HADRONIC LIGHT-BY-LIGHT
FOR THE MUON ANOMALY
RENORMALIZATION GROUP

Johan Bijnens
Lund University

bijnens@thep.lu.se
http://www.thep.lu.se/~bijnens

Various ChPT: http://www.thep.lu.se/~bijnens/chpt.html
Overview

Part I: Muon $g - 2$
- Overview
- QED, Electroweak, Hadronic
- Light-by-Light: the various contributions
- Overall properties
- The leading in $N_c$ exchanges and quark-loop
  - $\pi$ and $K$ loop
- Summary Light-by-light

Part II: Renormalization group and leading logarithms
- Leading logarithms: principle
- $O(N)$ model: mass
- Large $N$
- Anomaly
Muon $g - 2$: overview

- in terms of the anomaly $a_\mu = (g - 2)/2$

- Data dominated by BNL E821 (statistics)(systematic)
  
  $a_{\mu^+}^{\text{exp}} = 11659204(6)(5) \times 10^{-10}$
  
  $a_{\mu^-}^{\text{exp}} = 11659215(8)(3) \times 10^{-10}$
  
  $a_{\mu}^{\text{exp}} = 11659208.9(5.4)(3.3) \times 10^{-10}$

- Theory is off somewhat (electroweak)(LO had)(HO had)
  
  $a_{\mu}^{\text{SM}} = 11659180.2(2)(4.2)(2.6) \times 10^{-10}$

- $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 28.7(6.3)(4.9) \times 10^{-10}$ (PDG)

- E821 goes to Fermilab, expect factor of four in precision

- Many BSM models CAN predict a value in this range (often a lot more or a lot less)
Muon $g - 2$: QED

\[
a_{\mu}^{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857410(27) \left( \frac{\alpha}{\pi} \right)^2 + 24.0505964(43) \left( \frac{\alpha}{\pi} \right)^3 \\
+ 130.8055(80) \left( \frac{\alpha}{\pi} \right)^4 + 663(20) \left( \frac{\alpha}{\pi} \right)^5 + \cdots
\]

- First three loops known analytically
- Four-loops fully done numerically
- Five loops estimate
- Kinoshita, Laporta, Remiddi, Schwinger, . . .
- $\alpha$ fixed from the electron $g - 2$: $\alpha = 1/137.035999084(51)$
- $a_{\mu}^{\text{QED}} = 11658471.809(0.015) \times 10^{-10}$

\[e = 20.95, \mu = 0.37, \tau = 0.002\]
Muon $g - 2$: Electroweak

- $a_{\mu}^{\text{EW}}[1-\text{loop}] = 19.48 \times 10^{-10}$
- $a_{\mu}^{\text{EW}}[2-\text{loop}] = -4.07(0.10)(0.18) \times 10^{-10}$
- $a_{\mu}^{\text{EW}} = 15.4(0.1)(0.2) \times 10^{-10}$ (triangle)(Higgs mass)
Muon $g - 2$: LO hadronic

\[ a_{\mu}^{\text{LOhad}} = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_{m_{\pi}^2}^{\infty} ds \frac{K(s)}{s} R^{(0)}(s) \]

- $R^{(0)}(s)$ bare cross-section ratio $\frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$

- Bare, many different evaluations, \ldots

- $a_{\mu}^{\text{LOHad}} = 692.3(4.2)(0.3) \times 10^{-10}$ (exp)(pert. QCD)
Muon $g-2$: HO hadronic

- Two main types of contributions

- HO HVP is like LO Had but a more complicated function
  \[ K(s) \ a_{\mu}^{\text{HO HVP}} = -9.84(0.06) \times 10^{-10} \]

- HLbL is the real problem: best estimate now:
  \[ a_{\mu}^{\text{HLbL}} = 10.5(2.6) \times 10^{-10} \]
## Summary of Muon $g - 2$ contributions

<table>
<thead>
<tr>
<th></th>
<th>$10^{10} a_\mu$</th>
<th></th>
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<tbody>
<tr>
<td>exp</td>
<td>11 659 208.9</td>
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<tr>
<td>LO Had</td>
<td>692.3</td>
<td>4.2</td>
<td></td>
</tr>
<tr>
<td>HO HVP</td>
<td>-9.8</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>HLbL</td>
<td>10.5</td>
<td>2.6</td>
<td></td>
</tr>
<tr>
<td>difference</td>
<td>28.7</td>
<td>8.1</td>
<td></td>
</tr>
</tbody>
</table>

- Error on LO had all $e^+e^-$ based OK
- $\tau$ based 2 $\sigma$
- Error on HLbL
- Errors added quadratically
- 3.5 $\sigma$
- Difference: 4% of LO Had
Our object

Muon line and photons: well known

The blob: fill in with hadrons/QCD

Trouble: low and high energy very mixed

Double counting needs to be avoided: hadron exchanges versus quarks
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
- $p^4$, order 1: pion-loop
- $p^8$, order $N_c$: quark-loop and heavier meson exchanges
- $p^6$, order $N_c$: pion exchange

Does not fully solve the problem
only short-distance quark-loop is really $p^8$
but it’s a start
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
- $p^4$, order 1: pion-loop
- $p^8$, order $N_c$: quark-loop and heavier meson exchanges
- $p^6$, order $N_c$: pion exchange

- Hayakawa, Kinoshita, Sanda: meson models, pion loop using hidden local symmetry, quark-loop with VMD, calculation in Minkowski space
- JB, Pallante, Prades: Try using as much as possible a consistent model-approach, calculation in Euclidean space
Papers: BPP and HKS

- JB, E. Pallante and J. Prades

- Hayakawa, Kinoshita, (Sanda)
Differences

- HK(S)
  - Purely hadronic exchanges
  - quark-loop with hadronic VMD
  - Studied dependence of everything on $m_V$

- BPP
  - Use the ENJL as an overall model to have a similar uncertainty on all low-energy parts
  - repair some of the worst short-comings
  - Add the short-distance quark-loop
  - Study of cut-off dependence
Differences

