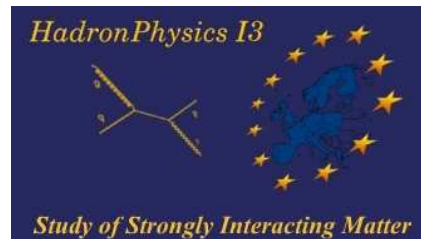




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# $K_{\ell 3}$ AT TWO LOOPS IN CHIRAL PERTURBATION THEORY

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<http://www.thep.lu.se/~bijnens>

Various ChPT: <http://www.thep.lu.se/~bijnens/chpt.html>

# Overview

- Motivation
- Chiral Perturbation Theory
- $K_{\ell 3}$
- What is new when including isospin breaking?
- First *very preliminary* results with isospin breaking at two loops

# Motivation

Basically two: Test ChPT and determine  $V_{us}$

From Moulson, hep-ex/0703013, Flavianet working group result

Mode	$\Delta(SU(2))$	$\Delta(EM)$	Decay	$V_{us}f_+(0)$
$K_{e3}^0$	0	+0.52%	$K_{Le3}$	0.21639(55)
$K_{\mu3}^0$	0	+0.95%	$K_{L\mu3}$	0.21649(68)
$K_{e3}^{\pm}$	2.31%	+0.52%	$K_{e3}^{\pm}$	0.21844(101)
$K_{\mu3}^{\pm}$	2.31%	+0.95%	$K_{\mu3}^{\pm}$	0.21809(125)

About 1% discrepancy: **is this real or a fluke**

The 2.31% is from on loop ChPT

Actual value of  $f_+(0)$ : nothing new

# Chiral Perturbation Theory

Degrees of freedom: Goldstone Bosons from Chiral  
Symmetry Spontaneous Breakdown

Power counting: Dimensional counting

Expected breakdown scale: Resonances, so  $M_\rho$  or higher  
depending on the channel

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## Chiral Symmetry

QCD: 3 light quarks: equal mass: interchange:  $SU(3)_V$

But  $\mathcal{L}_{QCD} = \sum_{q=u,d,s} [i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q (\bar{q}_R q_L + \bar{q}_L q_R)]$

So if  $m_q = 0$  then  $SU(3)_L \times SU(3)_R$ .

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So if  $m_q = 0$  then  $SU(3)_L \times SU(3)_R$ .

Can also see that via



$$\begin{aligned} v < c, m_q \neq 0 &\implies \\ v = c, m_q = 0 &\not\implies \end{aligned}$$



# Chiral Perturbation Theory

$$\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$$

$SU(3)_L \times SU(3)_R$  broken spontaneously to  $SU(3)_V$

8 generators broken  $\implies$  8 massless degrees of freedom  
**and** interaction vanishes at zero momentum

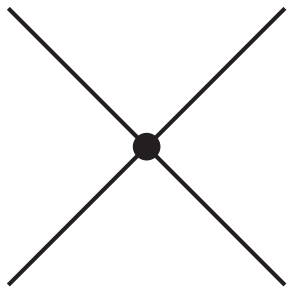
# Chiral Perturbation Theory

$$\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$$

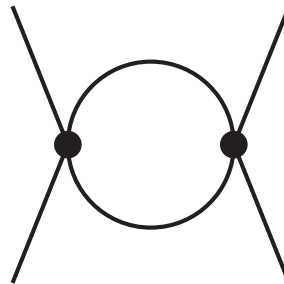
$SU(3)_L \times SU(3)_R$  broken spontaneously to  $SU(3)_V$

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Power counting in momenta:



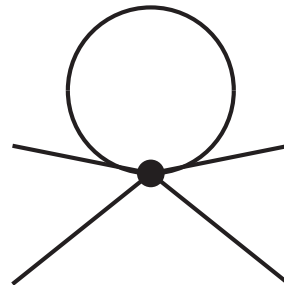
$$p^2$$



$$(p^2)^2 (1/p^2)^2 p^4 = p^4$$



$$1/p^2$$



$$(p^2) (1/p^2) p^4 = p^4$$

$$\int d^4 p$$

$$p^4$$



# Chiral Perturbation Theory

For lectures, review articles: see

<http://www.thep.lu.se/~bijmens/chpt.html>

Review paper on Two-Loops: JB, LU TP 06-16  
[hep-ph/0604043](http://arxiv.org/abs/hep-ph/0604043)

# Two Loop: Lagrangians

## Lagrangian Structure:

	2 flavour		3 flavour		3+3 PQChPT	
$p^2$	$F, B$	2	$F_0, B_0$	2	$F_0, B_0$	2
$p^4$	$l_i^r, h_i^r$	7+3	$L_i^r, H_i^r$	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
$p^6$	$c_i^r$	53+4	$C_i^r$	90+4	$K_i^r$	112+3

