Analytical results for hadronic contributions to the muon $g - 2$

Johan Bijnens

Lund University

bijnens@thep.lu.se
http://thep.lu.se/~bijnens
Why do we do this?

The muon $a_\mu = \frac{g_\mu - 2}{2}$ will be measured more precisely.

J-PARC

Fermilab
Why do we do this?

- Experiment dominated by BNL, FNAL error down by four
- Theory taken from PDG2018

\[ a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{Had}} \]

\[ a_{\mu}^{\text{Had}} = a_{\mu}^{\text{LO-HVP}} + a_{\mu}^{\text{HO-HVP}} + a_{\mu}^{\text{HLbL}} \]

- Impressive agreement with \( g_{\mu} \) to \( 2 \times 10^{-9} \)

<table>
<thead>
<tr>
<th>Part</th>
<th>value</th>
<th>errors</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{\mu}^{\text{EXP}} )</td>
<td>116 592 091.x</td>
<td>(54)(33)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( a_{\mu}^{\text{SM}} )</td>
<td>116 591 823.x</td>
<td>(1)(34)(26)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( \Delta a_{\mu} )</td>
<td>268.x</td>
<td>(63)(43)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( a_{\mu}^{\text{QED}} )</td>
<td>116 584 719.0</td>
<td>(0.1)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( a_{\mu}^{\text{EW}} )</td>
<td>153.6</td>
<td>(1.0)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( a_{\mu}^{\text{LO-HVP}} )</td>
<td>6 931.x</td>
<td>(33)(7)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( a_{\mu}^{\text{HO-HVP}} )</td>
<td>−86.3</td>
<td>(0.9)</td>
<td>( \times 10^{-11} )</td>
</tr>
<tr>
<td>( a_{\mu}^{\text{HLbL}} )</td>
<td>105.x</td>
<td>(26)</td>
<td>( \times 10^{-11} )</td>
</tr>
</tbody>
</table>
Hadronic contributions

The blobs are hadronic contributions

- I will present some results that are useful for LO-HVP and HLbL
My recent work related to the muon $g - 2$


[1,2,3]: HLbL, [4,5]: HVP
I will concentrate on [2,5]
Analytical results for hadronic contributions to the muon $g-2$

Johan Bijnens

Introduction

HLbL: the main object to calculate

- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks
General properties

\[ \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \]

Actually we really need

\[ \frac{\delta \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}} \bigg|_{q_4=0} \]
General properties

\[ \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3): \]

- In general 138 Lorentz structures (136 in 4 dimensions)
- Using \( q_{1\mu} \Pi^{\mu\nu\lambda\sigma} = q_{2\nu} \Pi^{\mu\nu\lambda\sigma} = q_{3\lambda} \Pi^{\mu\nu\lambda\sigma} = q_{4\sigma} \Pi^{\mu\nu\lambda\sigma} = 0 \)
- 43 (41) gauge invariant structures
- 41 helicity amplitudes
- Bose symmetry relates some of them
- Compare HVP: one function, one variable
- General calculation from experiment via dispersion relations: recent progress
  - Colangelo, Hoferichter, Kubis, Procura, Stoffer, . . .
- Well defined separation between different contributions
- Theory initiative: paper under preparation
- One remaining problem: intermediate- and short-distances
Short-distance

- Use (constituent) quark loop
- Used for full estimates since the beginning (1970s)
- Used for short-distance estimates
  - JB, Pallante, Prades, 1996
- Is it a first term in a systematic OPE?
- OPE has been used as constraints on specific contributions
  - $\pi^0\gamma^*\gamma^*$ asymptotic behaviour
  - Constraints on many other hadronic formfactors
  - $q_1^2 \approx q_2^2 \gg q_3^2$ Melnikhov, Vainshtein 2003
Short-distance: first attempt

\[ \Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \left\langle T \left( j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \right) \right\rangle \]

- Usual OPE: \( x, y, z \) all small
- First term in the expansion is the quark-loop
  no problem with \( \partial / \partial q_4^\rho \) and \( q_4 \to 0 \)
  \( p \) in loop \( \Rightarrow \) no singular propagators:

- Next term problems: no loop momentum;
  \( q_4 \to 0 \) propagator diverges:
Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, 1984
- For the $q_4$-leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge: $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$
  - whole calculation is immediately with $q_4 = 0$.
- First term is exactly the usual quark loop
  (even including quark masses)
Short-distance: next term(s)

- Do the usual QCD sum rule expansion in terms of vacuum condensates
- There are new condensates, induced by the constant magnetic field: \[ \langle \bar{q} \sigma_{\alpha\beta} q \rangle \equiv e_q F_{\alpha\beta} X_q \]
- Lattice QCD Bali et al., arXiv:1206.4205
  \[ X_u = 40.7 \pm 1.3 \text{ MeV}, \]
  \[ X_d = 39.4 \pm 1.4 \text{ MeV}, \]
  \[ X_s = 53.0 \pm 7.2 \text{ MeV} \]
- Could have started at order \(1/Q\), only starts at \(1/Q^2\) via \(m_q X_q\) corrections to the leading quark-loop result
- \(X_q\) and \(m_q\) are very small, only a very small correction
- Next order: very many condensates contribute, work in progress
Short-distance

- **Formalism of** Colangelo et al., 1702.07347
  \[ a_\mu = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} \sum_{i=1,12} \hat{T}_i \Pi_i \]

- The $\Pi_i$ are related to 6 $\hat{\Pi}_i$;

- **Bare quark loop derived from both**

- **In agreement with quark loop formulae from** Hoferichter, Stoffer, private communication
For $N_c = 3$ with $e_q = 1$ and one quark we get ($q_i^2 = -Q_i^2$) (preliminary)

\[
\hat{\Pi}_1 = m_q X_q \frac{2(q_3^2 - q_1^2 - q_2^2)}{q_1^2 q_2^2 q_3^2}, \quad \hat{\Pi}_4 = m_q X_q \frac{4}{q_1^2 q_2^2 q_3^2}, \\
\hat{\Pi}_7 = 0, \quad \hat{\Pi}_{17} = m_q X_q \frac{-4}{q_1^2 q_2^2 q_3^2} \\
\hat{\Pi}_{39} = 0, \quad \hat{\Pi}_{54} = m_q X_q \frac{2(q_1^2 - q_2^2)}{q_1^4 q_2^4 q_3^2}
\]
Short-distance: numerical results

- preliminary
- $Q_1, Q_2, Q_3 \geq Q_{\text{min}}$
- $m_u = m_d = m_s = 0$ for quark-loop
- $m_u = m_d = 5$ MeV and $m_s = 100$ MeV for $m_qX_q$

<table>
<thead>
<tr>
<th>$Q_{\text{min}}$</th>
<th>quarkloop</th>
<th>$m_uX_u + m_dX_d$</th>
<th>$m_sX_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 GeV</td>
<td>$17 \times 10^{-11}$</td>
<td>$-2.7 \times 10^{-13}$</td>
<td>$-4.1 \times 10^{-13}$</td>
</tr>
<tr>
<td>2 GeV</td>
<td>$4.3 \times 10^{-11}$</td>
<td>$-1.7 \times 10^{-14}$</td>
<td>$-2.6 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

Above 1 GeV still 15% of total value of HLbL

- Quarkloop goes roughly as $1/Q_{\text{min}}^2$
- $m_qX_q$ goes roughly as $1/Q_{\text{min}}^4$
- Naive suppression is $m_qX_q/Q_{\text{min}}^2 \sim 2 \times 10^{-3}$
- Observed $10^{-3}$
Short-distance: Conclusions

- We have shown that the quarkloop really is the first term of a proper OPE expansion for the HLbL.
- We have calculated the next term which is suppressed by quark masses and a small $X_q$: negligible.
- The next term contains both the usual vacuum and a large number of induced condensates but will not be suppressed by small quark masses.
- Why do this: matching of the sum over hadronic contributions to the expected short distance domain.
- Finding the onset of the asymptotic domain.
HVP: Summary of results