- **HK(S)**
  - Purely hadronic exchanges
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- **Sign mistake**
  - HKS: Euclidean versus Minkowski $\varepsilon^{\mu \nu \alpha \beta}$
  - BPP: notes all correct sign, program had wrong sign, probably minus sign from fermion loop not removed
The overall

\[ a^{\text{HLbL}}_\mu = \frac{-1}{48 m_\mu} \text{tr}[(\not{p} + m_\mu) M^{\lambda \beta}(0) (\not{p} + m_\mu)[\gamma_\lambda, \gamma_\beta]]. \]

\[ M^{\lambda \beta}(p_3) = |e|^6 \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \frac{1}{q^2 p_1^2 p_2^2 (p_4^2 - m_\mu^2)(p_5^2 - m_\mu^2)} \]

\[ \times \left[ \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_3 \lambda} \right] \gamma_\alpha(\not{p}_4 + m_\mu) \gamma_\nu(\not{p}_5 + m_\mu) \gamma_\rho. \]

- We used: \( \Pi^{\rho \nu \alpha \lambda}(p_1, p_2, p_3) = -p_3 \delta \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_3 \lambda}. \)

- Can calculate at \( p_3 = 0 \) but must take derivative

- derivative makes in quark-loop each permutation finite

- Four point function of \( V_i^\mu(x) \equiv \sum_i Q_i [\bar{q}_i(x) \gamma^\mu q_i(x)] \)

\[ \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) \equiv i^3 \int d^4 x \int d^4 y \int d^4 z e^{i(p_1 \cdot x + p_2 \cdot y + p_3 \cdot z)} \langle 0 | T \left( V_\rho^\alpha(0) V_\nu^\beta(x) V_\gamma^\alpha(y) V_\delta^\beta(z) \right) | 0 \rangle. \]
General properties

\[ \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3): \]

- In general 138 Lorentz structures (but only 32 contribute to \( g - 2 \))
- Using \( q_\rho \Pi^{\rho \nu \alpha \beta} = p_{1\nu} \Pi^{\rho \nu \alpha \beta} = p_{2\alpha} \Pi^{\rho \nu \alpha \beta} = p_{3\beta} \Pi^{\rho \nu \alpha \beta} = 0 \)
- 43 gauge invariant structures
- Bose symmetry relates some of them
- All depend on \( p_1^2, p_2^2 \) and \( q^2 \), but before derivative and \( p_3 \to 0 \) there are more
- Compare HVP: one function, one variable
- General calculation from experiment difficult to see how
General properties

$$\int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4}$$ plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
- 5 remain: $p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu$
- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.
- More recent: can do two more using Gegenbauer techniques Knecht-Nyffeler, Jegerlehner-Nyffeler,JB–Zahiri-Abyaneh
- $P_1^2, P_2^2$ and $Q^2$ remain
General properties

\[ \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \quad \text{plus loops inside the hadronic part} \]

- 8 dimensional integral, three trivial,

- 5 remain: \( p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu \)

- Rotate to Euclidean space:
  - Easier separation of long and short-distance
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- More recent: can do two more using Gegenbauer techniques Knecht-Nyffeler, Jegerlehner-Nyffeler, JB–Zahiri-Abyaneh

- \( P_1^2, P_2^2 \) and \( Q^2 \) remain

- study \( a^X_\mu = \int dl_{P_1} dl_{P_2} a^\text{XLL}_\mu = \int dl_{P_1} dl_{P_2} dl_Q a^\text{XLLQ}_\mu \)

\[ l_P = \ln \left( \frac{P}{\text{GeV}} \right), \text{to see where the contributions are} \]
ENJL: our main model

\[ \mathcal{L}_{\text{ENJL}} = \bar{q}^\alpha \left\{ i \gamma^\mu \left( \partial_\mu - iv_\mu - ia_\mu \gamma_5 \right) - (\mathcal{M} + s - ip\gamma_5) \right\} q^\alpha + 2g_S \left( \bar{q}_R^\alpha q_L^\beta \right) \left( \bar{q}_L^\beta q_R^\alpha \right) \\
- g_V \left[ \left( \bar{q}_L^\alpha \gamma^\mu q_L^\beta \right) \left( \bar{q}_L^\beta \gamma_\mu q_L^\alpha \right) + \left( \bar{q}_R^\alpha \gamma^\mu q_R^\beta \right) \left( \bar{q}_R^\beta \gamma_\mu q_R^\alpha \right) \right] \]

- \( \bar{q} \equiv (\bar{u}, \bar{d}, \bar{s}) \)
- \( v_\mu, a_\mu, s, p \): external vector, axial-vector, scalar and pseudoscalar matrix sources
- \( \mathcal{M} \) is the quark-mass matrix.

- \( g_V \equiv \frac{8\pi^2 G_V(\Lambda)}{N_c \Lambda^2} \quad , \quad g_S \equiv \frac{4\pi^2 G_S(\Lambda)}{N_c \Lambda^2} \).
- \( G_V, G_S \) are dimensionless and valid up to \( \Lambda \)
- No confinement but has good pion, vector meson and OK axial vector-meson phenomenology
ENJL: our main model


- Gap equation: chiral symmetry spontaneously broken

  \[ \longrightarrow = \longrightarrow + \odot \]

- Generates poles, i.e. mesons via bubble resummation
ENJL: our main model

- Can be thought of as a very simple rainbow and ladder approximation in the DSE equation with constant kernels for the one-gluon exchange
- Parameters fit via $F_\pi, L_i^r$, vector meson properties, ...
- $G_S = 1.216, G_V = 1.263, \Lambda = 1.16 \text{ GeV}$
- has $M_Q = 263 \text{ MeV}$
- Has a number of decent matchings to short-distance, e.g. $\Pi_V - \Pi_A$ but fails in others.
- Generates always VMD in external legs (but with a twist)
- Hook together general processes by one-loop vertices and bubble-chain propagators
Separation of contributions

- Quark loop with external bubble-chains
- \( \approx \) Quark-loop with VMD

- Also internal bubble chain
- \( \approx \) meson exchange
- Note that vertices have structure
- Off-shell effect in model included
$\pi^0$ exchange

$\pi^0 = 1/(p^2 - m_{\pi}^2)$

The blobs need to be modelled, and in e.g. ENJL contain corrections also to the $1/(p^2 - m_{\pi}^2)$

