CHIRAL PERTURBATION THEORY AND
\( \eta \rightarrow 3\pi \): AN INTRODUCTION

Johan Bijnens
Lund University

bijnens@thep.lu.se
http://thep.lu.se/~bijnens
http://thep.lu.se/~bijnens/chpt.html

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Overview

1. Chiral Perturbation Theory
2. Determination of LECs in the continuum
3. $\eta \to 3\pi$: Some model independent comments/results
   - Definitions
   - Experiment
   - Why?
4. $\eta \to 3\pi$ in ChPT
   - LO
   - LO and NLO
   - NNLO
5. Conclusions
Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
*On The Foundations Of Chiral Perturbation Theory*,

For references to lectures see:
http://www.thep.lu.se/~bijnens/chpt.html
Chiral Perturbation Theory

A general Effective Field Theory:
- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:
- **Degrees of freedom**: Goldstone Bosons from spontaneous breaking of chiral symmetry
- **Powercounting**: Dimensional counting in momenta/masses
- **Breakdown scale**: Resonances, so about $M_{\rho}$. 
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Chiral Symmetry

QCD: $N_f$ light quarks: equal mass: interchange: $SU(N_f)_V$

But \[ \mathcal{L}_{QCD} = \sum_{q=u,d,s} [i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q (\bar{q}_R q_L + \bar{q}_L q_R)] \]

So if $m_q = 0$ then $SU(3)_L \times SU(3)_R$.

Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- Mechanism: see talk by L. Giusti
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\implies$ 8 massless degrees of freedom and interaction vanishes at zero momentum
Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg power counting

Rules

\[ \int d^4p \]
\[ p^2 \]
\[ 1/p^2 \]
\[ p^4 \]

One loop example

\[(p^2)^2 (1/p^2)^2 p^4 = p^4\]

\[(p^2)(1/p^2)p^4 = p^4\]
Lagrangians: Lowest order

\[ U(\phi) = \exp(i\sqrt{2}\Phi/F_0) \] parametrizes Goldstone Bosons

\[ \Phi(\chi) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}. \]

LO Lagrangian: \[ \mathcal{L}_2 = \frac{F_0^2}{4} \left\{ \langle D_\mu U^\dagger D^\mu U \rangle + \langle \chi^\dagger U + \chi U^\dagger \rangle \right\}, \]

\[ D_\mu U = \partial_\mu U - ir_\mu U + iUL_\mu, \]
left and right external currents: \( r(l)_\mu = \nu_\mu + (-)a_\mu \)

Scalar and pseudoscalar external densities: \( \chi = 2B_0(s + ip) \) quark masses via scalar density: \( s = \mathcal{M} + \cdots \)

\[ \langle A \rangle = \text{Tr}_F \langle A \rangle \]
Lagrangians: Lagrangian structure

<table>
<thead>
<tr>
<th></th>
<th>2 flavour</th>
<th>3 flavour</th>
<th>PQChPT/$N_f$ flavour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^2$</td>
<td>$F, B$</td>
<td>$F_0, B_0$</td>
<td>$F_0, B_0$</td>
</tr>
<tr>
<td>$p^4$</td>
<td>$l_i^r, h_i^r$</td>
<td>$L_i^r, H_i^r$</td>
<td>$\hat{L}_i^r, \hat{H}_i^r$</td>
</tr>
<tr>
<td>$p^6$</td>
<td>$c_i^r$</td>
<td>$C_i^r$</td>
<td>$K_i^r$</td>
</tr>
</tbody>
</table>

$p^2$: Weinberg 1966
$p^4$: Gasser, Leutwyler 84,85
$p^6$: JB, Colangelo, Ecker 99,00

$L_i$ LEC = Low Energy Constants = ChPT parameters
$H_i$: contact terms: value depends on definition of currents/densities
Finite volume: no new LECs
Other effects: (many) new LECs
Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms
- includes Isospin and the eightfold way ($SU(3)_V$)
- Unitarity included perturbatively

\[
m^2_\pi = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2} \log \left(\frac{2B\hat{m}}{\mu^2}\right) + 2l'_3(\mu)\right] + \cdots
\]

\[
M^2 = 2B\hat{m}
\]
(Partial) History/References

- Original determination at $p^4$: Gasser, Leutwyler,

- $p^6$ 2 flavour: several papers (see later)

- $p^6$ 3 flavour: Amorós, JB, Talavera,

- Review article two-loops:

- Update of fits + new input:

- Recent review with more $p^6$ input: JB, Ecker,

- Review Kaon physics: Cirigliano, Ecker, Neufeld, Pich, Portoles,

- Lattice: FLAG reports:,
  Aoki et al., arXiv:1310.8555
Three flavour LECs: uncertainties

- $m_K^2, m_\eta^2 \gg m_\pi^2$
- Contributions from $\rho^6$ Lagrangian are larger
- Reliance on estimates of the $C_i$ much larger
- Typically: $C_i^r$: (terms with)
  - kinematical dependence $\equiv$ measurable
  - quark mass dependence $\equiv$ impossible (without lattice)
    - 100% correlated with $L_i^r$
- How suppressed are the $1/N_c$-suppressed terms?
- Are we really testing ChPT or just doing a phenomenological fit?
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Testing if ChPT works: relations


Systematic search for relations between observables that do not depend on the $C_i$

Included:

- $m_M^2$ and $F_M$ for $\pi, K, \eta$.
- 11 $\pi\pi$ threshold parameters
- 14 $\pi K$ threshold parameters
- 6 $\eta \rightarrow 3\pi$ decay parameters,
- 10 observables in $K_{\ell 4}$
- 18 in the scalar formfactors
- 11 in the vector formfactors
- Total: 76

We found 35 relations
Relations at NNLO: summary

- We did numerics for $\pi\pi$ (7), $\pi K$ (5) and $K_{\ell 4}$ (1) 13 relations
- $\pi\pi$: similar quality in two and three flavour ChPT
  The two involving $a_3^-$ significantly did not work well
- $\pi K$: relation involving $a_3^-$ not OK
  one more has very large NNLO corrections
- The relation with $K_{\ell 4}$ also did not work: related to that ChPT has trouble with curvature in $K_{\ell 4}$
- Conclusion: Three flavour ChPT “sort of” works
Fits: inputs


- $M_\pi, M_K, M_\eta, F_\pi, F_K/F_\pi$
- $\langle r^2 \rangle_\pi^S, c_\pi^S$ slope and curvature of $F_S$
- $\pi\pi$ and $\pi K$ scattering lengths $a^0_0, a^2_0, a^{1/2}_0$ and $a^{3/2}_0$.
- Value and slope of $F$ and $G$ in $K_{\ell 4}$
- $\frac{m_s}{\hat{m}} = 27.5$ (lattice)
- $\bar{l}_1, \ldots, \bar{l}_4$
- more variation with $C^r_i$, a penalty for a large $p^6$ contribution to the masses
- $17+3$ inputs and $8 L^r_i+34 C^r_i$ to fit
### Main fit

<table>
<thead>
<tr>
<th></th>
<th>ABC01</th>
<th>BJ12</th>
<th>$L'_4$ free</th>
<th>BE14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3L'_1$</td>
<td>old data</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^3L'_2$</td>
<td>0.39(12)</td>
<td>0.88(09)</td>
<td>0.64(06)</td>
<td>0.53(06)</td>
</tr>
<tr>
<td>$10^3L'_3$</td>
<td>0.73(12)</td>
<td>0.61(20)</td>
<td>0.59(04)</td>
<td>0.81(04)</td>
</tr>
<tr>
<td>$10^3L'_4$</td>
<td>−2.34(37)</td>
<td>−3.04(43)</td>
<td>−2.80(20)</td>
<td>−3.07(20)</td>
</tr>
<tr>
<td>$10^3L'_5$</td>
<td>0.97(11)</td>
<td>0.58(13)</td>
<td>0.50(07)</td>
<td>1.01(06)</td>
</tr>
<tr>
<td>$10^3L'_6$</td>
<td>≡ 0</td>
<td>0.75(75)</td>
<td>0.76(18)</td>
<td>≡ 0.3</td>
</tr>
<tr>
<td>$10^3L'_7$</td>
<td>0.97(11)</td>
<td>0.58(13)</td>
<td>0.50(07)</td>
<td>1.01(06)</td>
</tr>
<tr>
<td>$10^3L'_8$</td>
<td>−0.30(15)</td>
<td>−0.11(15)</td>
<td>−0.19(08)</td>
<td>−0.34(09)</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dof</td>
<td>1</td>
<td>4</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$F_0$ [MeV]</td>
<td>87</td>
<td>65</td>
<td>64</td>
<td>71</td>
</tr>
</tbody>
</table>

