LEADING LOGARITHMS IN EFFECTIVE FIELD THEORIES

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Overview

- Leading logarithms (LL)
- LL in renormalizable theories
- Weinberg's argument and proof to all loop orders
- Example: diagrams needed for the mass
- Massive $O(N)$ model
- Results: Mass, Decay Constant, Vacuum Expectation Value
- Large N
- Numerical results
- Other quantities

Work done with Lisa Carloni, arXiv:0909.5086 and to be published

Leading Logarithms

- Take a quantity with a single scale: $F(M)$
- The dependence on the scale will be logarithmic
- $L = \log(\mu/M)$
- $F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + F_3^3 L^3 + \cdots$
- Leading Logarithms: The terms $F_m^m L^m$

The F_m^m can be more easily calculated than the full result

- $\mu\,(dF/d\mu) \equiv 0$
- Ultraviolet divergences in Quantum Field Theory are always local

Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$
- f_i^j are pure numbers

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- f_i^j are pure numbers
- $\mu \frac{dF}{d\mu} \equiv F'$, $\mu \frac{d\alpha}{d\mu} \equiv \alpha'$, $\mu \frac{dL}{d\mu} = 1$
- $$F' = \alpha' + f_1^1 \alpha^2 + f_1^1 2\alpha' \alpha L + f_2^2 \alpha^3 2L + f_2^2 3\alpha' \alpha^2 L^2 + f_1^2 \alpha^3 + f_1^2 3\alpha' \alpha^2 L + f_0^2 3\alpha' \alpha^2 + f_3^3 \alpha^3 3L^2 + f_3^3 4\alpha' \alpha^3 L^3 + \cdots$$

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- $\alpha' = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \cdots$

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  $+ f_1^2 \alpha^3 + f_1^2 3\alpha' \alpha^2 L + f_0^2 3\alpha' \alpha^2 + f_3^3 \alpha^3 3L^2 + f_3^3 4\alpha' \alpha^3 L^3 + \cdots$
- $\alpha' = \beta_1 \alpha^2 + \beta_2 \alpha^3 + \cdots$
- $0 = F' = \left(\beta_0 + f_1^1\right) \alpha^2 + \left(2\beta_0 f_1^1 + 2 f_2^2\right) \alpha^3 L +$
  $\left(\beta_1 + 2\beta_0 f_0^1 + f_1^2\right) \alpha^3 + \left(3\beta_0 f_2^2 + 3 f_3^3\right) \alpha^4 L^2 + \cdots$

Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + f_1^1\alpha^2 L + f_0^1\alpha^2 + f_2^2\alpha^3 L^2 + f_1^2\alpha^3 L + f_0^2\alpha^3 + f_3^3\alpha^4 L^3 + \cdots$
- f_i^j are pure numbers
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- $F' = \alpha' + f_1^1\alpha^2 + f_1^1 2\alpha'\alpha L + f_2^2\alpha^3 2L + f_2^2 3\alpha'\alpha^2 L^2$
  $+ f_1^2\alpha^3 + f_1^2 3\alpha'\alpha^2 L + f_0^2 3\alpha'\alpha^2 + f_3^3\alpha^3 3L^2 + f_3^3 4\alpha'\alpha^3 L^3 + \cdots$
- $\alpha' = \beta_1\alpha^2 + \beta_2\alpha^3 + \cdots$
- $0 = F' = \left(\beta_0 + f_1^1\right)\alpha^2 + \left(2\beta_0 f_1^1 + 2f_2^2\right)\alpha^3 L +$
  $\left(\beta_1 + 2\beta_0 f_0^1 + f_1^2\right)\alpha^3 + \left(3\beta_0 f_2^2 + 3f_3^3\right)\alpha^4 L^2 + \cdots$
- $\boxed{\beta_0 = -f_1^1 = f_2^2 = -f_3^3 = \cdots}$

Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu\frac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–'t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the α the same in all orders
- LL one-loop β_0
- NLL two-loop β_1