Pointlike has a logarithmic divergence
\[ \pi^0 \text{ exchange} \]

<table>
<thead>
<tr>
<th>Cut-off ( \mu ) (GeV)</th>
<th>( a_\mu \times 10^{10} ) Point-like</th>
<th>( a_\mu \times 10^{10} ) ENJL–VMD</th>
<th>( a_\mu \times 10^{10} ) Point-Like-VMD</th>
<th>( a_\mu \times 10^{10} ) Transverse-VMD</th>
<th>( a_\mu \times 10^{10} ) Transverse-VMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.92(2)</td>
<td>3.29(2)</td>
<td>3.46(2)</td>
<td>3.60(3)</td>
<td>3.53(2)</td>
</tr>
<tr>
<td>0.7</td>
<td>7.68(4)</td>
<td>4.24(4)</td>
<td>4.49(3)</td>
<td>4.73(4)</td>
<td>4.57(4)</td>
</tr>
<tr>
<td>1.0</td>
<td>11.15(7)</td>
<td>4.90(5)</td>
<td>5.18(3)</td>
<td>5.61(6)</td>
<td>5.29(5)</td>
</tr>
<tr>
<td>2.0</td>
<td>21.3(2)</td>
<td>5.63(8)</td>
<td>5.62(5)</td>
<td>6.39(9)</td>
<td>5.89(8)</td>
</tr>
<tr>
<td>4.0</td>
<td>32.7(5)</td>
<td>6.22(17)</td>
<td>5.58(5)</td>
<td>6.59(16)</td>
<td>6.02(10)</td>
</tr>
</tbody>
</table>

BPP: All in reasonable agreement \( a_\mu^{\pi^0} = 5.9 \times 10^{-10} \)
\[ \pi^0 \text{ exchange} \]

- **BPP** \( a_{\mu}^{\pi^0} = 5.9 \times 10^{-10} \)

- **Nonlocal quark model:** \( a_{\mu}^{\pi^0} = 6.27 \times 10^{-10} \)  

- **DSE model:** \( a_{\mu}^{\pi^0} = 5.75 \times 10^{-10} \)  

- **LMD+V:** \( a_{\mu}^{\pi^0} = (5.8 - 6.3) \times 10^{-10} \)  

- **Formfactor inspired by AdS/QCD:** \( a_{\mu}^{\pi^0} = 6.54 \cdot 10^{-10} \)  
**MV short-distance: \( \pi^0 \) exchange**


- take \( p_1^2 \approx p_2^2 \gg q^2 \): Leading term in OPE of two vector currents is proportional to axial current

- These come from

  ![Diagram of quark loops]

- Are these part of the quark-loop? See also in

- Implemented via setting one blob = 1

- \( a_\mu^{\pi^0} = 7.7 \times 10^{-10} \)
MV short-distance: $\pi^0$ exchange


- take $p_1^2 \approx p_2^2 \gg q^2$: Leading term in OPE of two vector currents is proportional to axial current

- These come from

- Are these part of the quark-loop? See also in Dorokhov,Broniowski, phys.Rev. D78(2008)07301

- Implemented via setting one blob = 1

- $a_\mu^{\pi^0} = 7.7 \times 10^{-10}$

- A. Nyffeler: constraint via magnetic susceptibility $a_\mu^{\pi^0} = 7.2 \times 10^{-10}$

\( \pi^0 \) exchange


\[ a_\mu = \int dl_1 dl_2 a_{\mu}^{LL} \text{ with } l_i = \log(P_i/\text{GeV}) \]

Checking which momentum regions do what (but would need three dimensional)
Pseudoscalar exchange

- Point-like VMD: $\pi^0$, $\eta$ and $\eta'$ give 5.58, 1.38, 1.04.
- Models that include $U(1)_A$ breaking give similar ratios.
- Pure large $N_c$ models use this ratio.
- The MV argument should give some enhancement over the full VMD like models.
- Total pseudo-scalar exchange is about
  \[ a_{\mu}^{PS} = 8 - 10 \times 10^{-10} \]
- AdS/QCD estimate (includes excited pseudo-scalars)
  \[ a_{\mu}^{PS} = 10.7 \times 10^{-10} \]
  
Axial-vector exchange exchange

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ from Axial-Vector Exchange $\mathcal{O}(N_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.05(0.01)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.07(0.01)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.13(0.01)</td>
</tr>
<tr>
<td>2.0</td>
<td>0.24(0.02)</td>
</tr>
<tr>
<td>4.0</td>
<td>0.59(0.07)</td>
</tr>
</tbody>
</table>

There is some pseudo-scalar exchange piece here as well, off-shell not quite clear what is what.

- $a_\mu^{\text{axial}} = 0.6 \times 10^{-10}$
- MV: short distance enhancement + mixing (both enhance about the same)
  \[ a_\mu^{\text{axial}} = 2.2 \times 10^{-10} \]
### Pure quark loop

<table>
<thead>
<tr>
<th>Cut-off ( \Lambda ) (GeV)</th>
<th>( a_\mu \times 10^7 ) Electron Loop</th>
<th>( a_\mu \times 10^7 ) Muon Loop</th>
<th>( a_\mu \times 10^9 ) Constituent Quark Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.41(8)</td>
<td>2.41(3)</td>
<td>0.395(4)</td>
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<tr>
<td>0.7</td>
<td>2.60(10)</td>
<td>3.09(7)</td>
<td>0.705(9)</td>
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<tr>
<td>1.0</td>
<td>2.59(7)</td>
<td>3.76(9)</td>
<td>1.10(2)</td>
</tr>
<tr>
<td>2.0</td>
<td>2.60(6)</td>
<td>4.54(9)</td>
<td>1.81(5)</td>
</tr>
<tr>
<td>4.0</td>
<td>2.75(9)</td>
<td>4.60(11)</td>
<td>2.27(7)</td>
</tr>
<tr>
<td>8.0</td>
<td>2.57(6)</td>
<td>4.84(13)</td>
<td>2.58(7)</td>
</tr>
<tr>
<td>Known Results</td>
<td>2.6252(4)</td>
<td>4.65</td>
<td>2.37(16)</td>
</tr>
</tbody>
</table>