$p^2$ : Weinberg 1966

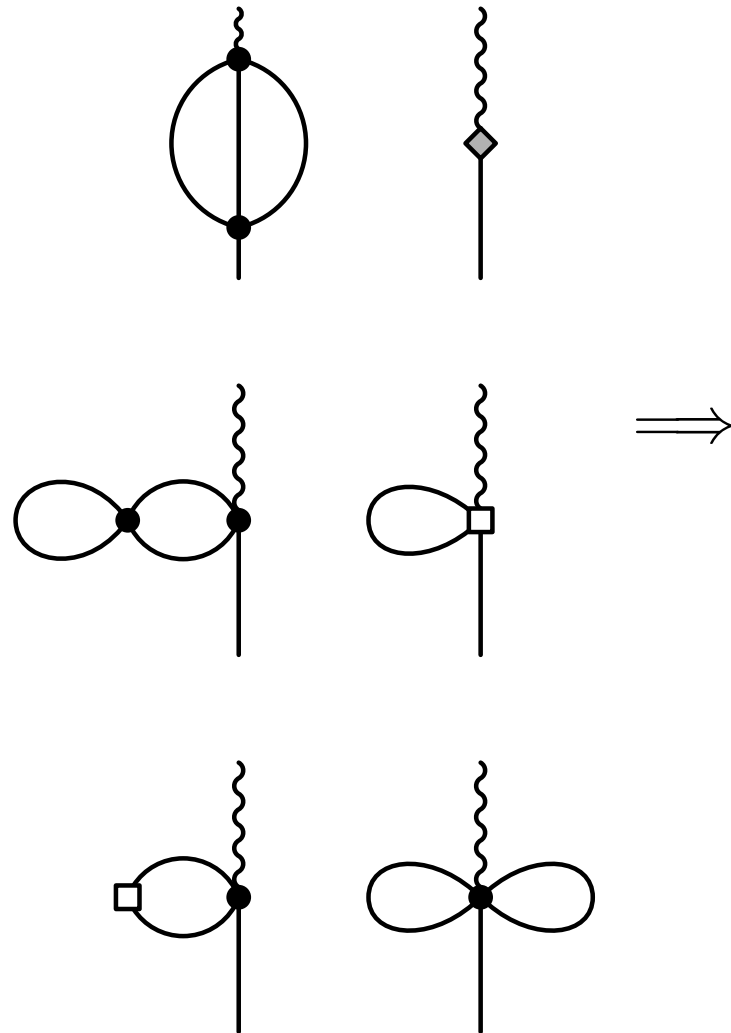
$p^4$ : Gasser, Leutwyler 84,85

$p^6$ : JB, Colangelo, Ecker 99,00

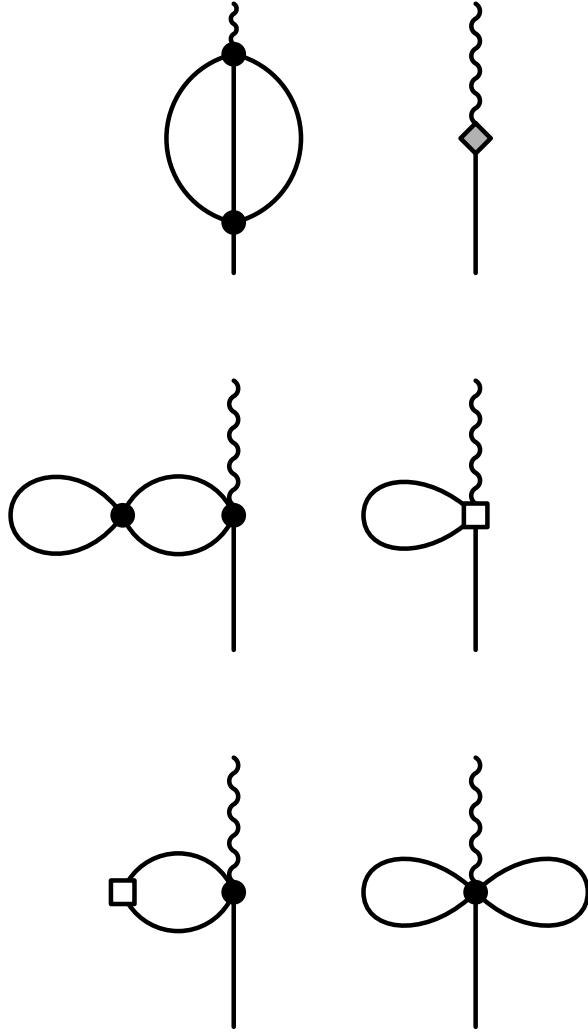
Note {

- ▀ replica method  $\implies$  PQ obtained from  $N_F$  flavour
- ▀ All infinities known
- ▀ 3 flavour is a special case of 3+3 PQ:  
 $\hat{L}_i^r, K_i^r \rightarrow L_i^r, C_i^r$

