceptibility
  $a_\mu^{\pi^0} = 7.2 \times 10^{-10}$
  A. Nyffeler, Phys. Rev. D **79** (2009) 073012 [arXiv:0901.1172 [hep-ph]].

π^0 exchange

- Which momentum regimes important studied: JB and J. Prades, Mod. Phys. Lett. A **22** (2007) 767 [hep-ph/0702170]
- $a_\mu = \int dl_1 dl_2 a_\mu^{LL}$ with $l_i = \log(P_i/GeV)$





Checking which momentum regions do what (but would need three dimensional)

Pseudoscalar exchange

- Point-like VMD: π^0 η and η' give 5.58, 1.38, 1.04.
- Models that include $U(1)_A$ breaking give similar ratios
- Pure large N_c models use this ratio
- The MV argument should give some enhancement over the full VMD like models
- Total pseudo-scalar exchange is about $a_\mu^{PS} = 8 - 10 \times 10^{-10}$
- AdS/QCD estimate (includes excited pseudo-scalars) $a_\mu^{PS} = 10.7 \times 10^{-10}$

  D. K. Hong and D. Kim, Phys. Lett. B **680** (2009) 480 [arXiv:0904.4042 [hep-ph]]

Axial-vector exchange exchange

Cut-off Λ (GeV)	$a_\mu \times 10^{10}$ from Axial-Vector Exchange $\mathcal{O}(N_c)$
0.5	0.05(0.01)
0.7	0.07(0.01)
1.0	0.13(0.01)
2.0	0.24(0.02)
4.0	0.59(0.07)

There is some pseudo-scalar exchange piece here as well, off-shell not quite clear what is what.

- $a_\mu^{\mathrm{axial}} = 0.6 \times 10^{-10}$
- MV: short distance enhancement + mixing (both enhance about the same)
  $a_\mu^{\mathrm{axial}} = 2.2 \times 10^{-10}$

Pure quark loop

Cut-off Λ (GeV)	$a_\mu \times 10^7$ Electron Loop	$a_\mu \times 10^9$ Muon Loop	$a_\mu \times 10^9$ Constituent Quark Loop
0.5	2.41(8)	2.41(3)	0.395(4)
0.7	2.60(10)	3.09(7)	0.705(9)
1.0	2.59(7)	3.76(9)	1.10(2)
2.0	2.60(6)	4.54(9)	1.81(5)
4.0	2.75(9)	4.60(11)	2.27(7)
8.0	2.57(6)	4.84(13)	2.58(7)
Known Results	2.6252(4)	4.65	2.37(16)

M_Q : 300 MeV

now all known analytically

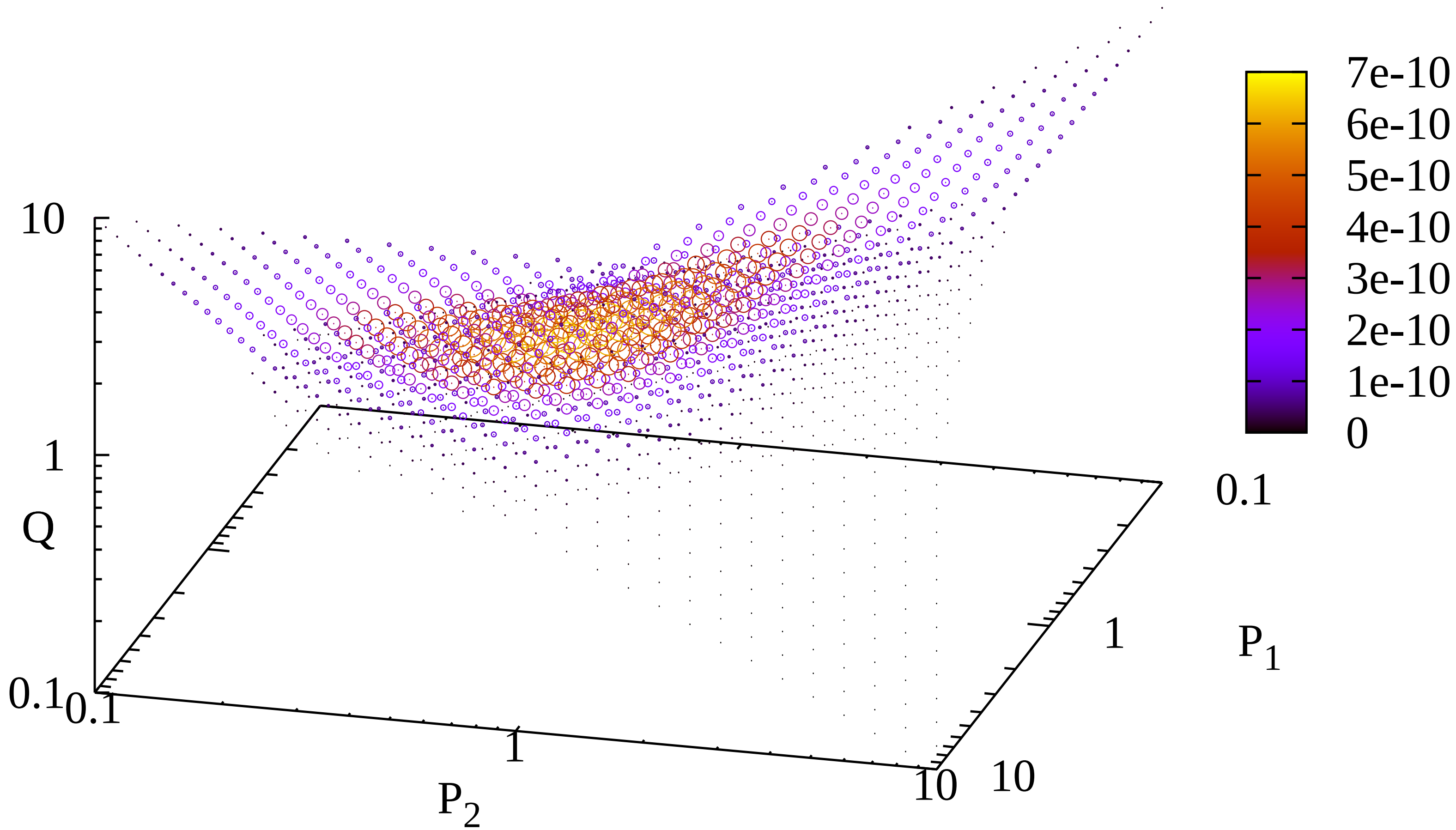
Us: 5+(3-1) integrals

extra are Feynman parameters

Slow convergence:

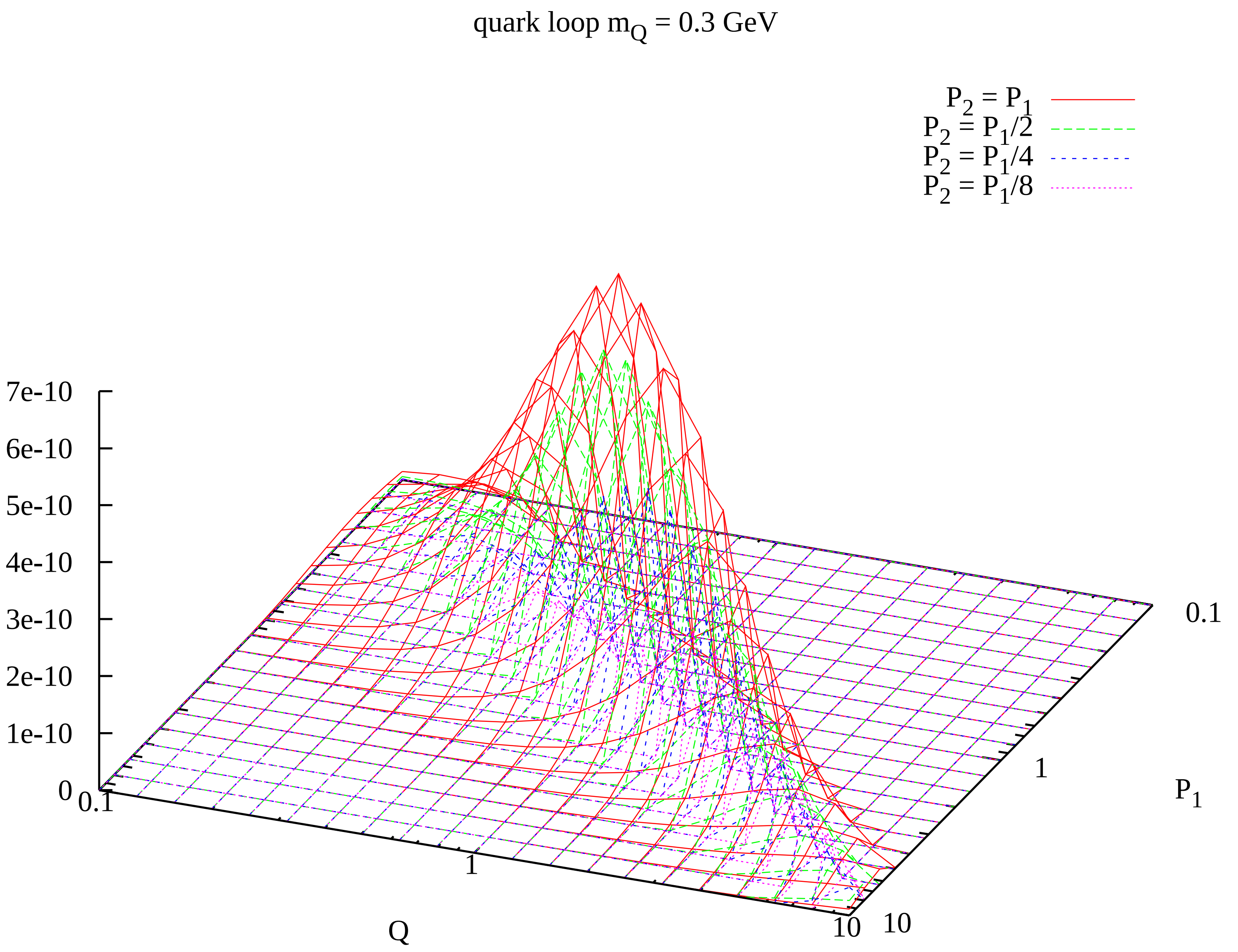
- electron: all at 500 MeV
- Muon: only half at 500 MeV, at 1 GeV still 20% missing
- 300 MeV quark: at 2 GeV still 25% missing