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- LL one-loop β_0
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- In effective field theories: different Lagrangian at each order
- The recursive argument does not work

Weinberg

- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop
- Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- General at 2 loop: JB, Colangelo, Ecker, hep-ph/9808421
- Proof at all orders using β-functions Büchler, Colangelo, hep-ph/0309049
- Proof with diagrams: present work

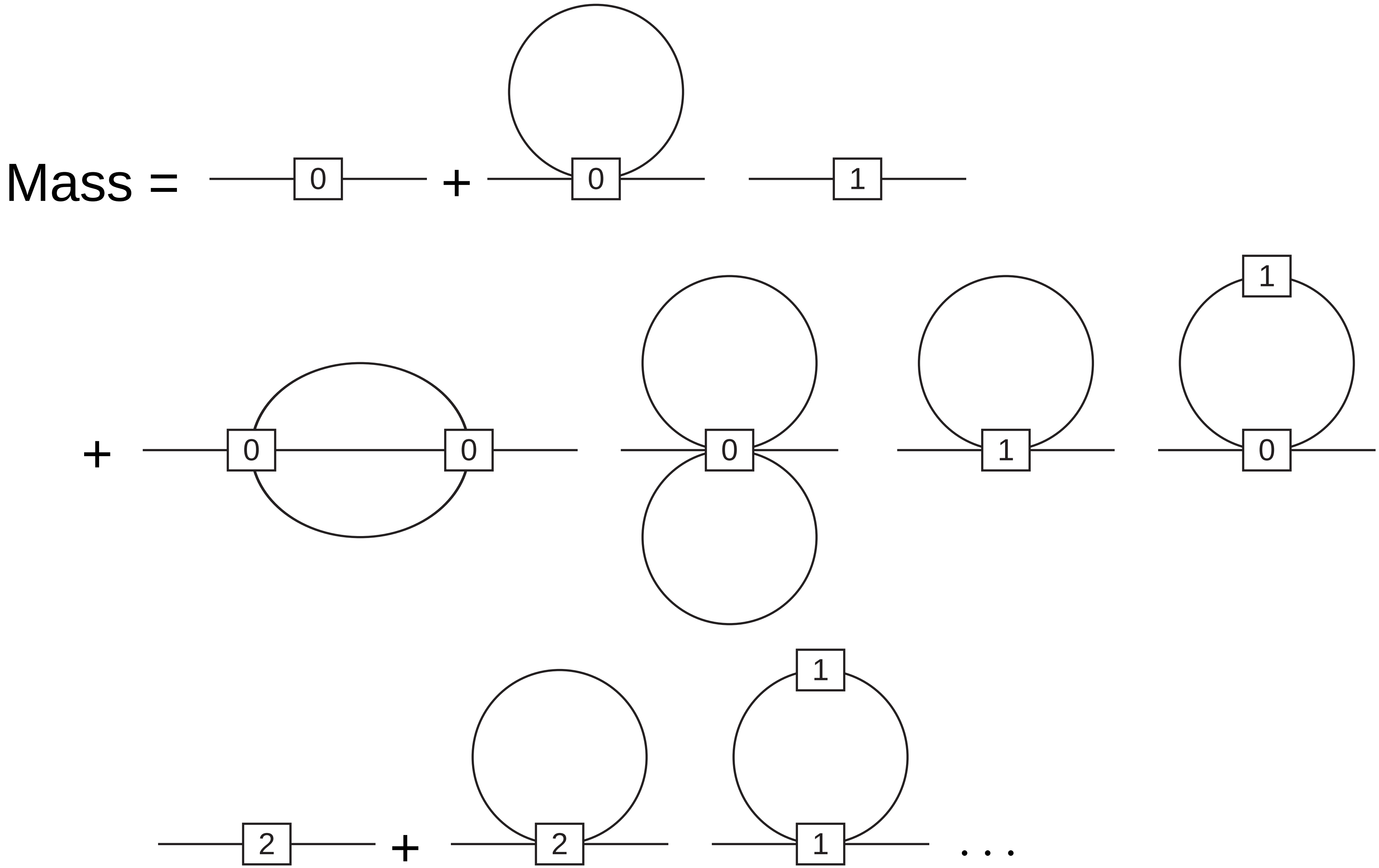
Weinberg's argument

- μ: dimensional regularization scale
- $d = 4 - w$
- loop-expansion $\equiv \hbar$-expansion
- $$\mathcal{L}^{\text{bare}} = \sum_{n\geq 0} \hbar^n \mu^{-nw} \mathcal{L}^{(n)}$$
- $$\mathcal{L}^{(n)} = \sum_i \left(\sum_{k=0,n} \frac{c_{ki}^{(n)}}{w^k} \right) \mathcal{O}_i^{(n)}$$
- $c_{0i}^{(n)}$ have a direct μ-dependence
- $c_{ki}^{(n)}$ $k \geq 1$ only depend on μ through $c_{0i}^{(m<n)}$

Weinberg's argument

- L_l^n l-loop contribution at order $\hbar^n$
- Expand in divergences from the loops (not from the counterterms) $L_l^n = \sum_{k=0,l} \frac{1}{w^k} L_{kl}^n$
- Neglected positive powers: not relevant here, but should be kept in general