Figure from 1807.09370

Some discrepancies, lattice accuracies improve
Status lattice results

- Dispersive has reached below 0.5%
- Lattice accuracies a few %
- Improvement continuous
- Finite volume corrections known to two-loop order in ChPT [4]
- Next: isospin breaking
  - $m_u - m_d$: “easy”
  - Electromagnetism: possibly large finite volume corrections since $1/L^n$ rather than $\exp(-m_\pi L)$


Presented at $g - 2$ meetings in KEK February 2018, Mainz June 2018
They are expected to be small
Conventions

- **Main object:**
  \[ \Pi_{ab}^{\mu\nu}(q) = i \int d^4xe^{i q \cdot x} \langle 0 | T(j_a^{\mu}(x)j_b^{\nu\dagger}(0)|0 \rangle \]

- **EM currents:**
  \[ j_{EM}^{\mu} = (2/3)j_{U}^{\mu} - (1/3)j_{D}^{\mu} - (1/3)j_{S}^{\mu} \]
  \[ j_{Q}^{\mu} = \bar{q}\gamma^{\mu}q \]

- **Continuum/infinite volume:**
  \[ \Pi_{EM}^{\mu\nu}(q) = (q^{\mu}q^{\nu} - q^2 g^{\mu\nu}) \Pi_{EM}(q^2) \]

- **Known positive weight functions \( v, w \) and \( Q^2 = -q^2 \):**
  \[ a_{\mu} = \int_{\text{threshold}}^{\infty} dq^2 w(q^2) \frac{1}{\pi} \text{Im} \Pi_{EM}(q^2) \]
  \[ a_{\mu} = \int_{0}^{\infty} dQ^2 v(Q^2) \left( -\Pi(Q^2) + \Pi(0) \right) \]

- **Dispersion relation:**
  \[ \Pi(q^2) = \Pi(0) + \frac{q^2}{\pi} \int_{\text{threshold}}^{\infty} ds \frac{1}{s(s - q^2)} \frac{1}{\pi} \text{Im} \Pi(s) \]
First: Scalar QED

- Pion loops: finite volume effects suppressed by \( e^{-m_\pi L} \) (if off-shell)
- Photon loops have suppression only by powers of \( 1/L \)
- Dynamical photons: large finite volume effects possible
- Scalar QED in usual \( \overline{MS} \)
  \[ (\mu_{\text{ChPT}}^2 = \mu_{\overline{MS}}^2 e, \; e = 2.71 \ldots) \]
- \( \mathcal{L} = (\partial_\mu \Phi^* + ieA_\mu \Phi^*)(\partial_\mu \Phi - ieA_\mu \Phi) - m_0^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \)
  \(-\lambda (\phi^* \phi)^2 \) not needed)
Finite volume

- Problem for photon: $\int d^d k \frac{1}{k^2} \rightarrow \int dk^0 \sum_{\vec{k}} \frac{1}{(k^0)^2 - \vec{k}^2}$
  - Singularity no longer has the $k^{d-1}$ to soften it
- Solution QED$_L$: drop all modes with $\vec{k} = 0$ Hayakawa-Uno
- We extend the arguments of Davoudi-Savage to two-loop order, use QED$_L$ and a lattice with infinite time extension
- We only calculate corrections suppressed by $1/L$ and powers, not exponentially suppressed contributions
- Below threshold so mesons (pions) are off-shell
Integrals at finite volume

- \( S = \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2-m^2)((k+l-p)^2-m^2)} \)
- do \( l^0, k^0 \) integrals via contour integration
- \( \vec{k} = \frac{2\pi}{L} \vec{n} \) and expand in \( 1/L \)
- Write the \( \vec{k} \) part as
  \[
  \frac{1}{L^{d-1}} \sum_{\vec{n} \neq \vec{0}} = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} + \left[ \frac{1}{L^{d-1}} \sum_{\vec{n} \neq \vec{0}} - \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \right]
  \]
- In the first term resum the series in \( 1/L \): infinite volume contribution
- Call the quantity in brackets \( (1/L^{d-1}) \Delta'_{\vec{n}} \)
- Define \( c_m = \Delta_{\vec{n}} \frac{1}{|\vec{n}|^m} \)
- These are known numerically
Example: mass

\[ \Delta_V m^2 = e^2 \left( \frac{4c_2}{16\pi^2 mL} + \frac{2c_1}{16\pi^2 m^2 L^2} + O \left( \frac{1}{L^4}, e^{-mL} \right) \right) \]