Pure quark loop: momentum area



- This plots $a_\mu^{\rm ql} = \int dl_{P_1} dl_{P_2} dl_Q a_\mu^{\rm LLQ}$
- Succeeded in 3D plot but was useless
- JB-Zahiri-Abyaneh, work in progress

Pure quark loop: momentum area



Most from $P_1 \approx P_2 \approx Q$, sizable large momentum part

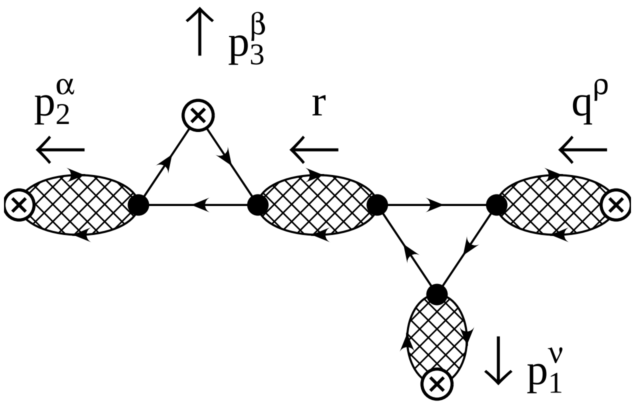
ENJL quark-loop

Cut-off Λ GeV	$a_\mu \times 10^{10}$ Quark-loop VMD	$a_\mu \times 10^{10}$ Quark-loop ENJL	$a_\mu \times 10^{10}$ Quark-loop masscut	$a_\mu \times 10^{10}$ sum ENL+masscut
0.5	0.48	0.78	2.46	3.2
0.7	0.72	1.14	1.13	2.3
1.0	0.87	1.44	0.59	2.0
2.0	0.98	1.78	0.13	1.9
4.0	0.98	1.98	0.03	2.0
8.0	0.98	2.00	.005	2.0

Very stable

- ENJL cuts off slower than pure VMD
- masscut: $M_Q = \Lambda$ to have short-distance and no problem with momentum regions
- Quite stable in region 1-4 GeV

ENJL: scalar



$$\Pi^{\rho\nu\alpha\beta}(p_1,p_2,p_3) \quad = \quad \overline{\Pi}^{VVS}_{ab}(p_1,r) g_S \left(1 + g_S \Pi^S(r)\right) \overline{\Pi}^{SVV}_{cd}(p_2,p_3) \mathcal{V}^{abcd\rho\nu\alpha\beta}(p_1,p_2,p_3)$$
$$+\text{permutations}$$

$$g_S\left(1+g_S\Pi_S\right) = \frac{g_A(q^2)(2M_Q)^2}{2f^2(q^2)}\frac{1}{M_S^2(q^2)-q^2}$$

$\mathcal{V}^{abcd\rho\nu\alpha\beta}$ was ENJL VMD legs

In ENJL only scalar+quark-loop properly chiral invariant

ENJL: scalar/QL

Cut-off Λ GeV	$a_\mu \times 10^{10}$ Quark-loop VMD	$a_\mu \times 10^{10}$ Quark-loop ENJL	$a_\mu \times 10^{10}$ Scalar Exchange
0.5	0.48	0.78	−0.22
0.7	0.72	1.14	−0.46
1.0	0.87	1.44	−0.60
2.0	0.98	1.78	−0.68
4.0	0.98	1.98	−0.68
8.0	0.98	2.00	−0.68

- Note: ENJL+scalar (BPP) $\approx$ Quark-loop VMD (HKS)
- $M_S \approx 620$ MeV certainly an overestimate for real scalars
- If scalar is σ: related to pion loop part?
- quark-loop: $a_\mu^{ql} \approx 1 \times 10^{-10}$ bare $a_\mu^{ql} = 2.37 \times 10^{-10}$

Quark loop DSE

- DSE model: $a_\mu^{ql} = 13.6(5.9) \times 10^{-10}$ T. Goecke, C. S. Fischer and R. Williams, Phys. Rev. D **83** (2011) 094006 [arXiv:1012.3886 [hep-ph]]
- Not a full calculation (yet) but includes an estimate of some of the missing parts
- Note: a lot larger than bare quark loop with constituent mass
- I am puzzled: this DSE model (Maris-Roberts) does reproduce a lot of low-energy phenomenology. I would have guessed that it would be very similar to ENJL in its results.
- Can one find something in between full DSE and ENJL that is easier to handle?

π and K-loop

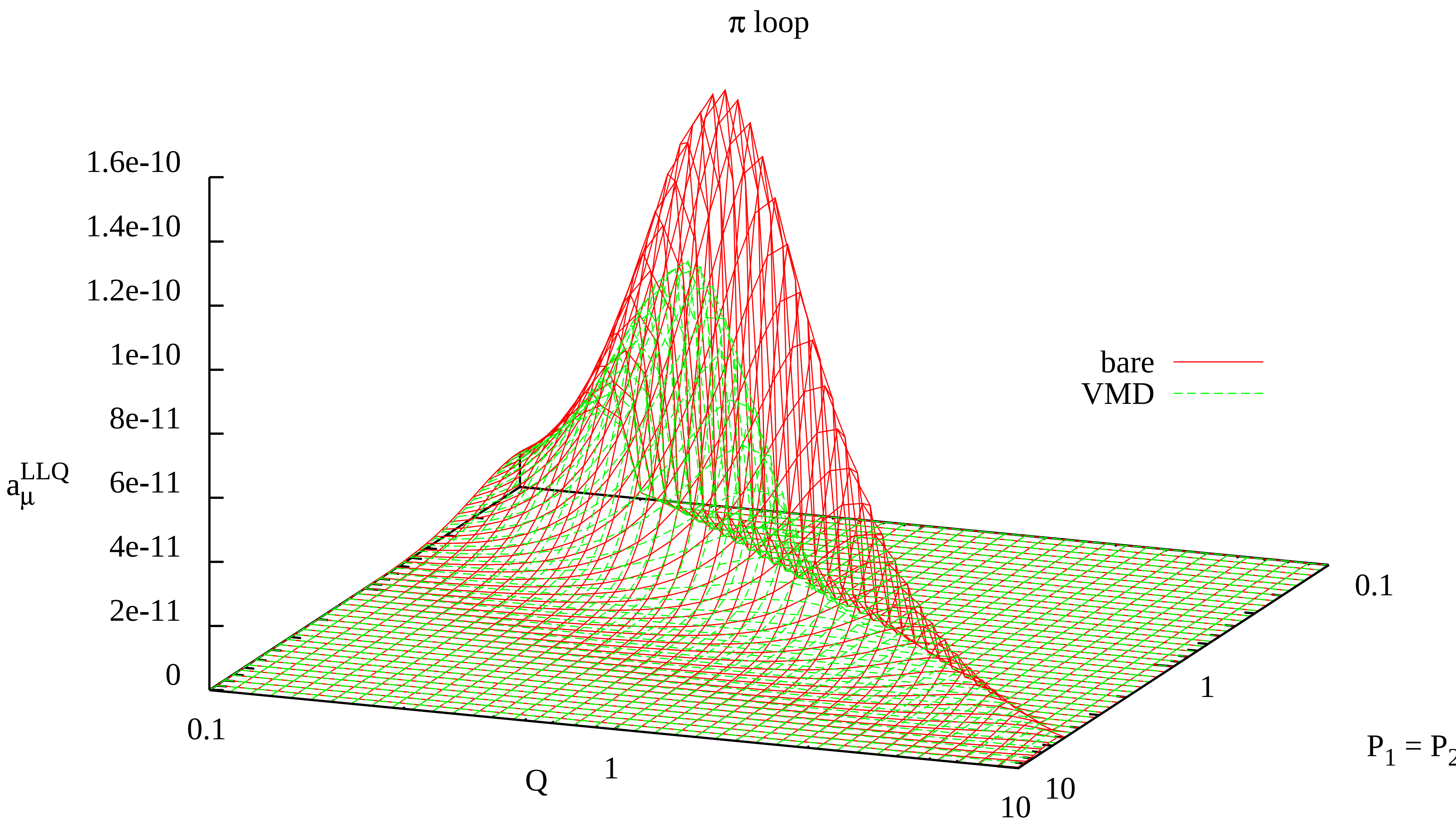
- The $\pi\pi\gamma^*$ vertex is always done using VMD
- $\pi\pi\gamma^*\gamma^*$ vertex two choices:
  - Hidden local symmetry model: only one γ has VMD
  - Full VMD
  - Both are chirally symmetric
  - Check if they live up to MV short distance (Full VMD does, HLS not checked yet)
  - The HLS model used has problems with π^+-π^0 mass difference (due not having an a_1)
- Final numbers quite different: -0.045 and -0.19
- For BPP stopped at 1 GeV but within 10% of higher Λ

π and K-loop

Cut-off	$10^{10} a_\mu$				
GeV	π bare	π VMD	π ENJL	π HLS	K ENJL
0.5	−1.71(7)	−1.16(3)	−1.20(0.03)	−1.05(0.01)	−0.020(0.001)
0.6	−2.03(8)	−1.41(4)	−1.42(0.03)	−1.15(0.01)	−0.026(0.001)
0.7	−2.41(9)	−1.46(4)	−1.56(0.03)	−1.17(0.01)	−0.034(0.001)
0.8	−2.64(9)	−1.57(6)	−1.67(0.04)	−1.16(0.01)	−0.042(0.001)
1.0	−2.97(12)	−1.59(15)	−1.81(0.05)	−1.07(0.01)	−0.048(0.002)
2.0	−3.82(18)	−1.70(7)	−2.16(0.06)	−0.68(0.01)	−0.087(0.005)
4.0	−4.12(18)	−1.66(6)	−2.18(0.07)	−0.50(0.01)	−0.099(0.005)