- $\{c\}_l^n$ all combinations $c_{k_1 j_1}^{(m_1)} c_{k_2 j_2}^{(m_2)} \dots c_{k_r j_r}^{(m_r)}$ with $m_i \geq 1$, such that $\sum_{i=1,r} m_i = n$ and $\sum_{i=1,r} k_i = l$.
- $\{c_n^n\} \equiv \{c_{ni}^{(n)}\}$, $\{c\}_2^2 = \{c_{2i}^{(2)}, c_{1i}^{(1)} c_{1k}^{(1)}\}$
- $\mathcal{L}^{(n)} = \boxed{\text{n}}$

Weinberg's argument



Weinberg's argument

- $\hbar^0$: L^0_0
- $\hbar^1$: $\frac{1}{w}\left(\mu^{-w}L^1_{00}(\{c\}^1_1) + L^1_{11}\right) + \mu^{-w}L^1_{00}(\{c\}^1_0) + L^1_{10}$

Weinberg's argument

- $\hbar^0$: L_0^0
- $\hbar^1$: $\frac{1}{w}\left(\mu^{-w}L_{00}^1(\{c\}_1^1) + L_{11}^1\right) + \mu^{-w}L_{00}^1(\{c\}_0^1) + L_{10}^1$
  - Expand $\mu^{-w} = 1 - w\log\mu + \frac{1}{2}w^2\log^2\mu + \cdots$
  - $1/w$ must cancel: $L_{00}^1(\{c\}_1^1) + L_{11}^1 = 0$
    this determines the c_{1i}^1
  - Explicit $\log\mu$: $-\log\mu\, L_{00}^1(\{c\}_0^1) \equiv \log\mu\, L_{11}^1$

Weinberg's argument

- $\hbar^2$:
$$\frac{1}{w^2}\left(\mu^{-2w}L^2_{00}(\{c\}^2_2)+\mu^{-w}L^2_{11}(\{c\}^1_1)+L^2_{22}\right)$$
$$+\frac{1}{w}\left(\mu^{-2w}L^2_{00}(\{c\}^2_1)+\mu^{-w}L^2_{11}(\{c\}^1_0)+\mu^{-w}L^2_{10}(\{c\}^1_1)+L^2_{21}\right)$$
$$+\left(\mu^{-2w}L^2_{00}(\{c\}^2_0)+\mu^{-w}L^2_{10}(\{c\}^1_0)+L^2_{20}\right)$$
- $1/w^2$ and $\log\mu/w$ must cancel
$$L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)+L^2_{22}=0$$
$$2L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)=0$$