- Agrees with earlier results
- \( c_2 \) shows up since BOTH propagators can be 'on-shell'
- For \( \Pi_{\mu\nu} \) below threshold only photon line goes 'on-shell'
  \[ \implies \text{corrections start only at } 1/L^2 \]
- Only true if infinite volume mass is used in the expressions at LO
Numerical results mass

Infinite volume is \( \frac{\delta m^2}{m^2} = 0.00852 \)

So very large finite volume corrections to electromagnetic part
Twopoint function

- Do the $k^0$, $l^0$ integral
- expand for large $L$ as explained earlier
- Rest can be expressed in terms of

$$Z_{ij}(m^2, p^2) = \int \frac{d^{d-1}l}{(2\pi)^{d-1}} \frac{1}{(l^2 + m^2)^{i/2}(4l^2 + 4m^2 - p^2)^j}$$

these correspond to one-loop integrals with masses
- $\Omega_{ij} \equiv Z_{ij}(m^2, p^2)m^{i+2j-d+1}$
- Calculate in center of mass frame: $p = (p^0, \vec{0})$
- $\text{QED}_L$: no disconnected contribution
- $t_{\mu\nu}$ spatial part of $g_{\mu\nu}$
- $\tilde{\Pi}((p^0)^2) \equiv \frac{-1}{3p^2} t_{\mu\nu} (\Pi^{\mu\nu}(p) - \Pi^{\mu\nu}(p = 0))$
- Infinite volume: $\tilde{\Pi}(p^2) = \Pi(p^2)$
Analytical results for hadronic contributions to the muon $g - 2$

Johan Bijnens

Introduction

HLbL

LO-HVP

sQED

Finite volume

Conclusions

Diagrams

- Lowest order:

- NLO:

contributions from counterterms and □□□□
Results

\[ \tilde{\Pi}(p^2) = \frac{c_1}{\pi m^2 L^2} \left( \frac{16}{3} \Omega_{-1,3} - \frac{1}{3} \Omega_{1,2} - \frac{32}{3} \Omega_{1,3} - \frac{2}{3} \Omega_{3,2} + \frac{16}{3} \Omega_{3,3} - \frac{1}{8} \Omega_{5,1} + \Omega_{5,2} \right) \]

\[ + \frac{c_0}{m^3 L^3} \left( - \frac{128}{3} \Omega_{-2,4} + \frac{256}{3} \Omega_{0,4} - \frac{5}{3} \Omega_{2,2} + \frac{8}{3} \Omega_{2,3} - \frac{128}{3} \Omega_{2,4} \right) \]

\[ - \frac{3}{8} \Omega_{4,1} + \frac{7}{6} \Omega_{4,2} - \frac{8}{3} \Omega_{4,3} \right) + O \left( \frac{1}{L^4}, e^{-mL} \right) \]

Simplify using relations of the $\Omega_{ij}$ to

\[ \tilde{\Pi}(p^2) = + \frac{c_0}{m^3 L^3} \left( - \frac{16}{3} \Omega_{0,3} - \frac{5}{3} \Omega_{2,2} + \frac{40}{9} \Omega_{2,3} - \frac{3}{8} \Omega_{4,1} + \frac{7}{6} \Omega_{4,2} + \frac{8}{9} \Omega_{4,3} \right) \]

\[ + O \left( \frac{1}{L^4}, e^{-mL} \right) \]

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect
Scalar QED on a lattice

Correction for $mL \sim 5$ less than 1%
Generality

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect

- The blob (hadrons) is a four-point function of electromagnetic currents and has no singularities in the regime needed
- The conclusion about $1/L^3$ is general
Conclusions HVP em finite volume

Showed you results for:

- Finite volume corrections to the electromagnetic contribution as estimated in scalar QED are small.
- This is a universal feature: the object under study is neutral and contains no hadronic on-shell propagators.
- At the level of precision needed for the next level: negligible.