$?= (17 + 3) - (8 + 34)$
Main fit: Comments

- All values of the $C_i^r$ we settled on are “reasonable”
- Leaving $L_4^r$ free ends up with $L_4^r \approx 0.76$
- Keeping $L_4^r$ small: also $L_6^r$ and $2L_1^r - L_2^r$ small (large $N_c$ relations)
- Compatible with lattice determinations
- Not too bad with resonance saturation both for $L_i^r$ and $C_i^r$
- Decent convergence (but enforced for masses)
- Many prejudices went in: large $N_c$, resonance model, quark model estimates,...
Some results of this fit

Mass:
\[
\begin{align*}
\frac{m_\pi^2}{m_{\pi \text{phys}}^2} &= 1.055(p^2) - 0.005(p^4) - 0.050(p^6), \\
\frac{m_K^2}{m_{K \text{phys}}^2} &= 1.112(p^2) - 0.069(p^4) - 0.043(p^6), \\
\frac{m_\eta^2}{m_{\eta \text{phys}}^2} &= 1.197(p^2) - 0.214(p^4) + 0.017(p^6),
\end{align*}
\]

Decay constants:
\[
\begin{align*}
\frac{F_\pi}{F_0} &= 1.000(p^2) + 0.208(p^4) + 0.088(p^6), \\
\frac{F_K}{F_\pi} &= 1.000(p^2) + 0.176(p^4) + 0.023(p^6).
\end{align*}
\]

Scattering:
\[
\begin{align*}
\frac{a_0}{m_\pi} &= 0.160(p^2) + 0.044(p^4) + 0.012(p^6), \\
\frac{a_{0}^{1/2}}{m_\pi} &= 0.142(p^2) + 0.031(p^4) + 0.051(p^6).
\end{align*}
\]
Chiral perturbation theory and $\eta \to 3\pi$: an introduction

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Definitions: $\eta \to 3\pi$


\[ s = (p_{\pi^+} + p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2 \]
\[ t = (p_{\pi^-} + p_{\pi^0})^2 = (p_\eta - p_{\pi^+})^2 \]
\[ u = (p_{\pi^+} + p_{\pi^0})^2 = (p_\eta - p_{\pi^-})^2 \]

\[ s + t + u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0. \]

\[ \langle \pi^0 \pi^+ \pi^- | \eta \rangle = i (2\pi)^4 \delta^4 (p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}) A(s, t, u) \]
\[ \langle \pi^0 \pi^0 \pi^0 | \eta \rangle = i (2\pi)^4 \delta^4 (p_\eta - p_1 - p_2 - p_3) \bar{A}(s_1, s_2, s_3) \]
\[ \bar{A}(s_1, s_2, s_3) = A(s_1, s_2, s_3) + A(s_2, s_3, s_1) + A(s_3, s_1, s_2) \]

Observables: $\Gamma(\eta \to \pi^+ \pi^- \pi^0)$ and $r = \frac{\Gamma(\eta \to \pi^0 \pi^0 \pi^0)}{\Gamma(\eta \to \pi^+ \pi^- \pi^0)}$
Definitions: Dalitz plot

\[ x = \sqrt{3} \frac{T^+ - T^-}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta} (u - t) \]

\[ y = \frac{3T_0}{Q_\eta} - 1 = \frac{3((m_\eta - m_{\pi^0})^2 - s)}{2m_\eta Q_\eta} - 1 \overset{\text{iso}}{=} \frac{3}{2m_\eta Q_\eta} (s_0 - s) \]

\[ Q_\eta = m_\eta - 2m_{\pi^0} - m_{\pi^0} \]

\( T^i \) is the kinetic energy of pion \( \pi^i \)

\[ z = \frac{2}{3} \sum_{i=1,3} \left( \frac{3E_i - m_\eta}{m_\eta - 3m_{\pi^0}} \right)^2 \quad E_i \text{ is the energy of pion } \pi^i \]

\[ |M|^2 = A_0^2 \left( 1 + ay + by^2 + dx^2 + fy^2 + gxy^2 + \cdots \right) \]

\[ |\overline{M}|^2 = \overline{A_0}^2 \left( 1 + 2\alpha z + \cdots \right) \]

Note: neutral, next order: \( x \) and \( y \) appear separately
Relations

Expand amplitudes and use isospin: JB, Ghorbani, arXiv:0709.0230

\[ M(s, t, u) = A \left( 1 + \tilde{a}(s - s_0) + \tilde{b}(s - s_0)^2 + \tilde{d}(u - t)^2 + \cdots \right) \]

\[ \overline{M}(s, t, u) = A \left( 3 + (\tilde{b} + 3\tilde{d}) \left( (s - s_0)^2 + (t - s_0)^2 + (u - s_0)^2 \right) \right) \]