# Long Expressions



# Long Expressions



$$\begin{aligned}
 \delta_{\text{loop}}^{(6)22} = & \pi_{16} L_0^r [4/9 \chi_\eta \chi_4 - 1/2 \chi_1 \chi_3 + \chi_{13}^2 - 13/3 \bar{\chi}_1 \chi_{13} - 35/18 \bar{\chi}_2] - 2 \pi_{16} L_1^r \chi_{13}^2 \\
 & - \pi_{16} L_2^r [11/3 \chi_\eta \chi_4 + \chi_{13}^2 + 13/3 \bar{\chi}_2] + \pi_{16} L_3^r [4/9 \chi_\eta \chi_4 - 7/12 \chi_1 \chi_3 + 11/6 \chi_{13}^2 - 17/6 \bar{\chi}_1 \chi_{13} - 43/36 \bar{\chi}_2] \\
 & + \pi_{16}^2 [-15/64 \chi_\eta \chi_4 - 59/384 \chi_1 \chi_3 + 65/384 \chi_{13}^2 - 1/2 \bar{\chi}_1 \chi_{13} - 43/128 \bar{\chi}_2] - 48 L_4^r L_5^r \bar{\chi}_1 \chi_{13} - 72 L_5^r \bar{\chi}_2^2 \\
 & - 8 L_5^r \chi_{13}^2 + \bar{A}(\chi_p) \pi_{16} [-1/24 \chi_p + 1/48 \bar{\chi}_1 - 1/8 \bar{\chi}_1 R_{\eta\eta}^p + 1/16 \bar{\chi}_1 R_p^c - 1/48 R_{\eta\eta}^p \chi_p - 1/16 R_{\eta\eta}^p \chi_\eta \\
 & + 1/48 R_{\eta\eta}^p \chi_\eta + 1/16 R_p^c \chi_{13}] + \bar{A}(\chi_p) L_6^r [8/3 R_{\eta\eta}^p \chi_p + 2/3 R_p^c \chi_p + 2/3 R_p^c] + \bar{A}(\chi_p) L_5^r [2/3 R_{\eta\eta}^p \chi_p \\
 & + 5/3 R_p^c \chi_p + 5/3 R_p^c] + \bar{A}(\chi_p) L_4^r [-2 \bar{\chi}_1 \bar{\chi}_{\eta\eta 0}^{pp} - 2 \bar{\chi}_1 R_{\eta\eta}^p + 3 \bar{\chi}_1 R_p^c] + \bar{A}(\chi_p) L_5^r [-2/3 \bar{\chi}_{\eta\eta 1}^{pp} - R_{\eta\eta}^p \chi_p \\
 & + 1/3 R_{\eta\eta}^p \chi_\eta + 1/2 R_p^c \chi_p - 1/6 R_p^c \chi_\eta] + \bar{A}(\chi_p)^2 [1/16 + 1/72 (R_{\eta\eta}^p)^2 - 1/72 R_{\eta\eta}^p R_p^c + 1/288 (R_p^c)^2] \\
 & + \bar{A}(\chi_p) \bar{A}(\chi_{ps}) [-1/36 R_{\eta\eta}^p - 5/72 R_{\eta\eta}^p + 7/144 R_p^c] - \bar{A}(\chi_p) \bar{A}(\chi_{qs}) [1/36 R_{\eta\eta}^p + 1/24 R_{\eta\eta}^p + 1/48 R_p^c] \\
 & + \bar{A}(\chi_p) \bar{A}(\chi_{ps}) [-1/72 R_{\eta\eta}^p R_{\eta\eta}^p + 1/144 R_p^c R_{\eta\eta}^p] + 1/8 \bar{A}(\chi_p) \bar{A}(\chi_{13}) + 1/12 \bar{A}(\chi_p) \bar{A}(\chi_{46}) R_{\eta\eta}^p \\
 & + \bar{A}(\chi_p) \bar{B}(\chi_p, \chi_p; 0) [1/4 \chi_p - 1/18 R_{\eta\eta}^p R_p^c \chi_p - 1/72 R_{\eta\eta}^p R_p^c + 1/18 (R_p^c)^2 \chi_p + 1/144 R_p^c R_p^c] \\
 & + \bar{A}(\chi_p) \bar{B}(\chi_p, \chi_\eta; 0) [1/18 R_{\eta\eta}^p R_p^c \chi_p - 1/18 R_{\eta\eta}^p R_p^c \chi_p] + \bar{A}(\chi_p) \bar{B}(\chi_\eta, \chi_\eta; 0) [-1/72 R_{\eta\eta}^p R_p^c + 1/144 R_p^c R_p^c] \\
 & - 1/12 \bar{A}(\chi_p) \bar{B}(\chi_{ps}, \chi_{ps}; 0) R_{\eta\eta}^p \chi_{ps} - 1/18 \bar{A}(\chi_p) \bar{B}(\chi_1, \chi_3; 0) R_{\eta\eta}^p R_p^c \chi_p \\
 & + 1/18 \bar{A}(\chi_p) \bar{C}(\chi_p, \chi_p; 0) R_p^c R_p^c \chi_p + \bar{A}(\chi_p; \varepsilon) \pi_{16} [1/8 \bar{\chi}_1 R_{\eta\eta}^p - 1/16 \bar{\chi}_1 R_p^c - 1/16 R_p^c \chi_p - 1/16 R_{\eta\eta}^p] \\
 & + \bar{A}(\chi_{ps}) \pi_{16} [1/16 \chi_{ps} - 3/16 \chi_{qs} - 3/16 \bar{\chi}_1] - 2 \bar{A}(\chi_{ps}) L_0^r \chi_{ps} - 5 \bar{A}(\chi_{ps}) L_3^r \chi_{ps} - 3 \bar{A}(\chi_{ps}) L_4^r \bar{\chi}_1 \\
 & + \bar{A}(\chi_{ps}) L_5^r \chi_{13} + \bar{A}(\chi_{ps}) \bar{A}(\chi_\eta) [7/144 R_{\eta\eta}^p - 5/72 R_{\eta\eta}^p - 1/48 R_{\eta\eta}^p + 5/72 R_{\eta\eta}^p - 1/36 R_{\eta\eta}^p] \\
 & + \bar{A}(\chi_{ps}) \bar{B}(\chi_p, \chi_p; 0) [1/24 R_{\eta\eta}^p \chi_p - 5/24 R_{\eta\eta}^p \chi_{ps}] + \bar{A}(\chi_{ps}) \bar{B}(\chi_\eta, \chi_\eta; 0) [-1/18 R_{\eta\eta}^p R_{\eta\eta}^p \chi_p \\
 & - 1/9 R_{\eta\eta}^p R_{\eta\eta}^p \chi_{ps}] - 1/48 \bar{A}(\chi_{ps}) \bar{B}(\chi_\eta, \chi_\eta; 0) R_p^c + 1/18 \bar{A}(\chi_{ps}) \bar{B}(\chi_1, \chi_3; 0) R_p^c \chi_s \\
 & + 1/9 \bar{A}(\chi_{ps}) \bar{B}(\chi_1, \chi_3; 0, k) R_{\eta\eta}^p + 3/16 \bar{A}(\chi_{ps}; \varepsilon) \pi_{16} [\chi_s + \bar{\chi}_1] - 1/8 \bar{A}(\chi_{p4})^2 - 1/8 \bar{A}(\chi_{p4}) \bar{A}(\chi_{p6}) \\
 & + 1/8 \bar{A}(\chi_{p4}) \bar{A}(\chi_{46}) - 1/32 \bar{A}(\chi_{p6})^2 + \bar{A}(\chi_\eta) \pi_{16} [1/16 \bar{\chi}_1 R_{\eta\eta}^p - 1/48 R_{\eta\eta}^p \chi_\eta + 1/16 R_{\eta\eta}^p \chi_{13}] \\
 & + \bar{A}(\chi_\eta) L_0^r [4 R_{\eta\eta}^p \chi_\eta + 2/3 R_{\eta\eta}^p \chi_\eta] - 8 \bar{A}(\chi_\eta) L_1^r \chi_\eta - 2 \bar{A}(\chi_\eta) L_2^r \chi_\eta + \bar{A}(\chi_\eta) L_3^r [4 R_{\eta\eta}^p \chi_\eta + 5/3 R_{\eta\eta}^p \chi_{13}] \\
 & + \bar{A}(\chi_\eta) L_4^r [4 \chi_\eta + \bar{\chi}_1 R_{\eta\eta}^p] - \bar{A}(\chi_\eta) L_5^r [1/6 R_{\eta\eta}^p \chi_\eta + R_{\eta\eta}^p \chi_{13} + 1/6 R_{\eta\eta}^p \chi_\eta] + 1/288 \bar{A}(\chi_\eta)^2 (R_{\eta\eta}^p)^2 \\
 & + 1/12 \bar{A}(\chi_\eta) \bar{A}(\chi_{46}) R_{\eta\eta}^p + \bar{A}(\chi_\eta) \bar{B}(\chi_p, \chi_p; 0) [-1/36 \bar{\chi}_{\eta\eta 1}^{pp} - 1/18 R_{\eta\eta}^p R_{\eta\eta}^p \chi_p + 1/18 R_{\eta\eta}^p R_p^c \chi_p \\
 & + 1/144 R_{\eta\eta}^p R_{\eta\eta}^p] + \bar{A}(\chi_\eta) \bar{B}(\chi_\eta, \chi_\eta; 0) [-1/18 \bar{\chi}_{\eta\eta 1}^{pp} + 1/18 \bar{\chi}_{\eta\eta 1}^{pp} + 1/18 (R_{\eta\eta}^p)^2 R_{\eta\eta}^p \chi_p] \\
 & - 1/12 \bar{A}(\chi_\eta) \bar{B}(\chi_{ps}, \chi_{ps}; 0) R_{\eta\eta}^p \chi_{ps} - \bar{A}(\chi_\eta) \bar{B}(\chi_\eta, \chi_\eta; 0) [1/216 R_{\eta\eta}^p \chi_4 + 1/27 R_{\eta\eta}^p \chi_6] \\
 & - 1/18 \bar{A}(\chi_\eta) \bar{B}(\chi_1, \chi_3; 0) R_{\eta\eta}^p R_{\eta\eta}^p \chi_\eta + 1/18 \bar{A}(\chi_\eta) \bar{C}(\chi_p, \chi_p, \chi_p; 0) R_{\eta\eta}^p R_p^c \chi_p + \bar{A}(\chi_\eta; \varepsilon) \pi_{16} [1/8 \chi_\eta \\
 & - 1/16 \bar{\chi}_1 R_{\eta\eta}^p - 1/8 R_{\eta\eta}^p \chi_\eta - 1/16 R_{\eta\eta}^p \chi_\eta] + \bar{A}(\chi_1) \bar{A}(\chi_3) [-1/72 R_{\eta\eta}^p R_p^c + 1/36 R_{\eta\eta}^p R_p^c + 1/144 R_{\eta\eta}^p R_p^c] \\
 & - 4 \bar{A}(\chi_{13}) L_1^r \chi_{13} - 10 \bar{A}(\chi_{13}) L_2^r \chi_{13} + 1/8 \bar{A}(\chi_{13})^2 - 1/2 \bar{A}(\chi_{13}) \bar{B}(\chi_1, \chi_3; 0, k) \\
 & + 1/4 \bar{A}(\chi_{13}; \varepsilon) \pi_{16} \chi_{13} + 1/4 \bar{A}(\chi_{14}) \bar{A}(\chi_{34}) + 1/16 \bar{A}(\chi_{16}) \bar{A}(\chi_{36}) - 24 \bar{A}(\chi_4) L_1^r \chi_4 - 6 \bar{A}(\chi_4) L_2^r \chi_4 \\
 & + 12 \bar{A}(\chi_4) L_4^r \chi_4 + 1/12 \bar{A}(\chi_4) \bar{B}(\chi_p, \chi_p; 0) (R_{\eta\eta}^p)^2 \chi_4 + 1/6 \bar{A}(\chi_4) \bar{B}(\chi_p, \chi_p; 0) [R_{\eta\eta}^p R_{\eta\eta}^p \chi_4 - R_{\eta\eta}^p R_{\eta\eta}^p \chi_4] \\
 & - 1/24 \bar{A}(\chi_4) \bar{B}(\chi_\eta, \chi_\eta; 0) R_{\eta\eta}^p \chi_4 - 1/6 \bar{A}(\chi_4) \bar{B}(\chi_1, \chi_3; 0) R_{\eta\eta}^p R_{\eta\eta}^p \chi_4 + 3/8 \bar{A}(\chi_4; \varepsilon) \pi_{16} \chi_4 \\
 & - 32 \bar{A}(\chi_{46}) L_1^r \chi_{46} - 8 \bar{A}(\chi_{46}) L_2^r \chi_{46} + 16 \bar{A}(\chi_{46}) L_4^r \chi_{46} + \bar{A}(\chi_{46}) \bar{B}(\chi_p, \chi_p; 0) [1/9 \chi_{46} + 1/12 R_{\eta\eta}^p \chi_p \\
 & + 1/36 R_{\eta\eta}^p \chi_4 + 1/9 R_{\eta\eta}^p \chi_6] + \bar{A}(\chi_{46}) \bar{B}(\chi_p, \chi_p; 0) [-1/18 R_{\eta\eta}^p \chi_4 - 1/9 R_{\eta\eta}^p \chi_6 + 1/9 R_{\eta\eta}^p \chi_4 + 1/18 R_{\eta\eta}^p \chi_4] \\
 & - 1/6 \bar{A}(\chi_{46}) \bar{B}(\chi_p, \chi_\eta; 0, k) [R_{\eta\eta}^p - R_{\eta\eta}^p] + 1/9 \bar{A}(\chi_{46}) \bar{B}(\chi_\eta, \chi_\eta; 0) R_{\eta\eta}^p \chi_{46} - \bar{A}(\chi_{46}) \bar{B}(\chi_1, \chi_3; 0) [2/9 \chi_{46} \\
 & + 1/9 R_{\eta\eta}^p \chi_6 + 1/18 R_{\eta\eta}^p \chi_4] - 1/6 \bar{A}(\chi_{46}) \bar{B}(\chi_1, \chi_3; 0, k) R_{\eta\eta}^p + 1/2 \bar{A}(\chi_{46}; \varepsilon) \pi_{16} \chi_{46} \\
 & + \bar{B}(\chi_p, \chi_p; 0) \pi_{16} [1/16 \bar{\chi}_1 R_{\eta\eta}^p + 1/96 R_{\eta\eta}^p \chi_p + 1/32 R_{\eta\eta}^p \chi_\eta] + 2/3 \bar{B}(\chi_p, \chi_p; 0) L_0^r R_p^c \chi_p \\
 & + 5/3 \bar{B}(\chi_p, \chi_p; 0) L_3^r R_p^c \chi_p + \bar{B}(\chi_p, \chi_p; 0) L_4^r [-2 \bar{\chi}_1 \bar{\chi}_{\eta\eta 0}^{pp} \chi_p - 4 \bar{\chi}_1 R_{\eta\eta}^p \chi_p + 4 \bar{\chi}_1 R_p^c \chi_p + 3 \bar{\chi}_1 R_p^c] \\
 & + \bar{B}(\chi_p, \chi_p; 0) L_5^r [-2/3 \bar{\chi}_{\eta\eta 1}^{pp} \chi_p - 4/3 R_{\eta\eta}^p \chi_p^2 + 4/3 R_{\eta\eta}^p \chi_p^2 + 1/2 R_{\eta\eta}^p \chi_p - 1/6 R_{\eta\eta}^p \chi_\eta] \\
 & + \bar{B}(\chi_p, \chi_p; 0) L_6^r [4 \bar{\chi}_1 \bar{\chi}_{\eta\eta 1}^{pp} + 8 \bar{\chi}_1 R_{\eta\eta}^p \chi_p - 8 \bar{\chi}_1 R_p^c \chi_p] + 4 \bar{B}(\chi_p, \chi_p; 0) L_7^r (R_p^c)^2 \\
 & + \bar{B}(\chi_p, \chi_p; 0) L_8^r [4/3 \bar{\chi}_{\eta\eta 2}^{pp} + 8/3 R_{\eta\eta}^p \chi_p^2 - 8/3 R_p^c \chi_p^2] + \bar{B}(\chi_p, \chi_p; 0)^2 [-1/18 R_{\eta\eta}^p R_p^c \chi_p + 1/18 R_{\eta\eta}^p R_p^c \chi_p \\
 & + 1/288 (R_p^c)^2] + 1/18 \bar{B}(\chi_p, \chi_p; 0) \bar{B}(\chi_\eta, \chi_\eta; 0) [R_{\eta\eta}^p R_p^c \chi_p - R_{\eta\eta}^p R_p^c \chi_p]
 \end{aligned}$$