### MQ: 300 MeV
now all known analytically

### Us: 5+(3-1) integrals
extra are Feynman parameters

**Slow convergence:**

- **electron:** all at 500 MeV
- **Muon:** only half at 500 MeV, at 1 GeV still 20% missing
- **300 MeV quark:** at 2 GeV still 25% missing
Pure quark loop: momentum area

This plots $a_{\mu}^{q1} = \int dl_{P_1} dl_{P_2} dl_Q a_{LLQ}^{\mu}$

Succeeded in 3D plot but was useless

JB-Zahiri-Abyaneh, work in progress
Pure quark loop: momentum area

Quark loop $m_Q = 0.3$ GeV

Most from $P_1 \approx P_2 \approx Q$, sizable large momentum part
ENJL quark-loop

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ GeV</th>
<th>$a_\mu \times 10^{10}$ Quark-loop</th>
<th>$a_\mu \times 10^{10}$ Quark-loop</th>
<th>$a_\mu \times 10^{10}$ Quark-loop</th>
<th>$a_\mu \times 10^{10}$ sum</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>VMD</td>
<td>ENJL</td>
<td>masscut</td>
<td>ENL+masscut</td>
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<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>2.46</td>
<td>3.2</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>1.13</td>
<td>2.3</td>
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<td>1.0</td>
<td>0.87</td>
<td>1.44</td>
<td>0.59</td>
<td>2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>0.13</td>
<td>1.9</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>0.03</td>
<td>2.0</td>
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<tr>
<td>8.0</td>
<td>0.98</td>
<td>2.00</td>
<td>0.05</td>
<td>2.0</td>
</tr>
</tbody>
</table>

- ENJL cuts off slower than pure VMD
- masscut: $M_Q = \Lambda$ to have short-distance and no problem with momentum regions
- Quite stable in region 1-4 GeV

Very stable
\[
\Pi^{\nu\alpha\beta}(p_1, p_2, p_3) = \Pi^{SV}_{ab}(p_1, r) g_S \left(1 + g_S \Pi^S(r)\right) \Pi^{SV}_{cd}(p_2, p_3) \mathcal{V}^{abcd\rho\nu\alpha\beta}(p_1, p_2, p_3) + \text{permutations}
\]

\[
g_S \left(1 + g_S \Pi_S\right) = \frac{g_A(q^2)(2MQ)^2}{2f^2(q^2)} \frac{1}{M_S^2(q^2) - q^2}
\]

\[\mathcal{V}^{abcd\rho\nu\alpha\beta}\] was ENJL VMD legs

In ENJL only scalar+quark-loop properly chiral invariant
ENJL: scalar/QL

<table>
<thead>
<tr>
<th>Cut-off $\Lambda$ (GeV)</th>
<th>$a_\mu \times 10^{10}$ Quark-loop VMD</th>
<th>$a_\mu \times 10^{10}$ Quark-loop ENJL</th>
<th>$a_\mu \times 10^{10}$ Scalar Exchange</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.48</td>
<td>0.78</td>
<td>$-0.22$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.72</td>
<td>1.14</td>
<td>$-0.46$</td>
</tr>
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<td>1.0</td>
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<td>1.44</td>
<td>$-0.60$</td>
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<tr>
<td>2.0</td>
<td>0.98</td>
<td>1.78</td>
<td>$-0.68$</td>
</tr>
<tr>
<td>4.0</td>
<td>0.98</td>
<td>1.98</td>
<td>$-0.68$</td>
</tr>
<tr>
<td>8.0</td>
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</tr>
</tbody>
</table>

- Note: ENJL+scalar (BPP) $\approx$ Quark-loop VMD (HKS)
- $M_S \approx 620$ MeV certainly an overestimate for real scalars
- If scalar is $\sigma$: related to pion loop part?
- Quark-loop: $a_{\mu}^{ql} \approx 1 \times 10^{-10}$ bare $a_{\mu}^{ql} = 2.37 \times 10^{-10}$
DSE model: \( a_{\mu}^{ql} = 13.6(5.9) \times 10^{-10} \) T. Goecke, C. S. Fischer and R. Williams, Phys. Rev. D 83 (2011) 094006 [arXiv:1012.3886 [hep-ph]]

Not a full calculation (yet) but includes an estimate of some of the missing parts

Note: a lot larger than bare quark loop with constituent mass

I am puzzled: this DSE model (Maris-Roberts) does reproduce a lot of low-energy phenomenology. I would have guessed that it would be very similar to ENJL in its results.

Can one find something in between full DSE and ENJL that is easier to handle?
The $\pi\pi\gamma^*$ vertex is always done using VMD

$\pi\pi\gamma^*\gamma^*$ vertex two choices:
- Hidden local symmetry model: only one $\gamma$ has VMD
- Full VMD
- Both are chirally symmetric
- Check if they live up to MV short distance (Full VMD does, HLS not checked yet)
- The HLS model used has problems with $\pi^+ - \pi^0$ mass difference (due not having an $a_1$)

Final numbers quite different: $-0.045$ and $-0.19$

For BPP stopped at 1 GeV but within 10% of higher $\Lambda$
\[ \pi \text{ and } K \text{-loop} \]