Gives relations \((R_\eta = (2m_\eta Q_\eta)/3)\)

\[ a = -2R_\eta \text{Re}(\tilde{a}), \quad b = R_\eta^2 \left( |\tilde{a}|^2 + 2\text{Re}(\tilde{b}) \right), \quad d = 6R_\eta^2 \text{Re}(\tilde{d}). \]

\[ \alpha = \frac{1}{2} R_\eta^2 \text{Re} \left( \tilde{b} + 3\tilde{d} \right) = \frac{1}{4} \left( d + b - R_\eta^2 |\tilde{a}|^2 \right) \leq \frac{1}{4} \left( d + b - \frac{1}{4} a^2 \right) \]

equality if \(\text{Im}(\tilde{a}) = 0\)
Consequences:

- Relations between the charged and neutral decay
- Relations between $r$ and Dalitz plot
  (see also Gasser, Leutwyler, Nucl. Phys. B 250 (1985) 539)
- If you can calculate $\text{Im}(\tilde{a})$ then relation:
  nonrelativistic pion EFT

Schneider, Kubis and Ditsche, JHEP 1102 (2011) 028 [1010.3946].
Definitions: Dalitz plot

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dalitz_plot}
\caption{Dalitz plot for \( \eta \rightarrow 3\pi \) in ChPT.}
\end{figure}

- \textbf{x variation:} vertical
- \textbf{y variation:} parallel to \( t = u \)
Experiment: Decay rates

Width: determined from $\Gamma(\eta \rightarrow \gamma\gamma)$ and Branching ratios
Using the PDG12 partial update 2013 numbers

$$\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0) = 300 \pm 12 \text{ eV (in JB,Ghorbani 295 \pm 17 eV)}$$

$r$: 1.426 \pm 0.026 (our fit)
1.48 \pm 0.05 (our average)
Experiment: charged

<table>
<thead>
<tr>
<th>Exp.</th>
<th>a</th>
<th>b</th>
<th>d</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>KLOE (prel)</td>
<td>-1.104(3)</td>
<td>0.144(3)</td>
<td>0.073(3)</td>
<td>0.155(6)</td>
</tr>
<tr>
<td>WASA (prel)</td>
<td>-1.074(23)(3)</td>
<td>0.179(27)(8)</td>
<td>0.059(25)(10)</td>
<td>0.089(58)(110)</td>
</tr>
<tr>
<td>KLOE</td>
<td>-1.090(5)(+8)</td>
<td>0.124(6)(10)</td>
<td>0.057(6)(+7)</td>
<td>0.14(1)(2)</td>
</tr>
<tr>
<td>Crystal Barrel</td>
<td>-1.22(7)</td>
<td>0.22(11)</td>
<td>0.06(4) (input)</td>
<td></td>
</tr>
<tr>
<td>Layter et al.</td>
<td>-1.08(14)</td>
<td>0.034(27)</td>
<td>0.046(31)</td>
<td></td>
</tr>
<tr>
<td>Gormley et al.</td>
<td>-1.17(2)(21)</td>
<td>0.21(3)</td>
<td>0.06(4)</td>
<td></td>
</tr>
</tbody>
</table>

Crystal Barrel: $d$ input, but $a$ and $b$ insensitive to $d$

\[
\begin{align*}
a & & b & & d & & f \\
a & & 1 & & -0.226 & & -0.405 & & -0.795 \\
b & & & & 1 & & 0.358 & & 0.261 \\
d & & & & & & 1 & & 0.113 \\
f & & & & & & & & 1 \\
\end{align*}

Large correlations: KLOE:
Experiment: charged

But very good agreement:
### Experiment: neutral

<table>
<thead>
<tr>
<th>Exp.</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GAMS2000</td>
<td>$-0.022 \pm 0.023$</td>
</tr>
<tr>
<td>SND</td>
<td>$-0.010 \pm 0.021 \pm 0.010$</td>
</tr>
<tr>
<td>Crystal Barrel</td>
<td>$-0.052 \pm 0.017 \pm 0.010$</td>
</tr>
<tr>
<td>Crystal Ball (BNL)</td>
<td>$-0.031 \pm 0.004$</td>
</tr>
<tr>
<td>WASA/CELSIUS</td>
<td>$-0.026 \pm 0.010 \pm 0.010$</td>
</tr>
<tr>
<td>KLOE</td>
<td>$-0.0301 \pm 0.0035^{+0.0022}_{-0.0035}$</td>
</tr>
<tr>
<td>WASA@COSY</td>
<td>$-0.027 \pm 0.008 \pm 0.005$</td>
</tr>
<tr>
<td>Crystal Ball (MAMI-B)</td>
<td>$-0.032 \pm 0.002 \pm 0.002$</td>
</tr>
<tr>
<td>Crystal Ball (MAMI-C)</td>
<td>$-0.032 \pm 0.003$</td>
</tr>
</tbody>
</table>

All experiments in good agreement
Why is $\eta \to 3\pi$ interesting?