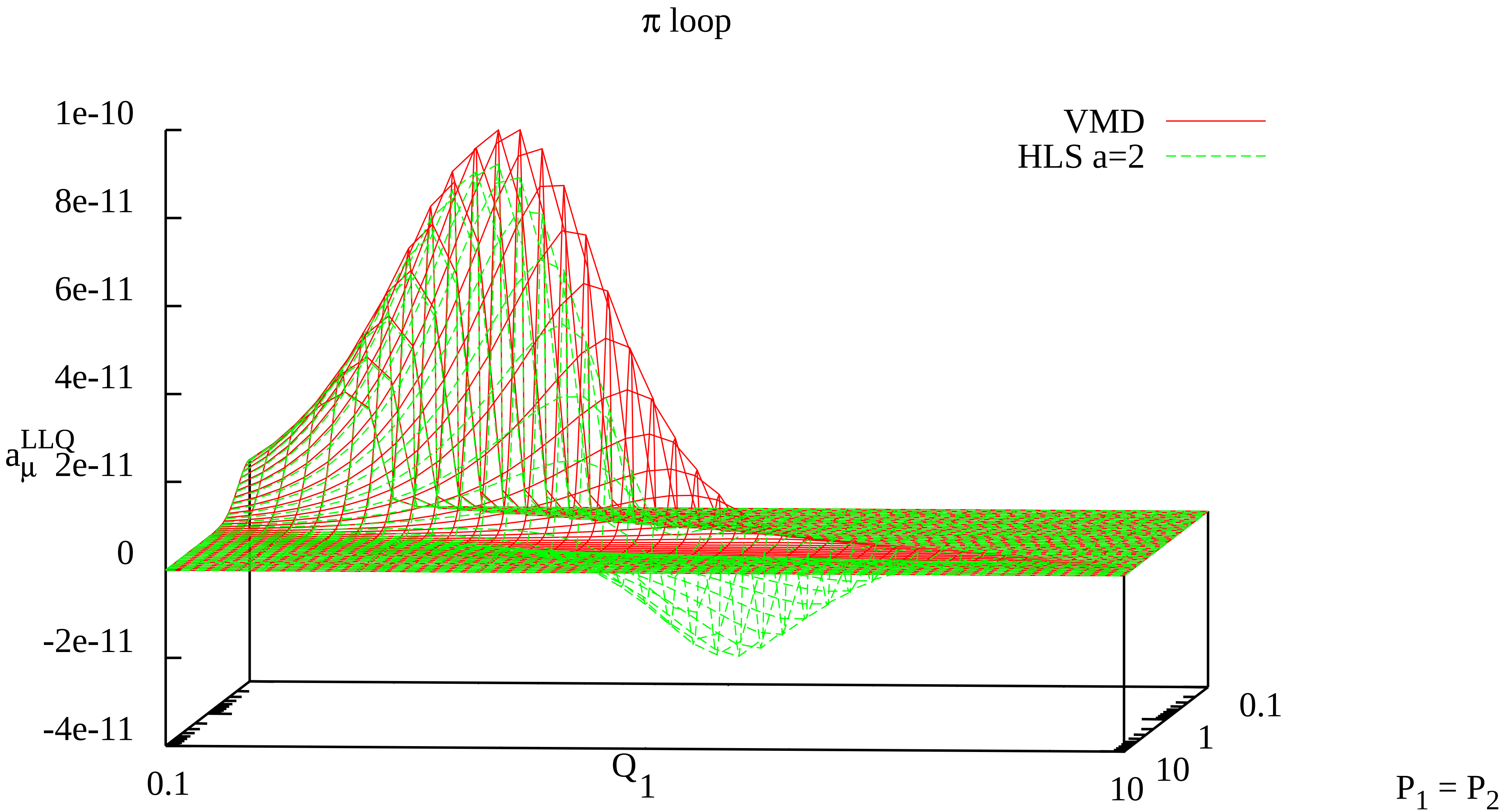
- HLS JB-Zahiri-Abyaneh
- note the suppression by the propagators

π loop: Bare vs VMD



Note: plotted $-a_\mu^{LLQ}$

π loop: VMD vs HLS



π loop

- $\pi\pi\gamma^*\gamma^*$ for $q_1^2 = q_2^2$ has a short-distance constraint from the OPE as well.
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- Ramsey-Musolf suggested to do pure ChPT for the π loop K. T. Engel, H. H. Patel and M. J. Ramsey-Musolf, arXiv:1201.0809 [hep-ph].
- So far ChPT at p^4 done for four-point function in limit $p_1, p_2, q \ll m_\pi$ (Euler-Heisenberg plus next order)
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- Both HLS and VMD have charge radius effect but not polarizability

π loop: L_9, L_{10}

- ChPT for muon $g-2$ at order p^6 is not powercounting finite so no prediction for a_μ exists.
- But can be used to study the low momentum end of the integral over P_1, P_2, Q
- The four-photon amplitude is finite still at two-loop order (counterterms start at order p^8)

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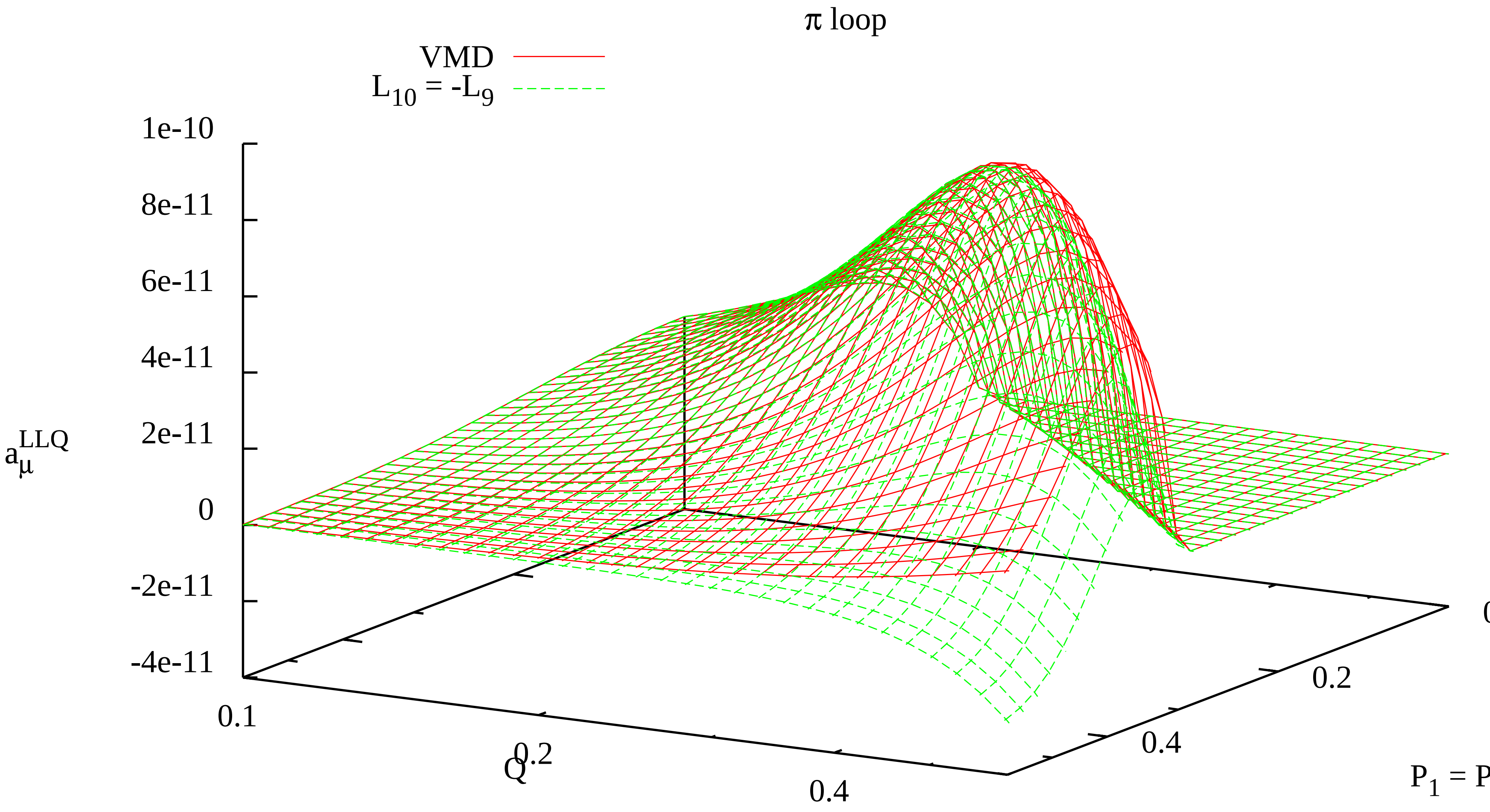
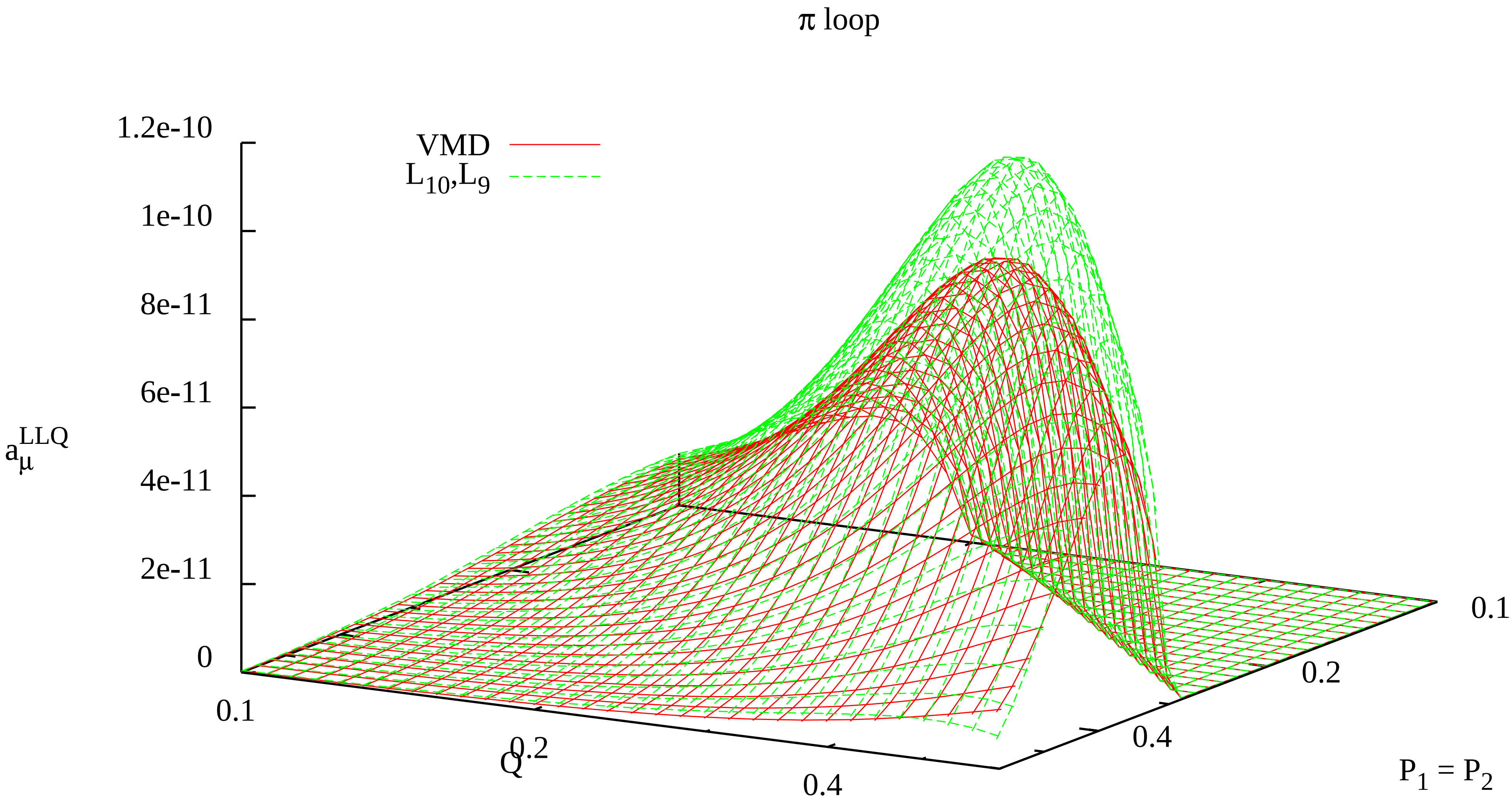
- But can be used to study the low momentum end of the integral over P_1, P_2, Q
- The four-photon amplitude is finite still at two-loop order (counterterms start at order p^8)
- Add L_9 and L_{10} vertices to the bare pion loop