<table>
<thead>
<tr>
<th>Cut-off (GeV)</th>
<th>[10^{10} a_{\mu}] (\pi) bare</th>
<th>(\pi) VMD</th>
<th>(\pi) ENJL</th>
<th>(\pi) HLS</th>
<th>(K) ENJL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>(-1.71(7))</td>
<td>(-1.16(3))</td>
<td>(-1.20(0.03))</td>
<td>(-1.05(0.01))</td>
<td>(-0.020(0.001))</td>
</tr>
<tr>
<td>0.6</td>
<td>(-2.03(8))</td>
<td>(-1.41(4))</td>
<td>(-1.42(0.03))</td>
<td>(-1.15(0.01))</td>
<td>(-0.026(0.001))</td>
</tr>
<tr>
<td>0.7</td>
<td>(-2.41(9))</td>
<td>(-1.46(4))</td>
<td>(-1.56(0.03))</td>
<td>(-1.17(0.01))</td>
<td>(-0.034(0.001))</td>
</tr>
<tr>
<td>0.8</td>
<td>(-2.64(9))</td>
<td>(-1.57(6))</td>
<td>(-1.67(0.04))</td>
<td>(-1.16(0.01))</td>
<td>(-0.042(0.001))</td>
</tr>
<tr>
<td>1.0</td>
<td>(-2.97(12))</td>
<td>(-1.59(15))</td>
<td>(-1.81(0.05))</td>
<td>(-1.07(0.01))</td>
<td>(-0.048(0.002))</td>
</tr>
<tr>
<td>2.0</td>
<td>(-3.82(18))</td>
<td>(-1.70(7))</td>
<td>(-2.16(0.06))</td>
<td>(-0.68(0.01))</td>
<td>(-0.087(0.005))</td>
</tr>
<tr>
<td>4.0</td>
<td>(-4.12(18))</td>
<td>(-1.66(6))</td>
<td>(-2.18(0.07))</td>
<td>(-0.50(0.01))</td>
<td>(-0.099(0.005))</td>
</tr>
</tbody>
</table>

- **HLS JB-Zahiri-Abyaneh**
- note the suppression by the propagators
Note: plotted $-a_{\mu}^{LLQ}$
\( \pi \) loop: VMD vs HLS

\[ a_\mu^{LLQ} \]

VMD

HLS \( a=2 \)

\[ P_1 = P_2 \]
\[ \pi \pi \gamma^* \gamma^* \text{ for } q_1^2 = q_2^2 \text{ has a short-distance constraint from the OPE as well.} \]

- HLS does not satisfy it
- full VMD does: so probably better estimate
\[ \pi \pi \gamma^* \gamma^* \text{ for } q_1^2 = q_2^2 \text{ has a short-distance constraint from the OPE as well.} \]

- HLS does not satisfy it
- full VMD does: so probably better estimate


- So far ChPT at \( p^4 \) done for four-point function in limit \( p_1, p_2, q \ll m_{\pi} \) (Euler-Heisenberg plus next order)
- Polarizability part \( (L_9 + L_{10}) \) could be 10\%, charge radius 30\%
\[ \pi \pi \gamma^* \gamma^* \text{ for } q_1^2 = q_2^2 \] has a short-distance constraint from the OPE as well.

HLS does not satisfy it

full VMD does: so probably better estimate


So far ChPT at \( p^4 \) done for four-point function in limit \( p_1, p_2, q \ll m_\pi \) (Euler-Heisenberg plus next order)

Polarizability part \( (L_9 + L_{10}) \) could be 10\%, charge radius 30\%

Both HLS and VMD have charge radius effect but not polarizability
\[ \pi \text{ loop: } L_9, L_{10} \]

- ChPT for muon \( g - 2 \) at order \( p^6 \) is not powercounting finite so no prediction for \( a_\mu \) exists.

- But can be used to study the low momentum end of the integral over \( P_1, P_2, Q \).

- The four-photon amplitude is finite still at two-loop order (counterterms start at order \( p^8 \)).
\( \pi \) loop: \( L_9, L_{10} \)

- ChPT for muon \( g - 2 \) at order \( p^6 \) is not powercounting finite so no prediction for \( a_\mu \) exists.

- But can be used to study the low momentum end of the integral over \( P_1, P_2, Q \).

- The four-photon amplitude is finite still at two-loop order (counterterms start at order \( p^8 \)).

- Add \( L_9 \) and \( L_{10} \) vertices to the bare pion loop

  JB-Zahiri-Abyaneh

- Program the Euler-Heisenberg plus NLO result of Ramsey-Musolf et al. into our programs for \( a_\mu \).

- Bare pion-loop and \( L_9, L_{10} \) part in limit \( p_1, p_2, q \ll m_\pi \) agree with Euler-Heisenberg plus next order analytically.

- Numerics very preliminary.
\( \pi \) loop: VMD vs charge radius

\[ VMD \]
\[ L_{10} = -L_9 \]
**π loop: VMD vs $L_9$ and $L_{10}$**

![Graph showing comparison between VMD and $L_9, L_{10}$ in the π loop](image)

- **VMD**
- **$L_{10}, L_9$**

Axes:
- $Q^{0.2}$
- $p_1 = p_2$
- $a_{\mu}^{LLQ}$

Values:
- $0.1$ to $1.2e-10$
- $1e-10$ to $1.2e-10$
- $2e-11$ to $1e-10$
- $4e-11$ to $8e-11$
- $6e-11$ to $1.2e-10$
- $8e-11$ to $1e-10$
- $10e-11$ to $1.2e-10$

**Note:** The graph visualizes the comparison between VMD and the combined $L_{10}, L_9$ in the π loop, showing the distribution of $a_{\mu}^{LLQ}$ as a function of $Q^{0.2}$ and $p_1 = p_2$. The graphs are colored to differentiate between VMD and the combined $L_{10}, L_9$ approaches.
### Summary: ENJL vs PdRV

<table>
<thead>
<tr>
<th></th>
<th>BPP</th>
<th>PdRV arXiv:0901.0306</th>
</tr>
</thead>
<tbody>
<tr>
<td>quark-loop</td>
<td>$\left(2.1 \pm 0.3\right) \cdot 10^{-10}$</td>
<td>$- $</td>
</tr>
<tr>
<td>pseudo-scalar</td>
<td>$\left(8.5 \pm 1.3\right) \cdot 10^{-10}$</td>
<td>$\left(11.4 \pm 1.3\right) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>axial-vector</td>
<td>$\left(0.25 \pm 0.1\right) \cdot 10^{-10}$</td>
<td>$\left(1.5 \pm 1.0\right) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>scalar</td>
<td>$\left(-0.68 \pm 0.2\right) \cdot 10^{-10}$</td>
<td>$\left(-0.7 \pm 0.7\right) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>$\pi K$-loop</td>
<td>$\left(-1.9 \pm 1.3\right) \cdot 10^{-10}$</td>
<td>$\left(-1.9 \pm 1.9\right) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>errors</td>
<td>linearly</td>
<td>quadratically</td>
</tr>
<tr>
<td>sum</td>
<td>$\left(8.3 \pm 3.2\right) \cdot 10^{-10}$</td>
<td>$\left(10.5 \pm 2.6\right) \cdot 10^{-10}$</td>
</tr>
</tbody>
</table>
What can we do more?