- Pions are in $I = 1$ state $\implies A \sim (m_u - m_d)$ or $\alpha_{em}$
- $\alpha_{em}$ effect is small
  - but is there via $(m_{\pi^+} - m_{\pi^0})$ in kinematics
  - Lowest order vanishes (current algebra)
  - $\alpha\hat{m}$ and $\alpha m_s$ small
- $\eta \to \pi^+ \pi^- \pi^0$ needs to be included directly
  - Estimates the corrections of $\alpha(m_u - m_d)$ as well
- Conclusion: at the precision I will discuss not relevant
- Exception: Cusps and Coulomb at $\pi^+ \pi^-$ thresholds

So $\eta \to 3\pi$ gives a handle on $m_u - m_d$
Most analysis use (i.e. almost all of mine): \( C_i^r \) from (single) resonance approximation

Motivated by large \( N_c \): large effort goes in this

Ananthanarayan, JB, Cirigliano, Donoghue, Ecker, Gamiz, Golterman, Kaiser, Knecht, Peris, Pich, Prades, Portoles, de Rafael, …
\[ C_i \]

\[
\mathcal{L}_V = -\frac{1}{4} \langle V_{\mu\nu} V^{\mu\nu} \rangle + \frac{1}{2} m_V^2 \langle V_\mu V^\mu \rangle - \frac{f_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle \\
- \frac{ig_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + f_\chi \langle V_\mu [u^\mu, \chi_-] \rangle
\]

\[
\mathcal{L}_A = -\frac{1}{4} \langle A_{\mu\nu} A^{\mu\nu} \rangle + \frac{1}{2} m_A^2 \langle A_\mu A^\mu \rangle - \frac{f_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle
\]

\[
\mathcal{L}_S = \frac{1}{2} \langle \nabla^\mu S \nabla_\mu S - M_S^2 S^2 \rangle + c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi_+ \rangle
\]

\[
\mathcal{L}_{\eta'} = \frac{1}{2} \partial_\mu P_1 \partial^\mu P_1 - \frac{1}{2} M_{\eta'}^2 P_1^2 + i\tilde{d}_m P_1 \langle \chi_- \rangle.
\]

\[ f_V = 0.20, \; f_\chi = -0.025, \; g_V = 0.09, \; c_m = 42 \text{ MeV}, \; c_d = 32 \text{ MeV}, \]

\[ \tilde{d}_m = 20 \text{ MeV}, \; m_V = m_\rho = 0.77 \text{ GeV}, \; m_A = m_{a_1} = 1.23 \text{ GeV}, \]

\[ m_S = 0.98 \text{ GeV}, \; m_{P_1} = 0.958 \text{ GeV} \]

\[ f_V, g_V, f_\chi, f_A: \text{ experiment} \]

\[ c_m \text{ and } c_d \text{ from resonance saturation at } O(p^4) \]
Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: $C_i^r$ in the masses/decay constants and how these effects correlate into the rest
- No $\mu$ dependence: obviously only estimate

What we do/did about it:

- Vary resonance estimate by factor of two
- Vary the scale $\mu$ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult
$L_i^{r}$ and $C_i^{r}$

Full NNLO fits of the $L_i^{r}$

- Amorós, JB, Talavera, 2000, 2001 (fit 10)
  simple $C_i^{r}$
- JB, Jemos, 2011 (BJ12)
  simple $C_i^{r}$
- JB, Ecker, 2014, (BE14)
  Continuum fit with more input for $C_i^{r}$
- Numerics presented for $\eta \rightarrow 3\pi$ is mostly with fit 10
  JB, Ghorbani, 2007
Lowest order

ChPT: Cronin 67: \[ A(s, t, u) = \frac{B_0 (m_u - m_d)}{3\sqrt{3}F_\pi^2} \left\{ 1 + \frac{3(s - s_0)}{m_\eta^2 - m_\pi^2} \right\} \]

with \( Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \) or \( R \equiv \frac{m_s - \hat{m}}{m_d - m_u} \)

\[ \hat{m} = \frac{1}{2}(m_u + m_d) \]

\[ A(s, t, u) = \frac{1}{Q^2} \frac{m_K^2}{m_\pi^2} (m_\pi^2 - m_K^2) \frac{M(s, t, u)}{3\sqrt{3}F_\pi^2} \]

\[ A(s, t, u) = \frac{\sqrt{3}}{4R} M(s, t, u) \]