  JB-Zahiri-Abyaneh
- Program the Euler-Heisenberg plus NLO result of Ramsey-Musolf et al. into our programs for a_μ
- Bare pion-loop and L_9, L_{10} part in limit $p_1, p_2, q \ll m_\pi$ agree with Euler-Heisenberg plus next order analytically
- Numerics very preliminary

π loop: VMD vs charge radius



π loop: VMD vs L_9 and L_{10}



Summary: ENJL vc PdRV

	BPP	PdRV arXiv:0901.0306
quark-loop	$(2.1 \pm 0.3) \cdot 10^{-10}$	—
pseudo-scalar	$(8.5 \pm 1.3) \cdot 10^{-10}$	$(11.4 \pm 1.3) \cdot 10^{-10}$
axial-vector	$(0.25 \pm 0.1) \cdot 10^{-10}$	$(1.5 \pm 1.0) \cdot 10^{-10}$
scalar	$(-0.68 \pm 0.2) \cdot 10^{-10}$	$(-0.7 \pm 0.7) \cdot 10^{-10}$
πK-loop	$(-1.9 \pm 1.3) \cdot 10^{-10}$	$(-1.9 \pm 1.9) \cdot 10^{-10}$
errors	linearly	quadratically
sum	$(8.3 \pm 3.2) \cdot 10^{-10}$	$(10.5 \pm 2.6) \cdot 10^{-10}$

What can we do more?

- Constraints from experiment: J. Bijnens and F. Persson, hep-ph/hep-ph/0106130 Studying three formfactors $P\gamma^*\gamma^*$ in $P \to \ell^+\ell^-\ell'^+\ell'^-$, $e^+e^- \to e^+e^- P$ exact tree level and for $g-2$ (but beware sign):
  - Conclusion: possible but VERY difficult
  - Two γ^* off-shell not so important for our choice of form-factor
- All information on hadrons and 1-2-3-4 off-shell photons is welcome: constrain the models
- More short-distance constraints: MV, Nyffeler integrate with all contributions, not just π^0-exchange
- Need a new overall evaluation with consistent approach.

What can we do more?

- The ENJL model can certainly be improved:
  - Chiral nonlocal quark-model (like nonlocal ENJL): so far only π^0-exchange done
  - DSE: π^0-exchange similar to everyone else, quark-loop very different, looking forward to final results
- More resonances models should be tried, AdS/QCD is one approach, RχT (Valencia *et al.*) possible,...
- Note short-distance matching must be done in many channels, there are theorems JB,Gamiz,Lipartia,Prades that with only a few resonances this requires compromises
- π-loop: HLS smaller than double VMD (understood) models with ρ and a_1 (in progress)

Leading Logarithms

- Take a quantity with a single scale: $F(M)$
- The dependence on the scale in field theory is typically logarithmic
- $L = \log(\mu/M)$
- $F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + F_3^3 L^3 + \cdots$
- Leading Logarithms: The terms $F_m^m L^m$

The F_m^m can be more easily calculated than the full result

- $\mu\left(dF/d\mu\right) \equiv 0$
- Ultraviolet divergences in Quantum Field Theory are always local

Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$
- f_i^j are pure numbers
- $\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$

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- f_i^j are pure numbers
- $\mu\dfrac{d\alpha}{d\mu} = \beta_0\alpha^2 + \beta_1\alpha^3 + \cdots$
- $\mu\dfrac{dF}{d\mu} = 0 \Longrightarrow \boxed{\beta_0 = -f_1^1 = f_2^2 = -f_3^3 = \cdots}$

Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu \frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–'t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the α the same in all orders
- LL one-loop β_0
- NLL two-loop β_1, f_0^1

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- LL one-loop β_0
- NLL two-loop β_1, f_0^1
- In effective field theories: different Lagrangian at each order
- The recursive argument does not work

Weinberg's argument

- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop
- Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- General at 2 loop: JB, Colangelo, Ecker, hep-ph/9808421
- Proof at all orders using β-functions
  Büchler, Colangelo, hep-ph/0309049
- Proof with diagrams: JB, Carloni, arXiv:0909.5086

Weinberg's argument

- μ: dimensional regularization scale
- $d = 4 - w$
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- goes on
  - $1/w^{n-1}, log\mu/w^{n-2}, \ldots, log\mu^{n-2}/w$
  - Get subleading logs $\log^{n-1}\mu$ from two-loop diagrams
  - subsubleading logs from 3-loop diagrams,. . .

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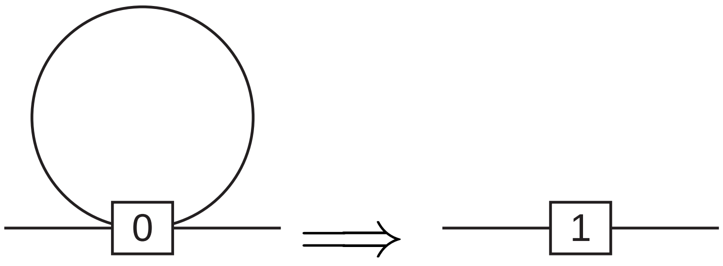
  - Get subleading logs $\log^{n-1} \mu$ from two-loop diagrams
  - subsubleading logs from 3-loop diagrams,. . .
- Many 1-loop diagrams (each harder for higher orders)

Mass to $\hbar^2$

- $\hbar^1$: 

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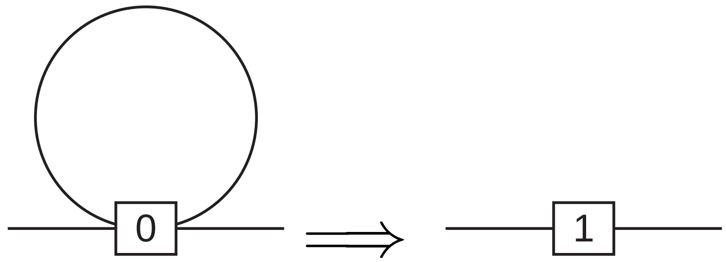
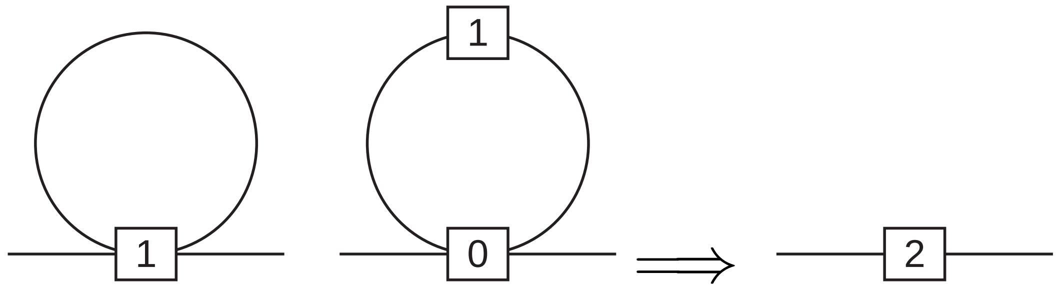
- $\hbar^1$: 