- **Constraints from experiment:** J. Bijnens and F. Persson, hep-ph/hep-ph/0106130

  Studying three formfactors $P \gamma^* \gamma^*$ in $P \rightarrow \ell^+ \ell^- \ell'^+ \ell'^-$, $e^+e^- \rightarrow e^+e^- P$ exact tree level and for $g = 2$ (but beware sign):

  - **Conclusion:** possible but VERY difficult
  - Two $\gamma^*$ off-shell not so important for our choice of form-factor

- All information on hadrons and 1-2-3-4 off-shell photons is welcome: constrain the models

- More short-distance constraints: MV, Nyffeler integrate with all contributions, not just $\pi^0$-exchange

- Need a new overall evaluation with consistent approach.
What can we do more?

- The ENJL model can certainly be improved:
  - Chiral nonlocal quark-model (like nonlocal ENJL): so far only $\pi^0$-exchange done
  - DSE: $\pi^0$-exchange similar to everyone else, quark-loop very different, looking forward to final results

- More resonances models should be tried, AdS/QCD is one approach, $R_\chi T$ (Valencia et al.) possible, …

- Note short-distance matching must be done in many channels, there are theorems JB,Gamiz,Lipartia,Prades that with only a few resonances this requires compromises

- $\pi$-loop: HLS smaller than double VMD (understood) models with $\rho$ and $a_1$ (in progress)
Leading Logarithms

- Take a quantity with a single scale: $F(M)$
- The dependence on the scale in field theory is typically logarithmic
  
  $L = \log (\mu/M)$

- $F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + F_3^3 L^3 + \cdots$

- Leading Logarithms: The terms $F_m^m L^m$

The $F_m^m$ can be more easily calculated than the full result

- $\mu (dF/d\mu) \equiv 0$

- Ultraviolet divergences in Quantum Field Theory are always local
Renormalizable theories

- Loop expansion \( \equiv \alpha \) expansion
- \( F = \alpha + f_1 \alpha^2 L + f_0 \alpha^2 + f_2 \alpha^3 L^2 + f_1 \alpha^3 L + f_0 \alpha^3 + f_3 \alpha^4 L^3 + \cdots \)
- \( f_i^j \) are pure numbers
- \( \mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots \)
Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$
- $f_i^j$ are pure numbers
- $\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$
- $\mu \frac{dF}{d\mu} = 0 \implies \beta_0 = -f_1^1 = f_2^2 = -f_3^3 = \cdots$
Can be extended to other operators as well

Underlying argument always \( \mu \frac{dF}{d\mu} = 0 \).

Gell-Mann–Low, Callan–Symanzik, Weinberg–’t Hooft

In great detail: J.C. Collins, *Renormalization*

Relies on the \( \alpha \) the same in all orders

LL one-loop \( \beta_0 \)

NLL two-loop \( \beta_1, f_0^1 \)
Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu \frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–‘t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the $\alpha$ the same in all orders
- LL one-loop $\beta_0$
- NLL two-loop $\beta_1, f_0^1$
- In effective field theories: different Lagrangian at each order
- The recursive argument does not work
Weinberg’s argument

- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop
- Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- Proof at all orders using $\beta$-functions
  Büchler, Colangelo, hep-ph/0309049
Weinberg’s argument

- $\mu$: dimensional regularization scale
- $d = 4 - w$
- at $n$-loop order ($\bar{\mathcal{H}}^n$) must cancel:
  - $1/w^n$, 

---

MESON 2012 Hadronic light-by-light and renormalization group Johan Bijnens p.49/73
Weinberg’s argument

- $\mu$: dimensional regularization scale
- $d = 4 - w$

at $n$-loop order ($\hbar^n$) must cancel:

- $1/w^n$, $\log \mu/w^{n-1}$, $\ldots$, $\log^{n-1} \mu/w$

This allows for relations between diagrams

All needed for $\log^n \mu$ coefficient can be calculated from one-loop diagrams
Weinberg’s argument

- $\mu$: dimensional regularization scale
- $d = 4 - w$

at $n$-loop order ($\hbar^n$) must cancel:
- $1/w^n$, $\log \mu/w^{n-1}$, $\ldots$, $\log^{n-1} \mu/w$

This allows for relations between diagrams
All needed for $\log^n \mu$ coefficient can be calculated from one-loop diagrams

goes on
- $1/w^{n-1}$, $\log\mu/w^{n-2}$, $\ldots$, $\log\mu^{n-2}/w$

Get subleading logs $\log^{n-1} \mu$ from two-loop diagrams
subsubleading logs from 3-loop diagrams, $\ldots$
Weinberg’s argument

- $\mu$: dimensional regularization scale
- $d = 4 - w$
- at $n$-loop order ($\hbar^n$) must cancel:
  - $1/w^n, \log \mu/w^{n-1}, \ldots, \log^{n-1} \mu/w$
  - This allows for relations between diagrams
  - All needed for $\log^n \mu$ coefficient can be calculated from one-loop diagrams
  - goes on
    - $1/w^{n-1}, \log \mu/w^{n-2}, \ldots, \log \mu^{n-2}/w$
    - Get subleading logs $\log^{n-1} \mu$ from two-loop diagrams
    - subsubleading logs from 3-loop diagrams, \ldots
  - Many 1-loop diagrams (each harder for higher orders)
Mass to $\bar{\hbar}^2$
Mass to $\bar{m}^2$
Mass to $\bar{\hbar}^2$

- $\bar{\hbar}^1$: $\begin{array}{c} 0 \end{array} \Rightarrow \begin{array}{c} 1 \end{array}$

- $\bar{\hbar}^2$: $\begin{array}{c} 1 \end{array} \Rightarrow \begin{array}{c} 0 \end{array} \Rightarrow \begin{array}{c} 2 \end{array}$