plus several more pages

# Usual ChPT two-loop: A list

## Two-Loop Two-Flavour

- Bellucci-Gasser-Sainio:  $\gamma\gamma \rightarrow \pi^0\pi^0$ : 1994
- Bürgi:  $\gamma\gamma \rightarrow \pi^+\pi^-$ ,  $F_\pi$ ,  $m_\pi$ : 1996
- JB-Colangelo-Ecker-Gasser-Sainio:  $\pi\pi$ ,  $F_\pi$ ,  $m_\pi$ : 1996-97
- JB-Colangelo-Talavera:  $F_{V\pi}(t)$ ,  $F_{S\pi}$ : 1998
- JB-Talavera:  $\pi \rightarrow \ell\nu\gamma$ : 1997
- Gasser-Ivanov-Sainio:  $\gamma\gamma \rightarrow \pi^0\pi^0$ ,  $\gamma\gamma \rightarrow \pi^+\pi^-$ : 2005-2006

## Two-Loops Three flavours

- $\Pi_{VV\pi}$ ,  $\Pi_{VV\eta}$ ,  $\Pi_{VVK}$  Kambor, Golowich; Kambor, Dürr; Amorós, JB, Talavera
- $\Pi_{VV\rho\omega}$  Maltman
- $\Pi_{AA\pi}$ ,  $\Pi_{AA\eta}$ ,  $F_\pi$ ,  $F_\eta$ ,  $m_\pi$ ,  $m_\eta$  Kambor, Golowich; Amorós, JB, Talavera
- $\Pi_{SS}$  Moussallam  $L_4^r, L_6^r$