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- $1/w^2$ and $\log\mu/w$ must cancel
$$L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)+L^2_{22}=0$$
$$2L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)=0$$
- Solution: $L^2_{00}(\{c\}^2_2)=-\frac{1}{2}L^2_{11}(\{c\}^1_1)\qquad L^2_{11}(\{c\}^1_1)=-2L^2_{22}$

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- Solution: $L^2_{00}(\{c\}^2_2)=-\frac{1}{2}L^2_{11}(\{c\}^1_1)$ $\quad L^2_{11}(\{c\}^1_1)=-2L^2_{22}$
- Explicit $\log\mu$ dependence (one-loop is enough)
$$\frac{1}{2}\log^2\mu\left(4L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)\right)=-\frac{1}{2}L^2_{11}(\{c\}^1_1)\log^2\mu\,.$$

All orders

- $\hbar^n$:

$$\frac{1}{w^n}\Big(\mu^{-nw}L^n_{00}(\{c\}^n_n)+\mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1})+\cdots$$
$$+\mu^{-w}L^n_{n-1\ n-1}(\{c\}^1_1)+L^n_{nn}\Big)+\frac{1}{w^{n-1}}\cdots$$

All orders

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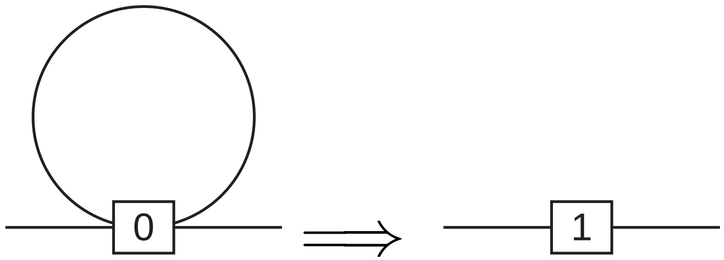
$$\frac{1}{w^n}\Big(\mu^{-nw}L^n_{00}(\{c\}^n_n)+\mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1})+\cdots$$
$$+\mu^{-w}L^n_{n-1\;n-1}(\{c\}^1_1)+L^n_{nn}\Big)+\frac{1}{w^{n-1}}\cdots$$
- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\;n-i}(\{c\}^i_i)=0 \qquad\qquad j=0,..,n-1.$$

All orders

- $\hbar^n$:
$$\frac{1}{w^n}\left(\mu^{-nw}L^n_{00}(\{c\}^n_n)+\mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1})+\cdots\right.$$
$$\left.+\mu^{-w}L^n_{n-1\;n-1}(\{c\}^1_1)+L^n_{nn}\right)+\frac{1}{w^{n-1}}\cdots$$
- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\;n-i}(\{c\}^i_i)=0 \qquad j=0,..,n-1.$$
- Solution: $L^n_{n-i\;n-i}(\{c\}^i_i)=(-1)^i\begin{pmatrix}n\\i\end{pmatrix}L^n_{nn}$
- explicit leading $\log\mu$ dependence and divergence
$$\log^n\mu\frac{(-1)^{n-1}}{n}L^n_{11}(\{c\}^{n-1}_{n-1}) \qquad L^n_{00}(\{c\}^n_n)=-\frac{1}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$

Mass to $\hbar^2$

- $\hbar^1$: 