- $\hbar^2$: 

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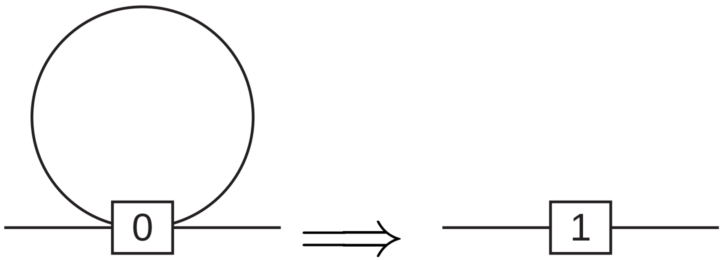
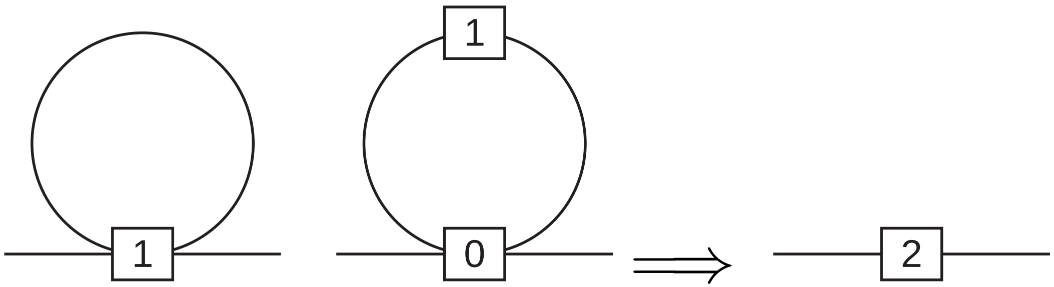
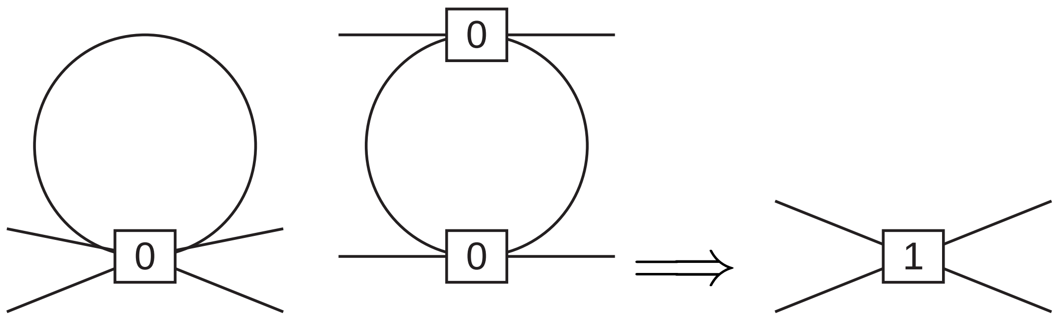
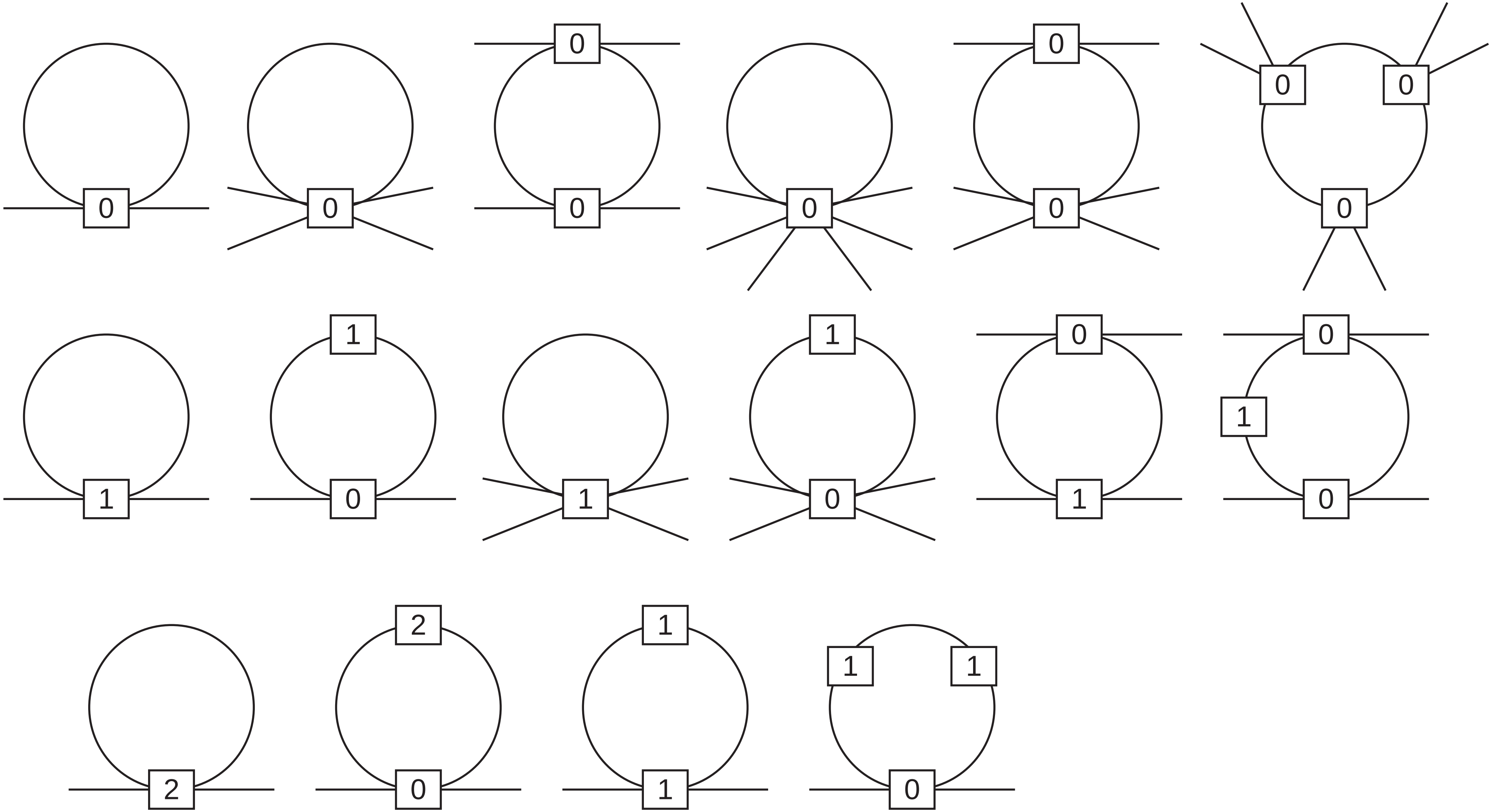
- $\hbar^1$: 

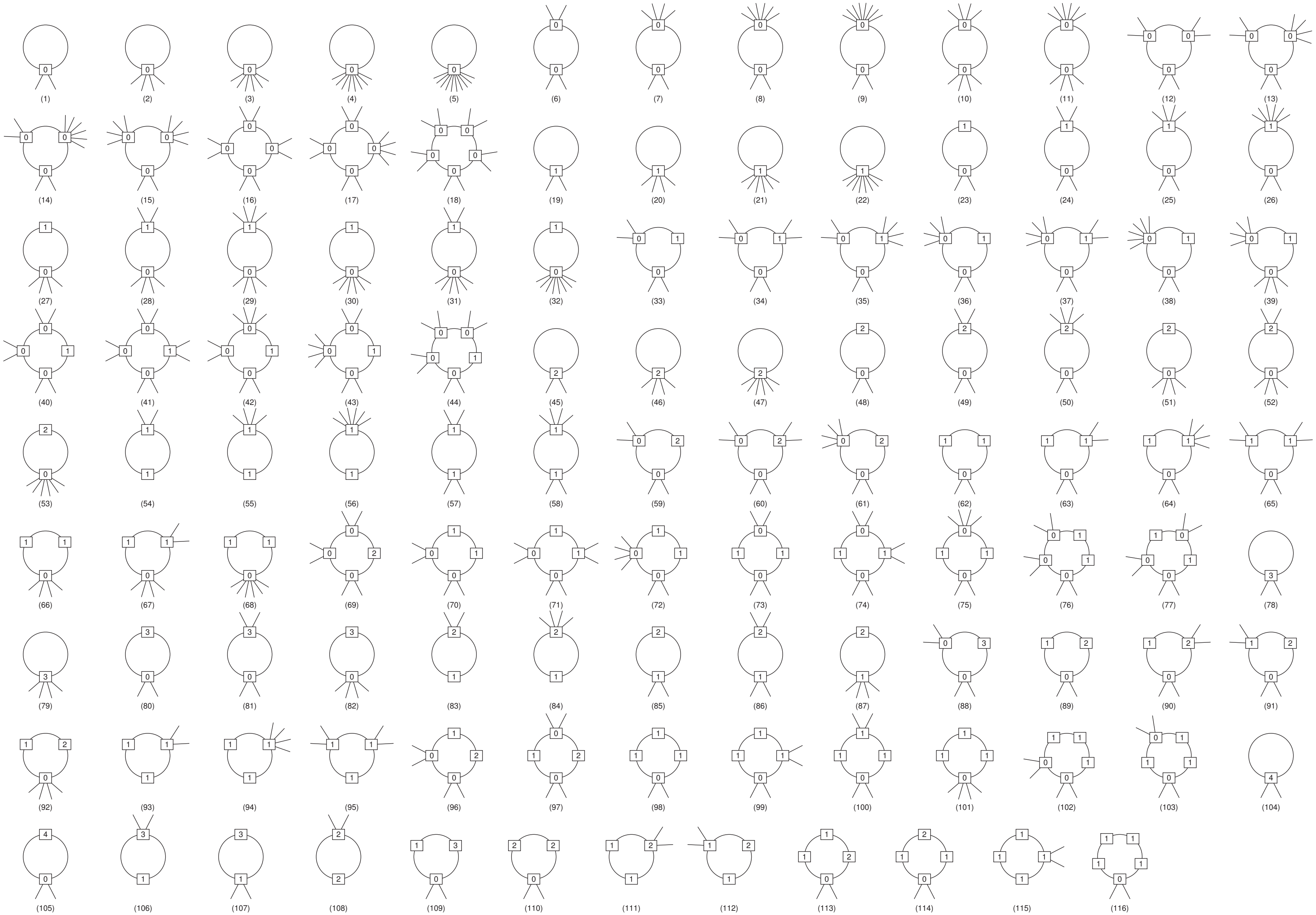
- $\hbar^2$: 