# Usual ChPT two-loop: A list

- $\Pi_{VVK}, \Pi_{AAK}, F_K, m_K$  Amorós, JB, Talavera
- $K_{\ell 4}, \langle \bar{q}q \rangle$  Amorós, JB, Talavera  $L_1^r, L_2^r, L_3^r$
- $F_M, m_M, \langle \bar{q}q \rangle$  ( $m_u \neq m_d$ ) Amorós, JB, Talavera  $L_{5,7,8}^r, m_u/m_d$
- $F_{V\pi}, F_{VK^+}, F_{VK^0}$  Post, Schilcher; JB, Talavera  $L_9^r$
- $K_{\ell 3}$  Post, Schilcher; JB, Talavera  $V_{us}$
- $F_{S\pi}, F_{SK}$  (includes  $\sigma$ -terms) JB, Dhonte  $L_4^r, L_6^r$
- $K, \pi \rightarrow \ell\nu\gamma$  Geng, Ho, Wu  $L_{10}^r$
- $\pi\pi$  JB, Dhonte, Talavera
- $\pi K$  JB, Dhonte, Talavera

# $K_{\ell 3}$

- H. Leutwyler and M. Roos, Z.Phys.C25:91,1984.
- J. Gasser and H. Leutwyler, Nucl.Phys.B250:517-538,1985.
- J. Bijnens and P. Talavera, hep-ph/0303103, Nucl. Phys. B669 (2003) 341-362
- V. Cirigliano et al., hep-ph/0110153, Eur.Phys.J.C23:121-133,2002.
- J. Bijnens and K. Ghorbani, to be published.

# $K_{\ell 3}$ Definitions

$$K_{\ell 3}^+ : \quad K^+(p) \rightarrow \pi^0(p') \ell^+(p_\ell) \nu_\ell(p_\nu)$$

$$K_{\ell 3}^0 : \quad K^0(p) \rightarrow \pi^-(p') \ell^+(p_\ell) \nu_\ell(p_\nu)$$

$$K_{\ell 3}^+ : \quad T = \frac{G_F}{\sqrt{2}} V_{us}^* \ell^\mu F_\mu^+(p', p)$$

$$\ell^\mu = \bar{u}(p_\nu) \gamma^\mu (1 - \gamma_5) v(p_\ell)$$

$$F_\mu^+(p', p) = \langle \pi^0(p') | V_\mu^{4-i5}(0) | K^+(p) \rangle$$

$$= \frac{1}{\sqrt{2}} [(p' + p)_\mu f_+^{K^+ \pi^0}(t) + (p - p')_\mu f_-^{K^+ \pi^0}(t)]$$

**Isospin:**  $f_+^{K^0 \pi^-}(t) = f_+^{K^+ \pi^0}(t) = f_+(t)$

$$f_-^{K^0 \pi^-}(t) = f_-^{K^+ \pi^0}(t) = f_-(t)$$



# $K_{\ell 3}$ Definitions and $V_{us}$

Scalar formfactor: 
$$f_0(t) = f_+(t) + \frac{t}{m_K^2 - m_\pi^2} f_-(t)$$

Usual parametrization: 
$$f_{+,0}(t) = f_+(0) \left( 1 + \lambda_{+,0} \frac{t}{m_\pi^2} \right)$$

- $|V_{us}|$ :
- Know theoretically  $f_+(0) = 1 + \dots$ 
    - Short distance correction to  $G_F$  from  $G_\mu$   
Marciano-Sirlin
    - Ademollo-Gatto-Behrends-Sirlin theorem:  
 $(m_s - \hat{m})^2$
    - Isospin Breaking Leutwyler-Roos: see later
    - Radiative corrections: Cirigliano et al
  - Know experimentally  $f_+(0)$

# $V_{us}$

## PDG2002:

$$|V_{ud}| = 0.9734 \pm 0.0008 \quad |V_{us}| = 0.2196 \pm 0.0026$$
$$|V_{ud}|^2 + |V_{us}|^2 = (0.9475 \pm 0.0016) + (0.0482 \pm 0.0011) =$$
$$0.9957 \pm 0.0019$$

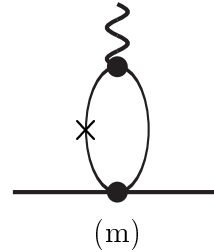
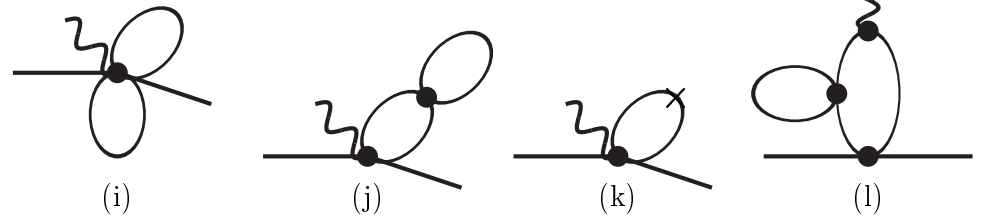
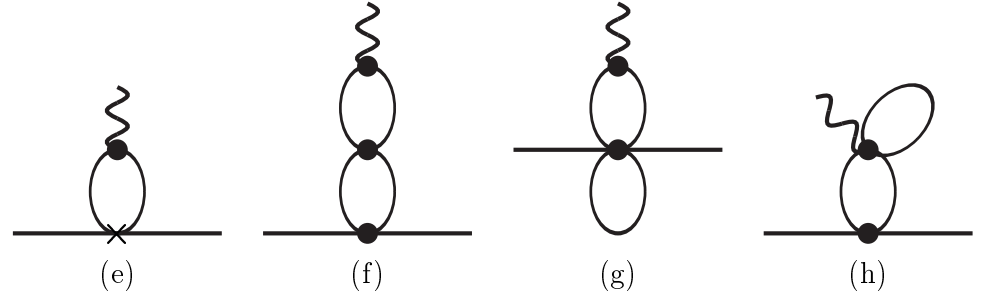
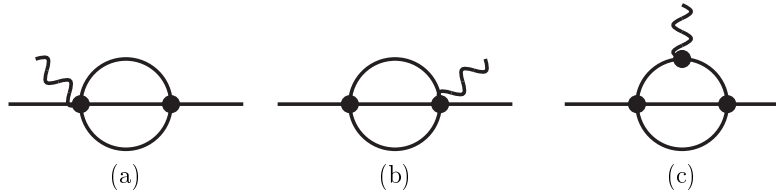
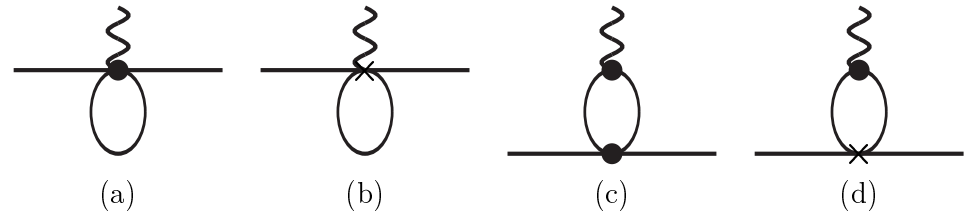
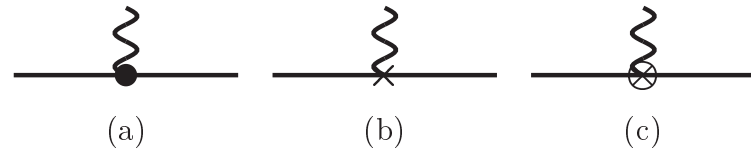
## PDG2006:

$$|V_{ud}| = 0.97377 \pm 0.00027 \quad |V_{us}| = 0.2257 \pm 0.0021$$
$$|V_{ud}|^2 + |V_{us}|^2 = (0.94823 \pm 0.00054) + (0.05094 \pm 0.00095) =$$
$$0.99917 \pm 0.00110$$