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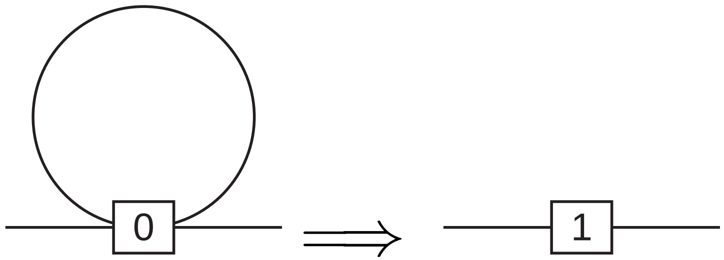
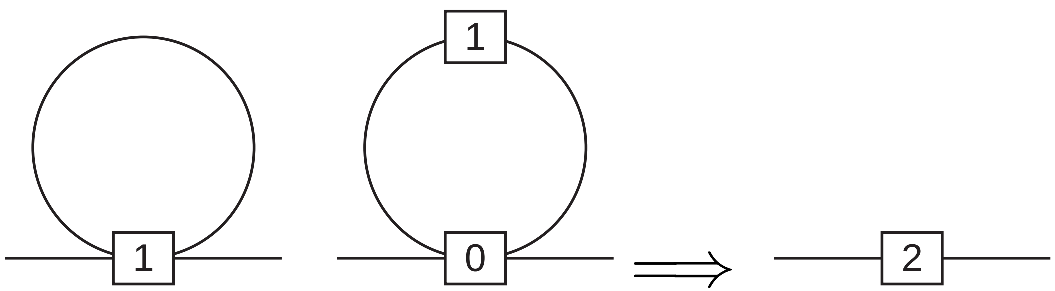
- $\hbar^1$: 

- $\hbar^2$: 