LO: \[ M(s, t, u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2} \]

\[ M(s, t, u) = \frac{1}{F_\pi^2} \left( \frac{4}{3}m_\pi^2 - s \right) \]
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$\eta \to 3\pi$: $p^2$ and $p^4$

- $\Gamma(\eta \to 3\pi) \propto |A|^2 \propto Q^{-4}$ allows a PRECISE measurement

- $Q^2$ form lowest order mass relation: $Q \approx 24$
  \[ \Rightarrow \Gamma(\eta \to \pi^+\pi^-\pi^0)_{\text{LO}} \approx 66 \text{ eV} \]

- $m_{K^+}^2 - m_{K^0}^2 \sim Q^{-2}$ at NNLO: $Q = 20.0 \pm 1.5$
  \[ \Rightarrow \Gamma(\eta \to \pi^+\pi^-\pi^0)_{\text{LO}} \approx 140 \text{ eV} \]

- At order $p^4$ Gasser-Leutwyler 1985:
  \[ \frac{\int \text{d}LIPS |A_2 + A_4|^2}{\int \text{d}LIPS |A_2|^2} = 2.4, \]
  \[(LIPS=\text{Lorentz invariant phase-space})\]

Major source: large $S$-wave final state rescattering

Experiment: $300 \pm 12$ eV (PDG 2012/13)
\[ \eta \to 3\pi: \ p^2 \ and \ p^4 \]

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  **Major source:** large $S$-wave final state rescattering

- **Experiment:** $300 \pm 12 \text{ eV (PDG 2012/13)}$
η → 3π: LO, NLO, NNLO, NNNLO, \ldots

- IN Gasser, Leutwyler, 1985 ($\sqrt{2.4} = 1.55$):
  - about half: $\pi\pi$-rescattering
  - other half: everything else

- $\pi\pi$-rescattering important Roiesnel, Truong, 1981

- Dispersive approach (talks: Passemar, Knecht, Szczepaniak): resum all $\pi\pi$

- assume rescattering + rest separable:

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\text{NNLO} & \cdots & \cdots & \cdots & \cdots \\
\text{NLO} & \text{NNLO} & \cdots & \cdots & \cdots \\
\text{LO} & \text{NLO} & \text{NNLO} & \cdots & \cdots \\
\end{array}
\]

\[\rightarrow \pi\pi\text{-rescattering}
\]

dispersive does this all the way
Why look at it this way?

$\delta_\pi = 0.3, \delta_O = 0.3$

LO = 1

NLO = $\delta_\pi + \delta_O = 0.6$

NNLO = $\delta_\pi^2 + \delta_\pi \delta_O + \delta_O^2 = 0.27$

Squared: 1 $\rightarrow$ 2.6 $\rightarrow$ 3.5

Underlying other is: 1 + 0.3 + 0.09

Goal: remove dispersive from ChPT, then add again via dispersion relations (but now all boxes)

Problem: Separation is not trivial
Why look at it this way?

\[ \delta_{\pi} = 0.3, \quad \delta_{O} = 0.3 \]

\[ \text{LO} = 1 \]

\[ \text{NLO} = \delta_{\pi} + \delta_{O} = 0.6 \]

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Goal: remove dispersive from ChPT, then add again via dispersion relations (but now all boxes)

Problem: Separation is not trivial
Why look at it this way?

\[ \uparrow \text{Other effects} \]

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\text{NNLO} & \cdots & \cdots & \cdots \\
\text{NLO} & \text{NNLO} & \cdots & \cdots \\
\text{LO} & \text{NLO} & \text{NNLO} & \cdots \\
\end{array}
\]

\[ \rightarrow \pi\pi\text{-rescattering} \]

\[ \text{dispersive does this all the way} \]

- \( \delta_\pi = 0.3, \delta_O = 0.3 \)
- \( \text{LO} = 1 \)
- \( \text{NLO} = \delta_\pi + \delta_O = 0.6 \)
- \( \text{NNLO} = \delta^2_\pi + \delta_\pi \delta_O + \delta^2_O = 0.27 \)
- \( \text{Squared: } 1 \rightarrow 2.6 \rightarrow 3.5 \)
- \( \text{Underlying other is: } 1 + 0.3 + 0.09 \)
- \( \text{Goal: remove dispersive from ChPT, then add again via dispersion relations (but now all boxes)} \)
- \( \text{Problem: Separation is not trivial} \)
Diagrams