## Problems:

- Ignores  $\Delta(0) = 0.0113$  from pure two-loop
- Comparison between experiments: we'll hear more next week

# $K_{\ell 3}$ Diagrams



● :  $p^2$  vertex  
 × :  $p^4$  vertex  
 ⊗ :  $p^6$  vertex

# $f_+(t)$ Theory

$$f_+(t) = 1 + f_+^{(4)}(t) + f_+^{(6)}(t)$$

$$f_+^{(4)}(t) = \frac{t}{2F_\pi^2} L_9^r + \text{loops}$$

$$f_+^{(6)}(t) = -\frac{8}{F_\pi^4} (C_{12}^r + C_{34}^r) (m_K^2 - m_\pi^2)^2 + \frac{t}{F_\pi^4} R_{+1}^{K\pi} \\ + \frac{t^2}{F_\pi^4} (-4C_{88}^r + 4C_{90}^r) + \text{loops}(L_i^r)$$

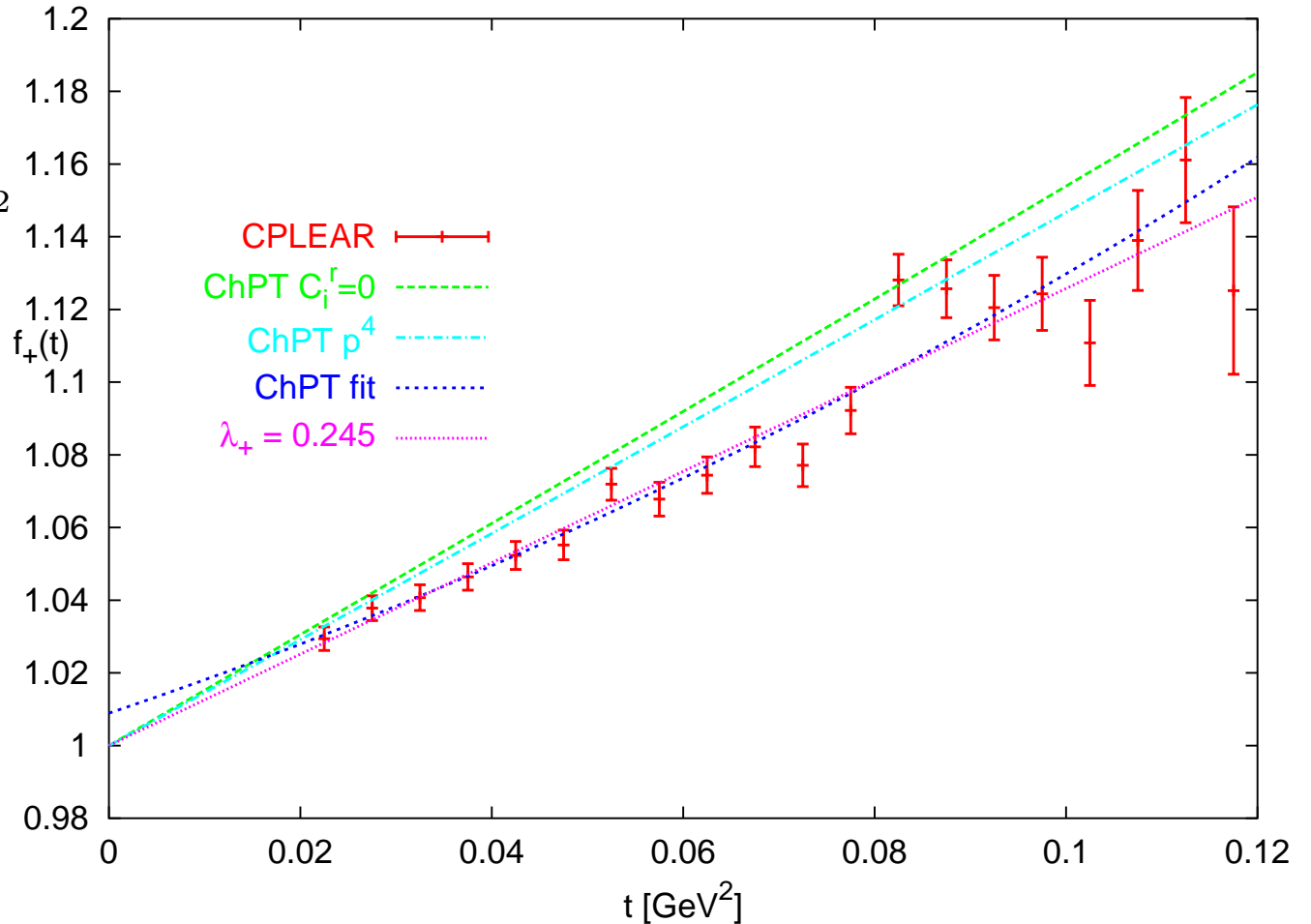
# ChPT fit to $f_+(t)$

$$\Rightarrow R_{+1}^{K\pi} = -(4.7 \pm 0.5) 10^{-5} \text{ GeV}^2$$

$$(c_+ = 3.2 \text{ GeV}^{-4})$$

$$\Rightarrow a_+ = 1.009 \pm 0.004$$

$$\Rightarrow \lambda_+ = 0.0170 \pm 0.0015$$



# $f_0(t)$

## Main Result:

$$f_0(t) = 1 - \frac{8}{F_\pi^4} (C_{12}^r + C_{34}^r) (m_K^2 - m_\pi^2)^2 \\ + 8 \frac{t}{F_\pi^4} (2C_{12}^r + C_{34}^r) (m_K^2 + m_\pi^2) + \frac{t}{m_K^2 - m_\pi^2} (F_K/F_\pi - 1) \\ - \frac{8}{F_\pi^4} t^2 C_{12}^r + \bar{\Delta}(t) + \Delta(0).$$

$\bar{\Delta}(t)$  and  $\Delta(0)$  contain **NO**  $C_i^r$  and only depend on the  $L_i^r$  at order  $p^6$