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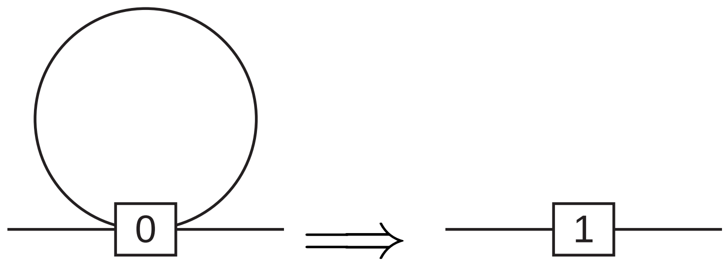
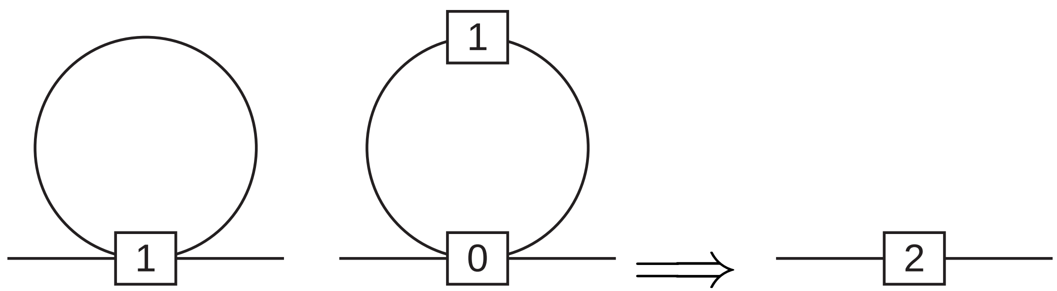
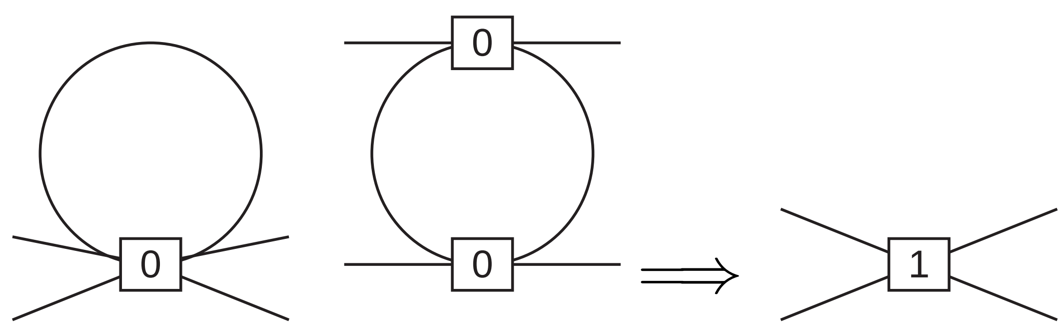
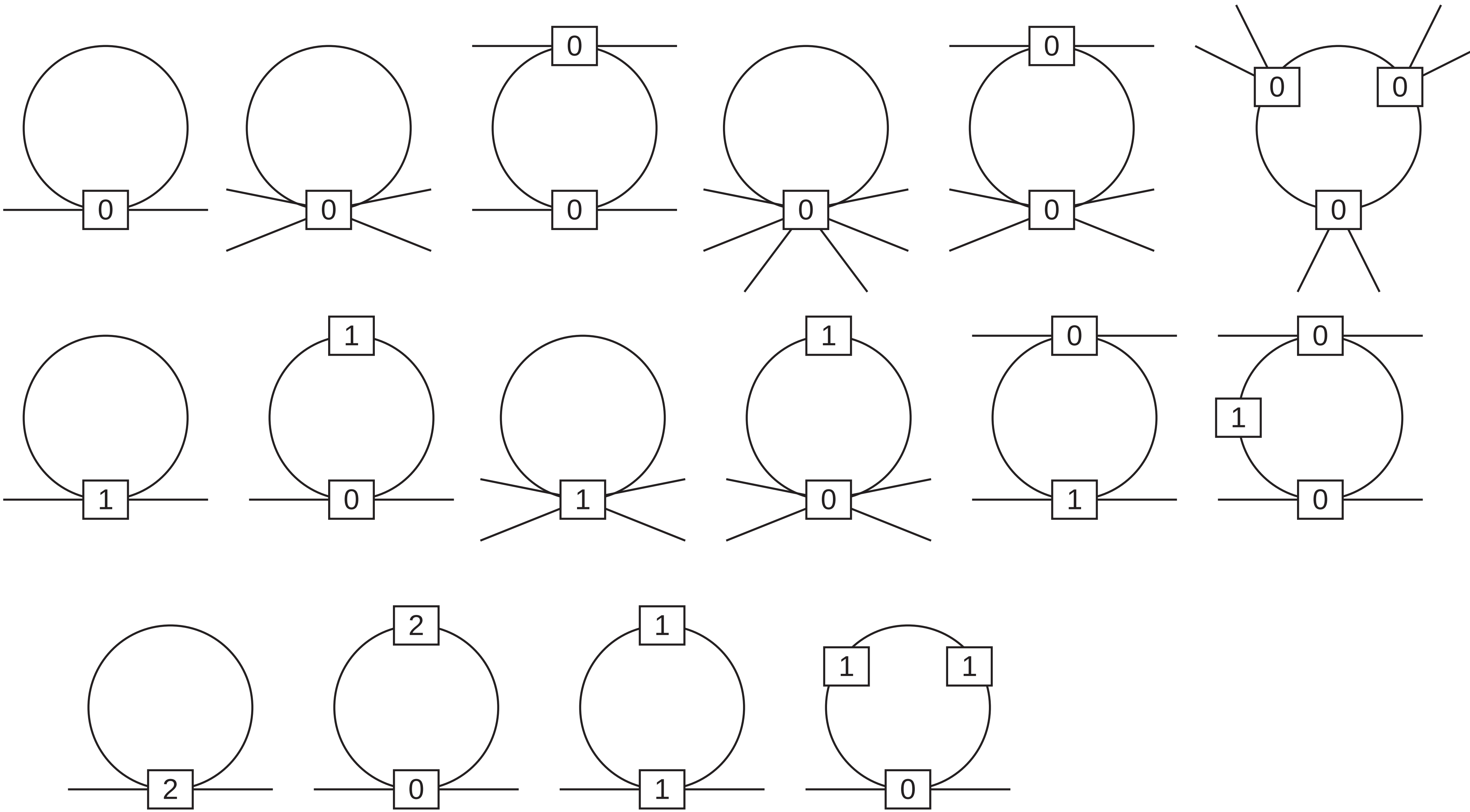
- $\hbar^1$: 

- $\hbar^2$: 

- but also needs $\hbar^1$: 