- but also needs $\hbar^1$: 

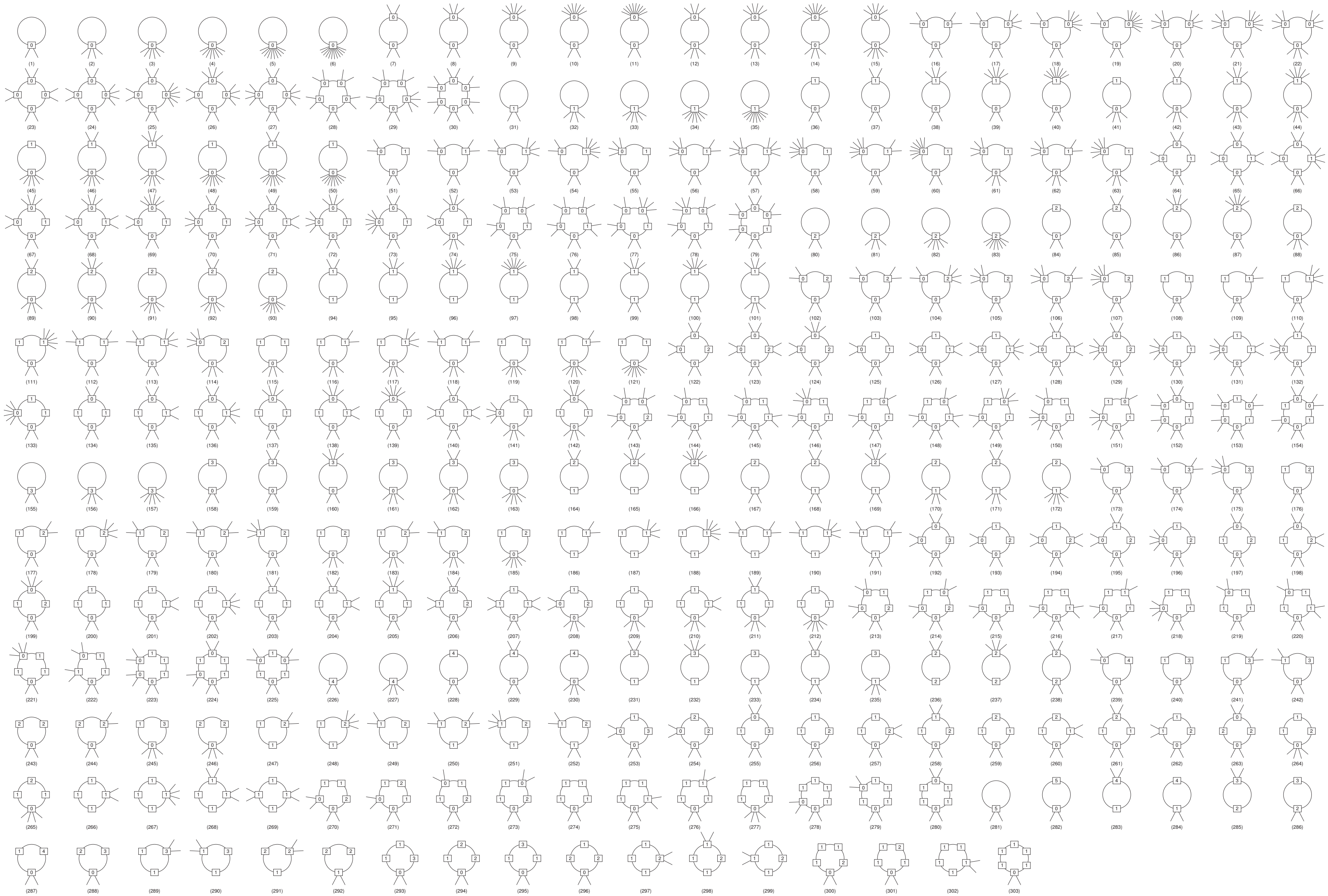
Mass to order $\hbar^3$



Mass to order $\hbar^5$



Mass to order $\hbar^6$



Mass+decay to $\hbar^5$

- $\hbar^1$: 18 + 27
- $\hbar^2$: 26 + 45
- $\hbar^3$: 33 + 51
- $\hbar^4$: 26 + 33
- $\hbar^5$: 13 + 13

- Calculate the divergence
- rewrite it in terms of a local Lagrangian
- Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
- We keep all terms to have all 1PI (one particle irreducible) diagrams finite

Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu\Phi^T\partial^\mu\Phi + F^2\chi^T\Phi\,.$
- Φ is a real $N+1$ vector; $\Phi \to O\Phi$; $\Phi^T\Phi = 1$.
- Vacuum expectation value $\langle\Phi^T\rangle = (1\;0\ldots 0)$
- Explicit symmetry breaking: $\chi^T = \left(M^2\;0\ldots 0\right)$
- Both spontaneous and explicit symmetry breaking
- N-vector ϕ
- N (pseudo-)Nambu-Goldstone Bosons
- $N = 3$ is two-flavour Chiral Perturbation Theory

Massive $O(N)$ sigma model: Φ vs ϕ

- $$\Phi_1 = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}$$ Gasser, Leutwyler

- $$\Phi_2 = \frac{1}{\sqrt{1+\frac{\phi^T\phi}{F^2}}} \begin{pmatrix} 1 \\ \frac{\phi}{F} \end{pmatrix}$$ similar to Weinberg

  $$\Phi_3 = \begin{pmatrix} 1-\frac{1}{2}\frac{\phi^T\phi}{F^2} \\ \sqrt{1-\frac{1}{4}\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{F} \end{pmatrix}$$ only mass term

- $$\Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$ CCWZ

Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use the four different parametrizations
- compare with known results:

  $$M^2_{phys} = M^2\left(1 - \frac{1}{2}L_M + \frac{17}{8}L_M^2 + \cdots\right),$$

  $$L_M = \frac{M^2}{16\pi^2F^2}\log\frac{\mu^2}{\mathcal{M}^2}$$

  Usual choice $\mathcal{M} = M$.
- large N (but known results only for massless case)
  Coleman, Jackiw, Politzer 1974
- large N massive later found partly in appendix of Kivel, Polyakov, Vladimirov on distribution functions.

Results

$$M^2_{\text{phys}} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

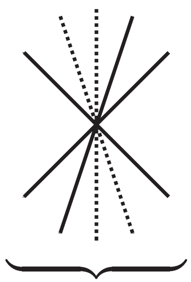
i	a_i, $N=3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$

F_{phys}, $\langle \bar{q}_i q_i \rangle$ as well done

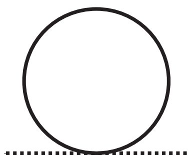
Anyone recognize any funny functions?

Large N

Power counting: pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N$, $M^2 \sim 1$

- 

  $\Leftrightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$

  $\Leftrightarrow N$

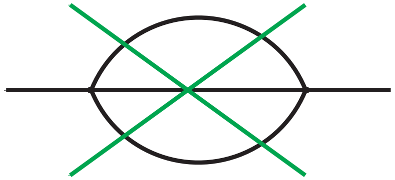
- 1PI diagrams:

$$\left.\begin{array}{l} N_L = N_I - \sum_n N_{2n} + 1 \\ 2N_I + N_E = \sum_n 2nN_{2n} \end{array}\right\} \Rightarrow N_L = \sum_n (n-1) N_{2n} - \frac{1}{2} N_E + 1$$

- diagram suppression factor: $\dfrac{N^{N_L}}{N^{N_E/2-1}}$

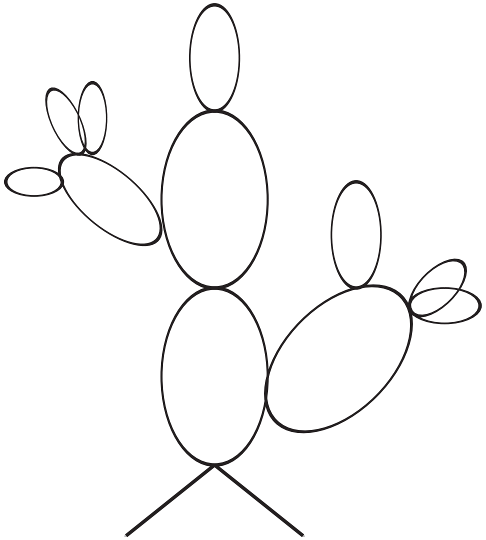
Large N

- diagrams with shared lines are suppressed



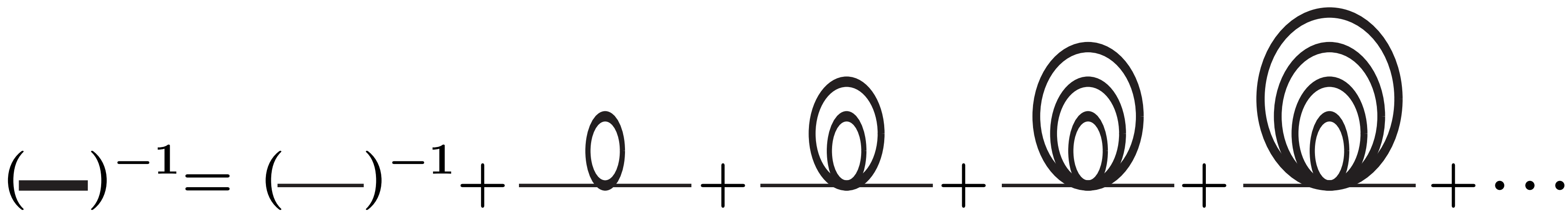
each new loop needs also a new flavour loop