$\implies$

**All needed parameters can be determined experimentally**

$$\Delta(0) = -0.0080 \pm 0.0057[\text{loops}] \pm 0.0028[L_i^r].$$

# Experiment

## Form Factor comparison

### KTeV [PRD 70(2004)]

$K_{e3}^0$  quadratic fit:  $\lambda''_+ \neq 0$  @  $4\sigma$  level

$K_{\mu 3}^0$  quadratic fit:  $\lambda_0 = (13.72 \pm 1.31) 10^{-3}$

Slopes consistent for  $K_{e3}^0$  and  $K_{\mu 3}^0$

### ISTRA+ [PLB 581(2004), PLB589(2004)]

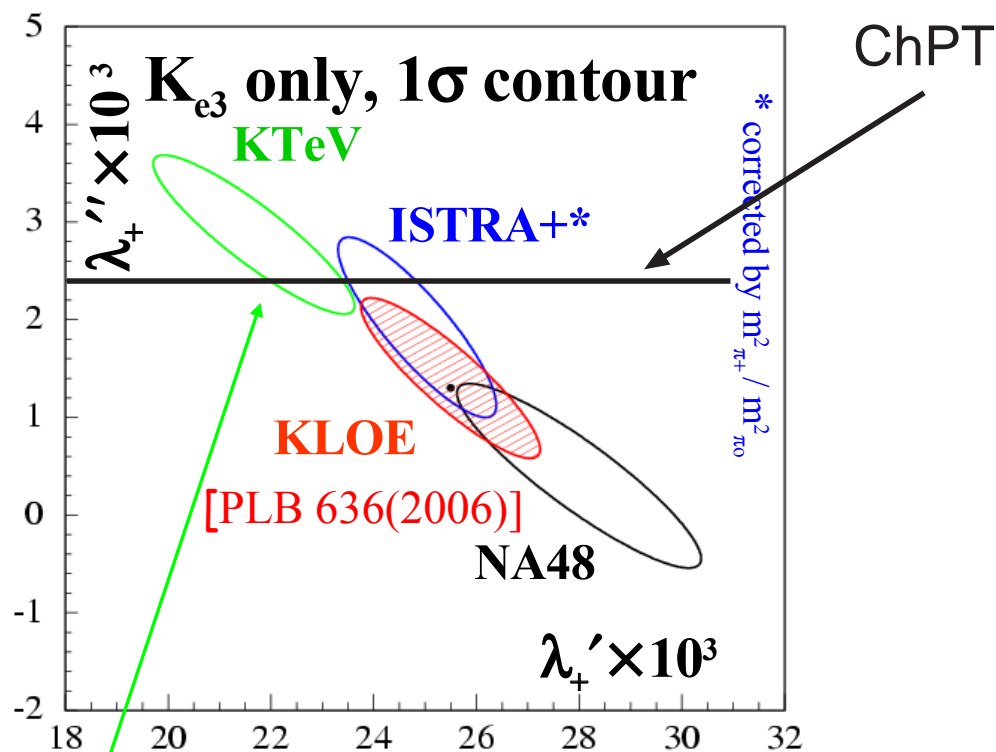
$K_{e3}^-$  quadratic fit:  $\lambda''_+ \neq 0$  @  $2\sigma$  level

$K_{\mu 3}^-$  quadratic fit:  $\lambda_0 = (17.11 \pm 2.31) 10^{-3}$

### NA48 [PLB 604(2004), HEP2005 289]

$K_{e3}^0$ : No evidence for quadratic term

$K_{\mu 3}^0$  linear fit:  $\lambda_0 = (12.0 \pm 1.7) 10^{-3}$

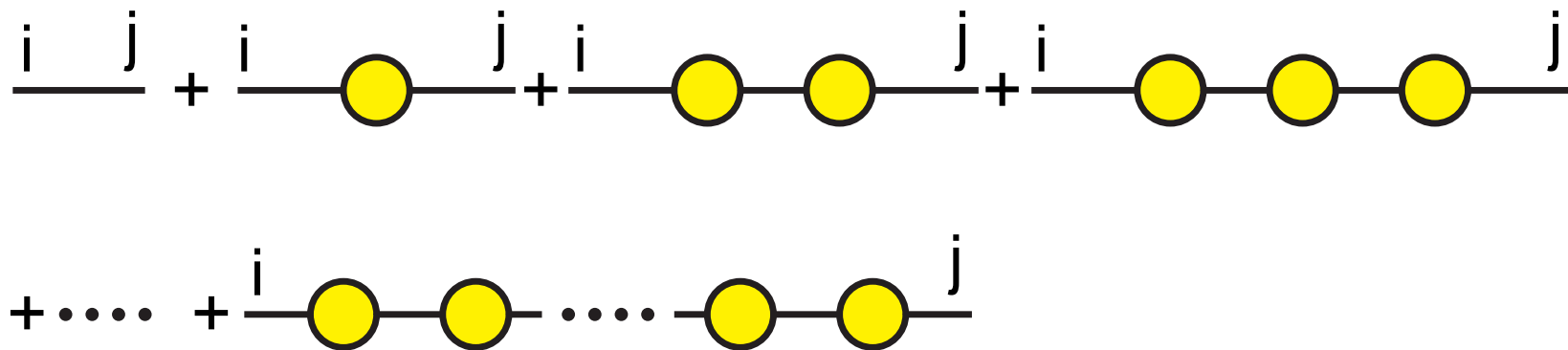


$\lambda''_+ \rightarrow -1\%$  phase space integral

$\rightarrow +0.5\%$  for  $V_{us}$

# New for isospin breaking at two loops

Two-point functions:



The diagrams for  $\langle 0 | \phi_i \phi_j | 0 \rangle \equiv G_{ij}$

**Problem: Mixing**

Take LSZ into account properly

Amoros, JB, Talavera, 2001



# Isospin Breaking

$$G = iP + iP(i\Pi iP) + iP(i\Pi iP)^2 + \dots = iP(1 + \Pi P)^{-1} .$$

No mixing:  $\Pi$ ,  $P$  and  $G$  all diagonal

$$\det(G^{-1}) = \det((1 + \Pi P) P^{-1}) = \det(P^{-1} + \Pi) = 0 .$$

Zeros give particle masses, residues can also be obtained.

Solve perturbatively for our case of  $\pi^0$ - $\eta$  mixing.