- Include mixing, renormalize, pull out factor $\frac{\sqrt{3}}{4R}$, …
- Two independent calculations (comparison lots of work)
- You have to carefully define which LO ($M$ or $M$)
- You have to carefully define which NLO
- Integrals only in numerical form: (g) is the hardest one
\[ \eta \rightarrow 3\pi: \quad M(s, t = u) \]

Along \( t = u \)

Along \( t = u \) parts
$\eta \rightarrow 3\pi$: $M(s, t = u)$

Along $t = u$

$L_i' = C_i' = 0$

I.e. where $C_i'(\mu)$ estimated

\[ M(s, t = u) \]

\[ L_i = C_i = 0 \]
Neutral decay

<table>
<thead>
<tr>
<th></th>
<th>$\bar{A}_0^2$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO</td>
<td>1090</td>
<td>0.000</td>
</tr>
<tr>
<td>NLO</td>
<td>2810</td>
<td>0.013</td>
</tr>
<tr>
<td>NLO ($L'_r = 0$)</td>
<td>2100</td>
<td>0.016</td>
</tr>
<tr>
<td>NNLO</td>
<td>4790</td>
<td>0.013</td>
</tr>
<tr>
<td>NNLOq</td>
<td>4790</td>
<td>0.014</td>
</tr>
<tr>
<td>NNLO ($C'_r = 0$)</td>
<td>4140</td>
<td>0.011</td>
</tr>
<tr>
<td>NNLO ($L'_r = C'_r = 0$)</td>
<td>2220</td>
<td>0.016</td>
</tr>
<tr>
<td>dispersive (KWW)</td>
<td>—</td>
<td>$-(0.007-0.014)$</td>
</tr>
<tr>
<td>tree dispersive</td>
<td>—</td>
<td>$-0.0065$</td>
</tr>
<tr>
<td>absolute dispersive</td>
<td>—</td>
<td>$-0.007$</td>
</tr>
<tr>
<td>Borasoy</td>
<td>—</td>
<td>$-0.031$</td>
</tr>
<tr>
<td>error</td>
<td>160</td>
<td>0.032</td>
</tr>
</tbody>
</table>

- experiment: $\alpha = -0.032$ with small error
- NNLO ChPT gets $a_0^0$ in $\pi\pi$ correct
Theory: charged

<table>
<thead>
<tr>
<th></th>
<th>$A_0^2$</th>
<th>a</th>
<th>b</th>
<th>d</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LO</strong></td>
<td>120</td>
<td>-1.039</td>
<td>0.270</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td><strong>NLO</strong></td>
<td>314</td>
<td>-1.371</td>
<td>0.452</td>
<td>0.053</td>
<td>0.027</td>
</tr>
<tr>
<td><strong>NLO ($L_i^r = 0$)</strong></td>
<td>235</td>
<td>-1.263</td>
<td>0.407</td>
<td>0.050</td>
<td>0.015</td>
</tr>
<tr>
<td><strong>NNLO</strong></td>
<td>538</td>
<td>-1.271</td>
<td>0.394</td>
<td>0.055</td>
<td>0.025</td>
</tr>
<tr>
<td><strong>NNLOp ($y$ from $T^0$)</strong></td>
<td>574</td>
<td>-1.229</td>
<td>0.366</td>
<td>0.052</td>
<td>0.023</td>
</tr>
<tr>
<td><strong>NNLOq (incl ($x, y)^4$)</strong></td>
<td>535</td>
<td>-1.257</td>
<td>0.397</td>
<td>0.076</td>
<td>0.004</td>
</tr>
<tr>
<td><strong>NNLO ($\mu = 0.6$ GeV)</strong></td>
<td>543</td>
<td>-1.300</td>
<td>0.415</td>
<td>0.055</td>
<td>0.024</td>
</tr>
<tr>
<td><strong>NNLO ($\mu = 0.9$ GeV)</strong></td>
<td>548</td>
<td>-1.241</td>
<td>0.374</td>
<td>0.054</td>
<td>0.025</td>
</tr>
<tr>
<td><strong>NNLO ($C_i^r = 0$)</strong></td>
<td>465</td>
<td>-1.297</td>
<td>0.404</td>
<td>0.058</td>
<td>0.032</td>
</tr>
<tr>
<td><strong>NNLO ($L_i^r = C_i^r = 0$)</strong></td>
<td>251</td>
<td>-1.241</td>
<td>0.424</td>
<td>0.050</td>
<td>0.007</td>
</tr>
<tr>
<td><strong>BJ12</strong></td>
<td>451</td>
<td>-1.303</td>
<td>0.406</td>
<td>0.060</td>
<td>0.031</td>
</tr>
<tr>
<td><strong>BE14</strong></td>
<td>614</td>
<td>-1.356</td>
<td>0.430</td>
<td>0.063</td>
<td>0.038</td>
</tr>
<tr>
<td><strong>BE14free</strong></td>
<td>552</td>
<td>-1.339</td>
<td>0.421</td>
<td>0.062</td>
<td>0.036</td>
</tr>
</tbody>
</table>