- in the large N limit only “*cactus*” diagrams survive:



large N: propagator

Generate recursively via a Gap equation



$\Rightarrow$ resume the series and look for the pole

$$M^2 = M^2_{\text{phys}}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

$$\overline{A}(m^2) = \frac{m^2}{16\pi^2}\log\frac{\mu^2}{m^2}.$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N

$$F_{\text{phys}} = F\sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

$$\langle\bar{q}q\rangle_{\text{phys}} = \langle\bar{q}q\rangle_0\sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

Comments:

- These are the full* leading N results, not just leading log
- But depends on the choice of N-dependence of higher order coefficients
- Assumes higher LECs zero ($< N^{n+1}$ for $\hbar^n$)
- Large N as in $O(N)$ not large N_c

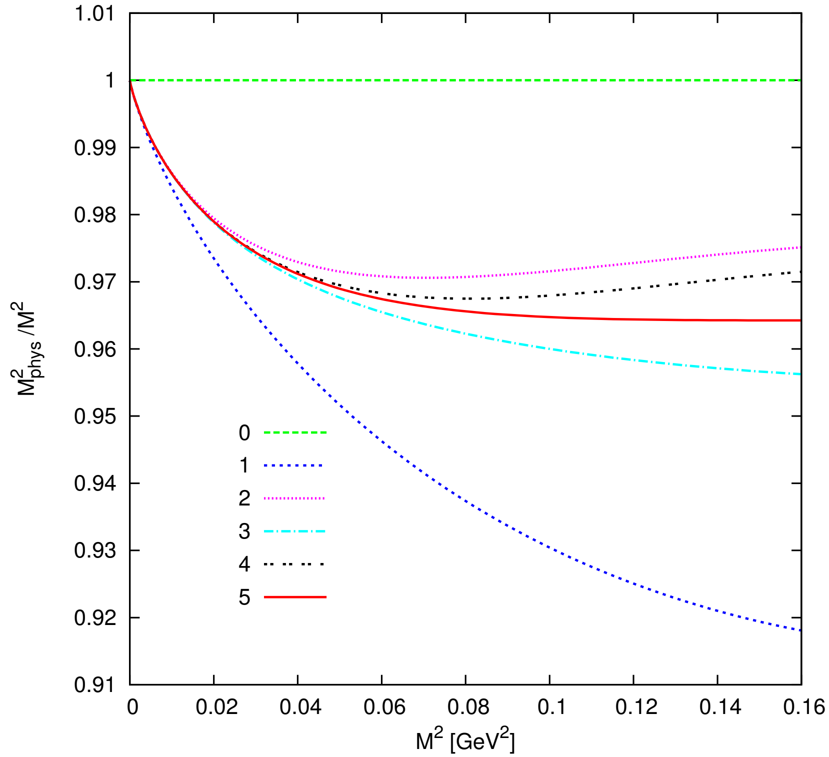
Large N: Checking expansions

$$M^2 = M^2_{\rm phys}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\rm phys})}$$

much smaller expansion coefficients than the table, try

$$M^2 = M^2_{\rm phys}(1 + d_1 L_{M_{\rm phys}} + d_2 L^2_{M_{\rm phys}} + d_3 L^3_{M_{\rm phys}} + \ldots)$$

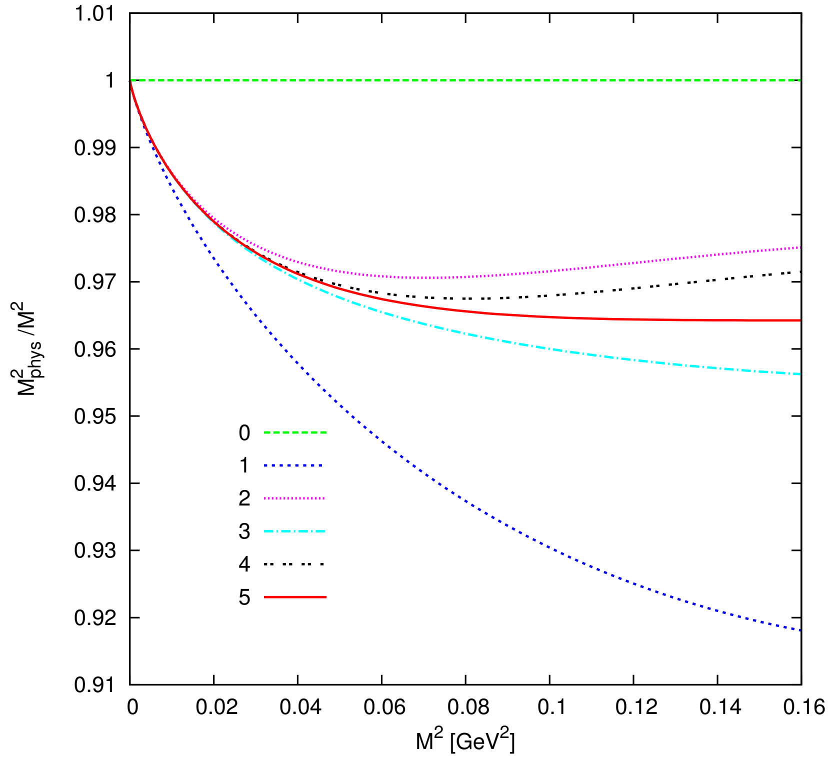
Numerical results

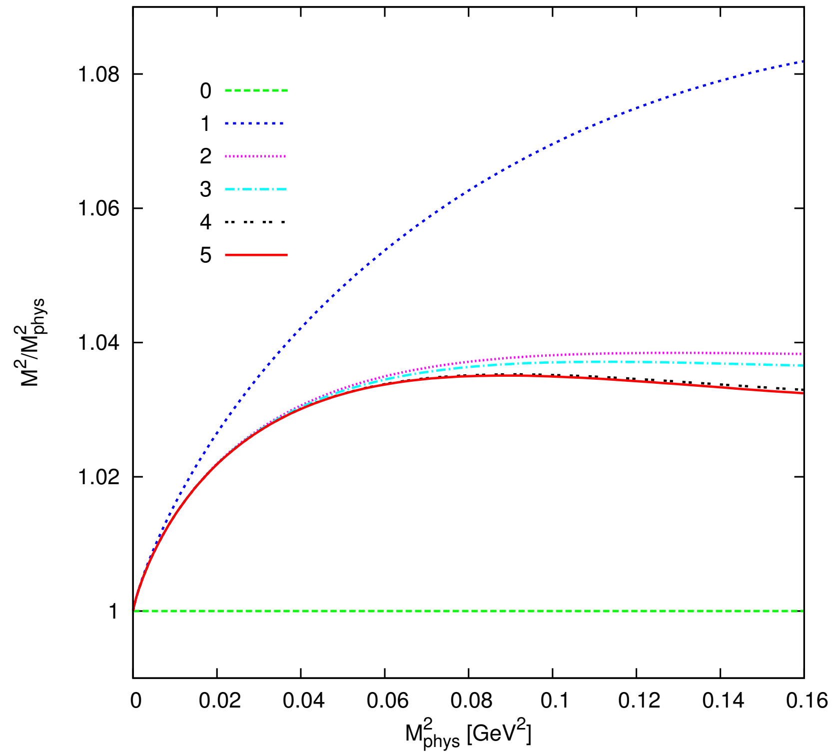


Left: $\frac{M^2_{\rm phys}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

F = 90 MeV, $\mu = 0.77$ GeV

Numerical results





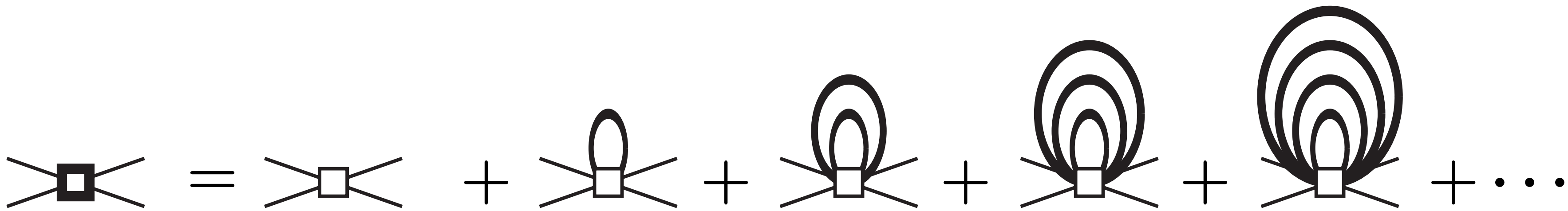
Left: $\frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

Right: $\frac{M^2}{M^2_{\text{phys}}} = 1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + \cdots$