Mass to order $\hbar^3$



Mass+decay to $\hbar^5$

- $\hbar^1$: 18 + 27
- $\hbar^2$: 26 + 45
- $\hbar^3$: 33 + 51
- $\hbar^4$: 26 + 33
- $\hbar^5$: 13 + 13

- Calculate the divergence
- rewrite it in terms of a local Lagrangian
- Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
- We keep all terms to have all 1PI (one particle irreducible) diagrams finite

Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu\Phi^T\partial^\mu\Phi + F^2\chi^T\Phi\,.$
- Φ is a real $N+1$ vector; $\Phi \to O\Phi$; $\Phi^T\Phi = 1$.
- Vacuum expectation value $\langle\Phi^T\rangle = (1\;0\ldots0)$
- Explicit symmetry breaking: $\chi^T = \left(M^2\;0\ldots0\right)$
- Both spontaneous and explicit symmetry breaking
- N-vector ϕ
- N (pseudo-)Nambu-Goldstone Bosons
- $N=3$ is two-flavour Chiral Perturbation Theory

Massive $O(N)$ sigma model: Φ vs ϕ

- $$\Phi_1 = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}$$ Gasser, Leutwyler

- $$\Phi_2 = \frac{1}{\sqrt{1+\frac{\phi^T\phi}{F^2}}}\begin{pmatrix} 1 \\ \frac{\phi}{F} \end{pmatrix}$$ Weinberg

  $$\Phi_3 = \begin{pmatrix} 1-\frac{1}{2}\frac{\phi^T\phi}{F^2} \\ \sqrt{1-\frac{1}{4}\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{F} \end{pmatrix}$$ only mass term

- $$\Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$ CCWZ

Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use the four different parametrizations
- compare with known results:

$$M^2_{phys} = M^2 \left(1 - \frac{1}{2} L_M + \frac{17}{8} L_M^2 + \cdots \right) ,$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{\mathcal{M}^2}$$

  Usual choice $\mathcal{M} = M$.

- large N (but known results only for massless case)

  Coleman, Jackiw, Politzer 1974

Results

$$M^2_{\text{phys}} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \dots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

i	a_i, $N = 3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$

Results

$$F_{\rm phys} = F(1 + b_1 L_M + b_2 L_M^2 + b_3 L_M^3 + \ldots)$$

i	b_i for $N = 3$	b_i for general N
1	1	$-\frac{1}{2} + \frac{N}{2}$
2	$-\frac{5}{4}$	$-\frac{1}{2} + \frac{7N}{8} - \frac{3N^2}{8}$
3	$\frac{83}{24}$	$-\frac{7}{24} + \frac{21N}{16} - \frac{73N^2}{48} + \frac{1N^3}{2}$
4	$-\frac{3013}{288}$	$\frac{47}{576} + \frac{1345N}{864} - \frac{14077N^2}{3456} + \frac{625N^3}{192} - \frac{105N^4}{128}$
5	$\frac{2060147}{51840}$	$-\frac{23087}{64800} + \frac{459413N}{172800} - \frac{189875N^2}{20736} + \frac{546941\ N^3}{43200} - \frac{1169\ N^4}{160} + \frac{3\ N^5}{2}$

Results

$$\langle \bar{q}_i q_i \rangle = -BF^2(1 + c_1 L_M + c_2 L_M^2 + c_3 L_M^3 + \dots)$$

$M^2 = 2B\hat{m}$ $\qquad \chi^T = 2B(s\ 0\ \ \dots 0)$

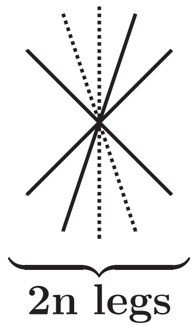
s corresponds to $\bar{u}u + \bar{d}d$ current

i	c_i for $N = 3$	c_i for general N
1	$\frac{3}{2}$	$\frac{N}{2}$
2	$-\frac{9}{8}$	$\frac{3N}{4} - \frac{3N^2}{8}$
3	$\frac{9}{2}$	$\frac{3N}{2} - \frac{3N^2}{2} + \frac{N^3}{2}$
4	$-\frac{1285}{128}$	$\frac{145N}{48} - \frac{55N^2}{12} + \frac{105N^3}{32} - \frac{105N^4}{128}$
5	46	$\frac{3007N}{480} - \frac{1471N^2}{120} + \frac{557\ N^3}{40} - \frac{1191\ N^4}{160} + \frac{3\ N^5}{2}$