- but also needs $\bar{\hbar}^1$: $\begin{array}{c} 0 \end{array} \Rightarrow \begin{array}{c} 1 \end{array}$
Mass to order $\hbar^3$
Mass to order $\hbar^5$
Mass to order $\hbar^6$
Mass+decay to $\bar{\hbar}^5$

- $\bar{\hbar}^1$: 18 + 27
- $\bar{\hbar}^2$: 26 + 45
- $\bar{\hbar}^3$: 33 + 51
- $\bar{\hbar}^4$: 26 + 33
- $\bar{\hbar}^5$: 13 + 13

Calculate the divergence
rewrite it in terms of a local Lagrangian

Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form

We keep all terms to have all 1PI (one particle irreducible) diagrams finite
Massive $O(N)$ sigma model

- $O(N + 1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2} \partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi$.
- $\Phi$ is a real $N + 1$ vector; $\Phi \rightarrow O\Phi; \Phi^T \Phi = 1$.
- Vacuum expectation value $\langle \Phi^T \rangle = (1 \ 0 \ldots 0)$
- Explicit symmetry breaking: $\chi^T = (M^2 \ 0 \ldots 0)$
- Both spontaneous and explicit symmetry breaking
- $N$-vector $\phi$
- $N$ (pseudo-)Nambu-Goldstone Bosons
- $N = 3$ is two-flavour Chiral Perturbation Theory
Massive $O(N)$ sigma model: $\Phi$ vs $\phi$

\[ \Phi_1 = \begin{pmatrix} \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1 - \frac{\phi^T \phi}{F^2}} \end{pmatrix} \]

Gasser, Leutwyler

\[ \Phi_2 = \frac{1}{\sqrt{1 + \frac{\phi^T \phi}{F^2}}} \begin{pmatrix} 1 \\ \frac{\phi}{F} \end{pmatrix} \]

similar to Weinberg

\[ \Phi_3 = \begin{pmatrix} 1 - \frac{1}{2} \frac{\phi^T \phi}{F^2} \\ \sqrt{1 - \frac{1}{4} \frac{\phi^T \phi \phi}{F^2}} \end{pmatrix} \]

only mass term

\[ \Phi_4 = \begin{pmatrix} \cos \sqrt{\frac{\phi^T \phi}{F^2}} \\ \sin \sqrt{\frac{\phi^T \phi}{F^2}} \end{pmatrix} \]

CCWZ
Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use the four different parametrizations
- compare with known results:
  \[ M_{phys}^2 = M^2 \left( 1 - \frac{1}{2} L_M + \frac{17}{8} L_M^2 + \cdots \right), \]
  \[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \]

Usual choice $\mathcal{M} = M$.

- large $N$ (but known results only for massless case)
  Coleman, Jackiw, Politzer 1974

- large $N$ massive later found partly in appendix of Kivel, Polyakov, Vladimirov on distribution functions.
Results

\[ M_{\text{phys}}^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots) \]

\[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \]

<table>
<thead>
<tr>
<th>i</th>
<th>(a_i, N = 3)</th>
<th>(a_i) for general (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\frac{1}{2})</td>
<td>(1 - \frac{N}{2})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{17}{8})</td>
<td>(\frac{7}{4} - \frac{7N}{4} + \frac{5 N^2}{8})</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{103}{24})</td>
<td>(\frac{37}{12} - \frac{113N}{24} + \frac{15 N^2}{4} - N^3)</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{24367}{1152})</td>
<td>(\frac{839}{144} - \frac{1601 N}{144} + \frac{695 N^2}{48} - \frac{135 N^3}{16} + \frac{231 N^4}{128})</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{8821}{144})</td>
<td>(\frac{33661}{2400} - \frac{1151407 N}{43200} + \frac{197587 N^2}{4320} - \frac{12709 N^3}{300} + \frac{6271 N^4}{320} - \frac{7 N^5}{2})</td>
</tr>
</tbody>
</table>

\[ F_{\text{phys}}, \langle \bar{q}iqi \rangle \] as well done

Anyone recognize any funny functions?
Large $N$

**Power counting:** pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N, M^2 \sim 1$

2n legs

$
\Leftrightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$

$
\Leftrightarrow N
$

**1PI diagrams:**

\[
\begin{align*}
N_L &= N_I - \sum_n N_{2n} + 1 \\
2N_I + N_E &= \sum_n 2nN_{2n}
\end{align*}
\]

$\Rightarrow N_L = \sum_n (n-1)N_{2n} - \frac{1}{2}N_E + 1$

**Diagram suppression factor:**

\[
\frac{N^{N_L}}{N^{N_E/2-1}}
\]
Large $N$

- diagrams with shared lines are suppressed

- each new loop needs also a new flavour loop

- in the large $N$ limit only “caactus” diagrams survive:
large $N$: propagator

Generate recursively via a Gap equation

\[
\begin{align*}
(\quad)^{-1} &= (\quad)^{-1} + 0 + \cdots \\
\Rightarrow \text{resum the series and look for the pole}
\end{align*}
\]

\[
M^2 = M^2_{\text{phys}} \sqrt{1 + \frac{N}{F^2} \overline{A}(M^2_{\text{phys}})}
\]

\[
\overline{A}(m^2) = \frac{m^2}{16\pi^2} \log \frac{\mu^2}{m^2}.
\]

Solve recursively, agrees with other result

Note: can be done for all parametrizations
\[ F_{\text{phys}} = F \sqrt{1 + \frac{N}{F^2} A(M^2_{\text{phys}})} \]

\[ \langle \bar{q}q \rangle_{\text{phys}} = \langle \bar{q}q \rangle_0 \sqrt{1 + \frac{N}{F^2} A(M^2_{\text{phys}})} \]

Comments:
- These are the full* leading \( N \) results, not just leading log
- But depends on the choice of \( N \)-dependence of higher order coefficients
- Assumes higher LECs zero (\( < N^{n+1} \text{ for } \hbar^n \))
- Large \( N \) as in \( O(N) \) not large \( N_c \)
Large $N$: Checking expansions

\[ M^2 = M^2_{\text{phys}} \sqrt{1 + \frac{N}{F^2} A(M^2_{\text{phys}})} \]

much smaller expansion coefficients than the table, try

\[ M^2 = M^2_{\text{phys}} (1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + ...) \]
Left: \[ \frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots \]

\[ F = 90 \text{ MeV}, \quad \mu = 0.77 \text{ GeV} \]
Left: \( \frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L^2_M + a_3 L^3_M + \cdots \)