Lowest order:  $\pi^0 \equiv \cos(\epsilon) \pi_3^0 + \sin(\epsilon) \eta_8$

lowest order mixing angle  $\epsilon$ :  $\tan(2\epsilon) = \frac{\sqrt{3} m_d - m_u}{2 m_s - \hat{m}}$ ,

# Isospin Breaking

Define lowest order diagonal basis  $P_{ij} = \frac{\delta_{ij}}{p^2 - m_{i0}^2}$

$\Pi$  starts at next-to-leading order

$$m_{phys}^2 = m_0^2 + (m^2)^{(4)} + (m^2)^{(6)} + \dots$$

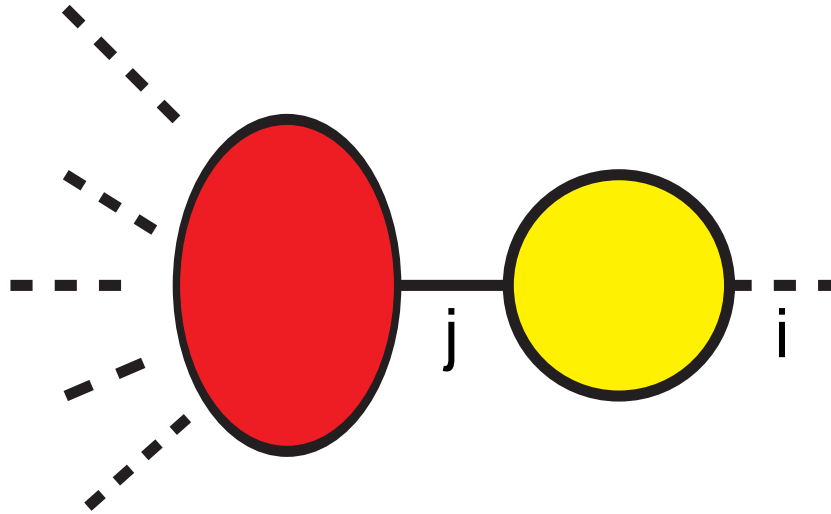
$$m_0^2 = m_{30}^2,$$

$$(m^2)^{(4)} = -\Pi_{33}^{(4)}(m_{30}^2, m_{i0}^2, F_0),$$

$$(m^2)^{(6)} = -\Pi_{33}^{(4)}(m_{30}^2, m_{i0}^2, F_0) - (m^2)^{(4)} \left. \frac{\partial}{\partial p^2} \Pi_{33}^{(4)}(p^2, m_{i0}^2, F_0) \right|_{p^2=m_{30}^2} \\ + \frac{1}{m_{30}^2 - m_{80}^2} \left( \Pi_{38}^{(4)}(m_{30}^2, m_{i0}^2, F_0) \right)^2$$

# Matrix element

Matrix element:



$$\mathcal{A}_{i_1 \dots i_n} = \left( \frac{(-i)^n}{\sqrt{Z_{i_1} \dots Z_{i_n}}} \right) \prod_{i=1}^n \lim_{k_i^2 \rightarrow m_i^2} (k_i^2 - m_i^2) G_{i_1 \dots i_n}(k_1, \dots, k_n)$$

# Isospin Breaking

$$G^{-1} = -i\mathcal{P}, \quad \mathcal{P} = P^{-1} + \Pi$$

$$Z_3 = \frac{1}{\left. \frac{\partial}{\partial p^2} (\det \mathcal{P}(p^2)) \right|_{p^2=m_\pi^2}} \mathcal{P}_{88}(m_\pi^2),$$

$$\text{Leading to } \mathcal{A}_3 = \frac{1}{\sqrt{\mathcal{P}_{88} \frac{\partial}{\partial p^2} (\mathcal{P}_{33} \mathcal{P}_{88} - \mathcal{P}_{38}^2)}} \{ \mathcal{P}_{88} \mathcal{G}_3 - \mathcal{P}_{38} \mathcal{G}_8 \} .$$

Now expand in chiral orders and ...

# Isospin Breaking

$$\begin{aligned}
 \mathcal{A}_3 = & \mathcal{G}_3^{(2)} + \left\{ \mathcal{G}_3^{(4)} - \frac{1}{2} Z_{33}^{(4)} \mathcal{G}_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(2)} \right\} \\
 & + \left[ \mathcal{G}_3^{(6)} - \frac{1}{2} Z_{33}^{(6)} \mathcal{G}_3^{(2)} - \frac{1}{2} Z_{33}^{(4)} \mathcal{G}_3^{(4)} + \frac{3}{8} \left( Z_{33}^{(4)} \right)^2 \mathcal{G}_3^{(2)} \right. \\
 & + \frac{Z_{38}^{(4)} \Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_3^{(2)} - \frac{1}{2} \left( \frac{\Pi_{38}^{(4)}}{\Delta m^2} \right)^2 \mathcal{G}_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(4)} \\
 & \left. - \frac{\Pi_{38}^{(6)}}{\Delta m^2} \mathcal{G}_8^{(2)} + \frac{\Pi_{38}^{(4)} \Pi_{88}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(2)} + \frac{1}{2} Z_{33}^{(4)} \frac{\Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(2)} \right]
 \end{aligned}$$

# Isospin Breaking

$$\begin{aligned}
 \mathcal{A}_3 = & \mathcal{G}_3^{(2)} + \left\{ \mathcal{G}_3^{(4)} - \frac{1}{2} Z_{33}^{(4)} \mathcal{G}_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(2)} \right\} \\
 & + \left[ \mathcal{G}_3^{(6)} - \frac{1}{2} Z_{33}^{(6)} \mathcal{G}_3^{(2)} - \frac{1}{2} Z_{33}^{(4)} \mathcal{G}_3^{(4)} + \frac{3}{8} \left( Z_{33}^{(4)} \right)^2 \mathcal{G}_3^{(2)} \right. \\
 & + \frac{Z_{38}^{(4)} \Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_3^{(2)} - \frac{1}{2} \left( \frac{\Pi_{38}^{(4)}}{\Delta m^2} \right)^2 \mathcal{G}_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(4)} \\
 & \left. - \frac{\Pi_{38}^{(6)}}{\Delta m^2} \mathcal{G}_8^{(2)} + \frac{\Pi_{38}^{(4)} \Pi_{88}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(2)} + \frac{1}{2} Z_{33}^{(4)} \frac{\Pi_{38}^{(4)}}{\Delta m^2} \mathcal{G}_8^{(2)} \right]
 \end{aligned}$$

Compute all diagrams, produce numerical programs, ...

# Numerical results for $f_+(0)$

**VERY PRELIMINARY**

Decay	$p^2$	$p^4$	pure 2-loop	$L_i^r$ at $p^6$	$C_i^r$	total
Iso conserving calculation						
$K^0$	1	-0.02266	0.01130	0.00332	???	0.99196
$K^+$	1	-0.02276	0.01104	0.00320	???	0.99154
$m_u/m_d = 0.45$						
$K^0$	1	-0.02310	0.01131	0.00325	???	0.99146
$K^+$	1.02465	-0.01741	0.00379	0.00648	???	1.01751
ratio	1.02465	1.0311				1.0262
$m_u/m_d = 0.58$						
$K^0$	1	-0.02299	0.01124	0.00325	???	0.99150
$K^+$	1.01702	-0.01897	0.00657	0.00551	???	1.01013
ratio	1.0170	1.0215				1.0188

# Some Comments

- $K^0$  only again on  $C_{12}^r + C_{34}^r$   
Ademollo-Gatto + Callan-Treiman as in isoconserving case  
but  $(m_s - m_u)^2$
- Not checked yet whether  $C_i^r$  in  $K^+$  decay can be determined
- $p^6$  lowers the isospin breaking
- $p^6$  fit has  $m_u/m_d = 0.45$  and not 0.58
- $0.58 \rightarrow 0.52$  from  $p^6$  and  $0.52 \rightarrow 0.45$  from violation of Dashen's theorem
- **Very preliminary numerics:** decrease discrepancy by about 0.5%.