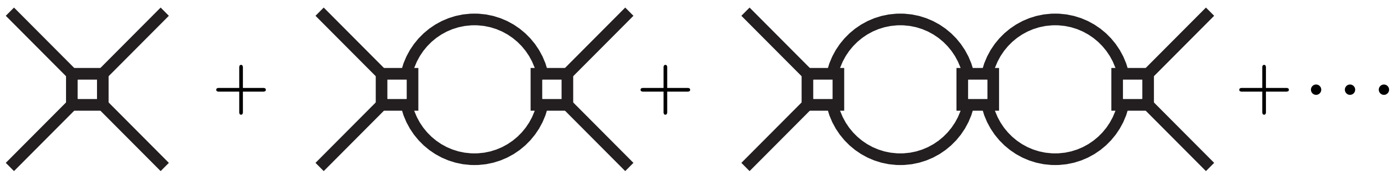
F = 90 MeV, $\mu = 0.77$ GeV

Large N: $\pi\pi$-scattering

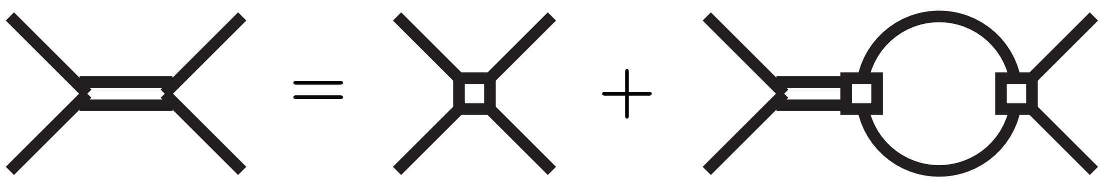
- Cactus diagrams for $A(s,t,u)$
- Branch with no momentum: resummed by —
- Branch starting at vertex: resum by



- The full result is then



- Can be summarized by a recursive equation



Large N: $\pi\pi$ scattering

$$y = \frac{N}{F^2}\overline{A}(M^2_{\rm phys})$$

$$A(s,t,u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{N}{2}\left(\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}\right)\overline{B}(M^2_{\rm phys}, M^2_{\rm phys}, s)}$$

or

$$A(s,t,u) = \frac{\frac{s-M^2_{\rm phys}}{F^2_{\rm phys}}}{1 - \frac{N}{2}\frac{s-M^2_{\rm phys}}{F^2_{\rm phys}}\overline{B}(M^2_{\rm phys}, M^2_{\rm phys}, s)}$$

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results

Anomaly for $O(4)/O(3)$

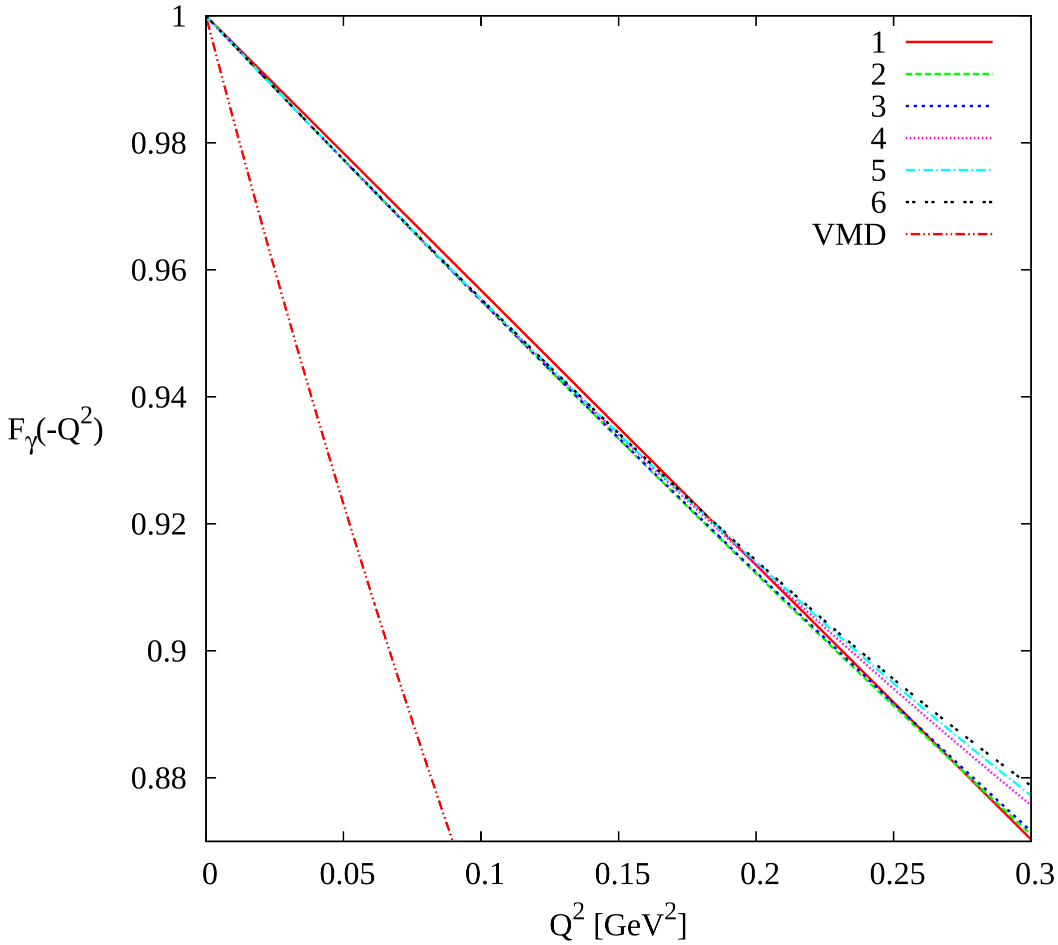
JB, Kampf, Lanz, arXiv:1201.2608

- $$\mathcal{L}_{WZW} = -\frac{N_c}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\Big\{\epsilon^{abc}\left(\frac{1}{3}\Phi^0\partial_\mu\Phi^a\partial_\nu\Phi^b\partial_\rho\Phi^c - \partial_\mu\Phi^0\partial_\nu\Phi^a\partial_\rho\Phi^b\Phi^c\right)v^0_\sigma$$
$$+(\partial_\mu\Phi^0\Phi^a - \Phi^0\partial_\mu\Phi^a)v^a_\nu\partial_\rho v^0_\sigma + \frac{1}{2}\epsilon^{abc}\Phi^0\Phi^a v^b_\mu v^c_\nu\partial_\rho v^0_\sigma\Big\}.$$
- $A(\pi^0 \to \gamma(k_1)\gamma(k_2)) =$
$\epsilon_{\mu\nu\alpha\beta}\,\varepsilon_1^{*\mu}(k_1)\varepsilon_2^{*\nu}(k_2)\,k_1^\alpha k_2^\beta\,F_{\pi\gamma\gamma}(k_1^2,k_2^2)$
- $F_{\pi\gamma\gamma}(k_1^2,k_2^2) = \dfrac{e^2}{4\pi^2F_\pi}\hat{F}F_\gamma(k_1^2)F_\gamma(k_2^2)F_{\gamma\gamma}(k_1^2,k_2^2)$
- $\hat{F}$: on-shell photon; $F_\gamma(k^2)$: formfactor;
$F_{\gamma\gamma}$ nonfactorizable

Anomaly for $O(4)/O(3)$

- Done to six-loops
- $\hat{F} = 1 + 0 - 0.000372 + 0.000088 + 0.000036 + 0.000009 + 0.0000002 + \ldots$
- Really good convergence
- $F_{\gamma\gamma}$ only starts at three-loop order (could have been two)
- $F_{\gamma\gamma}$ in the chiral limit only starts at four-loops.
- The leading logarithms thus predict this part to be fairly small.
- $F_\gamma(k^2)$: plot

Anomaly for $O(4)/O(3)$



Leading logs small, converge fast

$\gamma 3\pi$

- Experiment 1: $\bar{F}^{3\pi}_{\text{exp}} = 12.9 \pm 0.9 \pm 0.5$ GeV^{-3}
- Experiment 2: $F^{3\pi}_{0,\text{exp}} = 9.9 \pm 1.1$ GeV^{-3}
- Theory lowest order: $F^{3\pi}_0 = 9.8$ GeV^{-3}
- Theory (LL only)
  $F^{3\pi LL}_0 = (9.8 - 0.3 + 0.04 + 0.02 + 0.006 + 0.001 + \ldots)$ GeV^{-3}
- good convergence

Other results

- JB,Carloni, arXiv:1008.3499
  - massive case: $\pi\pi$, F_V and F_S to 4-loop order
  - large N for these cases also for massive $O(N)$.
  - done using bubble resummations or recursion eqation which can be solved analytically
- JB, Kampf, Lanz, arXiv:1201.2608
  - Mass, F_π, F_V to six loops
  - Anomaly: $\gamma^* 3\pi$ (five) and $\pi^0\gamma^*\gamma^*$ (six loops)
  - large N not relevant in this case
- JB, Kampf, Lanz, in preparation
  - $SU(N) \times SU(N)/SU(N)$

Other results

- Bissegger, Fuhrer, hep-ph/0612096 Dispersive methods, massless Π_S to five loops
- Kivel, Polyakov, Vladimirov, 0809.3236, 0904.3008, 1004.2197, 1012.4205
  - In the massless case tadpoles vanish
  - $\Longrightarrow$ number of external legs needed does not grow
  - All 4-meson vertices via Legendre polynomials
  - can do divergence of all one-loop diagrams analytically
  - algebraic (but quadratic) recursion relations
  - massless $\pi\pi$, F_V and F_S to arbitrarily high order
  - large N agrees with Coleman, Wess, Zumino
  - large N is not a good approximation

Conclusions Leading Logs

- Several quantities in massive $O(N)$ LL known to high loop order
- Large N in massive $O(N)$ model solved
- Had hoped: recognize the series also for general N
- Limited essentially by CPU time and size of intermediate files
- Some first studies on convergence etc.
- $\pi\pi$, F_V and F_S to four-loop order (F_V higher)
- The technique can be generalized to other models/theories
  - $SU(N) \times SU(N)/SU(N)$: under way
  - One nucleon sector: planned/hoped