Right: \( \frac{M^2}{M^2_{\text{phys}}} = 1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + \cdots \)

F = 90 MeV, \( \mu = 0.77 \text{ GeV} \)
Large $N$: $\pi\pi$-scattering

- Cactus diagrams for $A(s, t, u)$
- Branch with no momentum: resummed by $\text{Branch with no momentum}$
- Branch starting at vertex: resum by $\text{Branch starting at vertex}$

\[
\begin{align*}
&= \quad + \quad + \quad + \quad + \quad + \quad + \cdots \\
&= \quad + \quad + \quad + \quad + \cdots \\
&\text{Can be summarized by a recursive equation}
\end{align*}
\]
Large $\mathcal{N}$: $\pi\pi$ scattering

\[ y = \frac{N}{F^2} \overline{A}(M_{\text{phys}}^2) \]

\[ A(s, t, u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{N}{2} \left( \frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}} \right) \overline{B}(M_{\text{phys}}^2, M_{\text{phys}}^2, s)} \]

or

\[ A(s, t, u) = \frac{s - M_{\text{phys}}^2}{F_{\text{phys}}^2} \frac{1}{1 - \frac{N}{2} \frac{s - M_{\text{phys}}^2}{F_{\text{phys}}^2} \overline{B}(M_{\text{phys}}^2, M_{\text{phys}}^2, s)} \]

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results
Anomaly for $O(4)/O(3)$


$$\mathcal{L}_{WZW} = -\frac{N_c}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} \left\{ \epsilon^{abc} \left( \frac{1}{3} \Phi^0 \partial_\mu \Phi^a \partial_\nu \Phi^b \partial_\rho \Phi^c - \partial_\mu \Phi^0 \partial_\nu \Phi^a \partial_\rho \Phi^b \Phi^c \right) v^0_\sigma 
+ (\partial_\mu \Phi^0 \Phi^a - \Phi^0 \partial_\mu \Phi^a) v^a_\nu \partial_\rho v^0_\sigma + \frac{1}{2} \epsilon^{abc} \Phi^0 \Phi^a v^b_\mu v^c_\nu \partial_\rho v^0_\sigma \right\}. $$

$$A(\pi^0 \rightarrow \gamma(k_1)\gamma(k_2)) = 
\epsilon_{\mu\nu\alpha\beta} \varepsilon^\ast_1(\epsilon^\ast_1(k_1)) \varepsilon^\ast_2(k_2) k_1^\alpha k_2^\beta F_{\pi\gamma\gamma}(k_1^2, k_2^2)$$

$$F_{\pi\gamma\gamma}(k_1^2, k_2^2) = \frac{e^2}{4\pi^2 F_\pi} \hat{F} F_\gamma(k_1^2) F_\gamma(k_2^2) F_{\gamma\gamma}(k_1^2, k_2^2)$$

\(\hat{F}\): on-shell photon; \(F_\gamma(k^2)\): formfactor; \(F_{\gamma\gamma}\) nonfactorizable
Done to six-loops

\[ \hat{F} = 1 + 0.000372 + 0.000088 + 0.000036 + 0.000009 + 0.0000002 + \ldots \]

Really good convergence

\( F_{\gamma\gamma} \) only starts at three-loop order (could have been two)

\( F_{\gamma\gamma} \) in the chiral limit only starts at four-loops.

The leading logarithms thus predict this part to be fairly small.

\( F_{\gamma}(k^2) \): plot
Anomaly for $O(4)/O(3)$

Leading logs small, converge fast
Experiment 1: $\overline{F}_{\text{exp}}^{3\pi} = 12.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3}$

Experiment 2: $F_{0,\text{exp}}^{3\pi} = 9.9 \pm 1.1 \text{ GeV}^{-3}$

Theory lowest order: $F_{0}^{3\pi} = 9.8 \text{ GeV}^{-3}$

Theory (LL only)

$F_{0}^{3\pi LL} = (9.8 - 0.3 + 0.04 + 0.02 + 0.006 + 0.001 + \ldots) \text{ GeV}^{-3}$

good convergence
Other results

JB, Carloni, arXiv:1008.3499
- **massive case**: $\pi\pi$, $F_V$ and $F_S$ to 4-loop order
- large $N$ for these cases also for massive $O(N)$.
- done using bubble resummations or recursion equation which can be solved analytically

- Mass, $F_\pi$, $F_V$ to six loops
- Anomaly: $\gamma^* 3\pi$ (five) and $\pi^0 \gamma^* \gamma^*$ (six loops)
- large $N$ not relevant in this case

JB, Kampf, Lanz, in preparation
- $SU(N) \times SU(N)/SU(N)$
Other results

- Bissegger, Fuhrer, hep-ph/0612096 Dispersive methods, massless $\Pi_S$ to five loops
- Kivel, Polyakov, Vladimirov, 0809.3236, 0904.3008, 1004.2197, 1012.4205

- In the massless case tadpoles vanish
- $\implies$ number of external legs needed does not grow
- All 4-meson vertices via Legendre polynomials
- can do divergence of all one-loop diagrams analytically
- algebraic (but quadratic) recursion relations
- massless $\pi\pi$, $F_V$ and $F_S$ to arbitrarily high order
- large $N$ agrees with Coleman, Wess, Zumino
- large $N$ is not a good approximation
Several quantities in massive $O(N)$ LL known to high loop order

Large $N$ in massive $O(N)$ model solved

Had hoped: recognize the series also for general $N$

Limited essentially by CPU time and size of intermediate files

Some first studies on convergence etc.

$\pi\pi$, $F_V$ and $F_S$ to four-loop order ($F_V$ higher)

The technique can be generalized to other models/theories

$SU(N) \times SU(N)/SU(N)$: under way

One nucleon sector: planned/hoped