|                  |         |       |       |       |       |
| **error**        | 18      | 0.075 | 0.102 | 0.057 | 0.160 |
| **KLOE 08**      | -1.090  | 0.124 | 0.057 | 0.14  |

- NLO to NNLO changes, but no large ones
- Error: $\Delta |M(s, t, u)|^2 = |M^{(6)} M(s, t, u)|$
Experiment vs Theory: charged
Experiment: relative

Chiral Perturbation Theory and $\eta \to 3\pi$: an introduction
Johan Bijnens

Chiral Perturbation Theory
Determination of LECs in the continuum
Model independent
$\eta \to 3\pi$ in ChPT
LO
LO and NLO
NNLO
Conclusions
Experiment vs Theory: relative
$r$ and decay rates

$$\sin \epsilon = \frac{\sqrt{3}}{4R} + O(\epsilon^2)$$

$$\Gamma(\eta \to \pi^+ \pi^- \pi^0) = \begin{cases} \sin^2 \epsilon \cdot 0.572 \text{ MeV} & \text{LO}, \\
\sin^2 \epsilon \cdot 1.59 \text{ MeV} & \text{NLO}, \\
\sin^2 \epsilon \cdot 2.68 \text{ MeV} & \text{NNLO}, \\
\sin^2 \epsilon \cdot 2.33 \text{ MeV} & \text{NNLO } C_i^r = 0, \\
\end{cases}$$

$$\Gamma(\eta \to \pi^0 \pi^0 \pi^0) = \begin{cases} \sin^2 \epsilon \cdot 0.884 \text{ MeV} & \text{LO}, \\
\sin^2 \epsilon \cdot 2.31 \text{ MeV} & \text{NLO}, \\
\sin^2 \epsilon \cdot 3.94 \text{ MeV} & \text{NNLO}, \\
\sin^2 \epsilon \cdot 3.40 \text{ MeV} & \text{NNLO } C_i^r = 0. \end{cases}$$
$r$ and decay rates

$$r \equiv \frac{\Gamma(\eta \to \pi^0\pi^0\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$$

$$r_{\text{LO}} = 1.54$$
$$r_{\text{NLO}} = 1.46$$
$$r_{\text{NNLO}} = 1.47$$
$$r_{\text{NNLO } c_i=0} = 1.46$$

PDG 2013

$$r = 1.48 \pm 0.05 \text{ our average}.$$  
$$r = 1.426 \pm 0.026 \text{ our fit},$$

Reasonable agreement
\( R \) and \( Q \) from \( \eta \rightarrow 3\pi \)

<table>
<thead>
<tr>
<th>( R (\eta) )</th>
<th>LO</th>
<th>NLO</th>
<th>NNLO</th>
<th>NNLO ((C'_i = 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.9</td>
<td>31.5</td>
<td>40.9</td>
<td>38.2</td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>44</td>
<td>37</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>37</td>
<td>32</td>
<td>—</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( Q (\eta) )</th>
<th>LO</th>
<th>NLO</th>
<th>NNLO</th>
<th>NNLO ((C'_i = 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.5</td>
<td>21.3</td>
<td>24.3</td>
<td>23.4</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>24</td>
<td>22</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>22</td>
<td>20</td>
<td>—</td>
<td></td>
</tr>
</tbody>
</table>

LO from \( R = \frac{m_{K^0}^2 + m_{K^+}^2 - 2m_{\pi^0}^2}{2(m_{K^0}^2 - m_{K^+}^2)} \) (QCD part only)

NLO and NNLO from masses: Amorós, JB, Talavera 2001

\[ Q^2 = \frac{m_s + \hat{m}}{2\hat{m}} R = 14.4 R \]

\( m_s / \hat{m} = 27.8 \) used for \( \eta \rightarrow 3\pi \)
Conclusions

- Short introduction to ChPT
- New best fit for the $L_i^r$: BE14
- Overview of $\eta \to 3\pi$
- $\eta \to 3\pi$ at NNLO in ChPT plus some preliminary updates