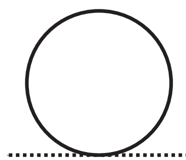
Anyone recognize any funny functions?

Large N

Power counting: pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N$, $M^2 \sim 1$

- 

$\Leftrightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$


$\Leftrightarrow N$

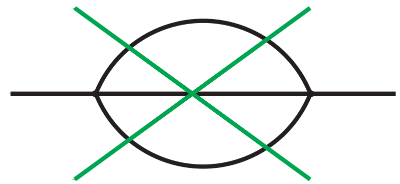
- 1PI diagrams:

$$\left.\begin{array}{l} N_L = N_I - \sum_n N_{2n} + 1 \\ 2N_I + N_E = \sum_n 2nN_{2n} \end{array}\right\} \Rightarrow N_L = \sum_n (n-1)N_{2n} - \frac{1}{2}N_E + 1$$

- diagram suppression factor: $\frac{N^{N_L}}{N^{N_E/2-1}}$

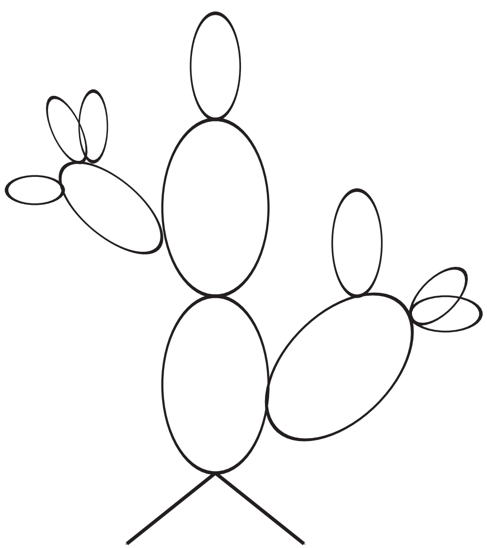
Large N

- diagrams with shared lines are suppressed



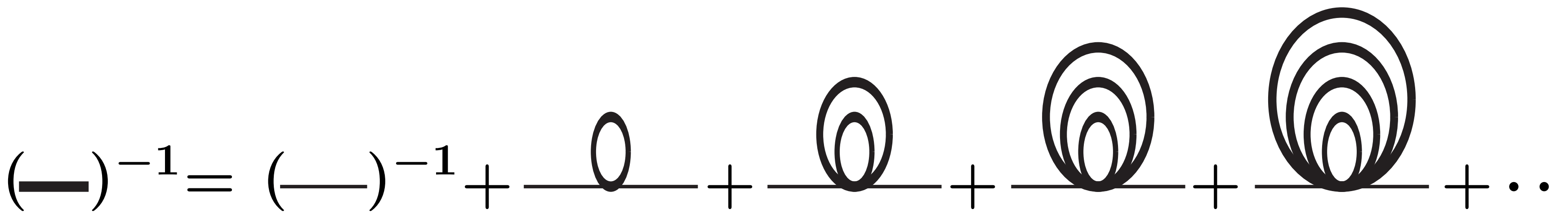
each new loop needs also a new flavour loop

- in the large N limit only "*cactus*" diagrams survive:



large N: propagator

Generate recursively via a Gap equation



$\Rightarrow$ resum the series and look for the pole

$$M^2 = M^2_{\text{phys}}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

$$\overline{A}(m^2) = \frac{m^2}{16\pi^2}\log\frac{\mu^2}{m^2}.$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N: Decay constant

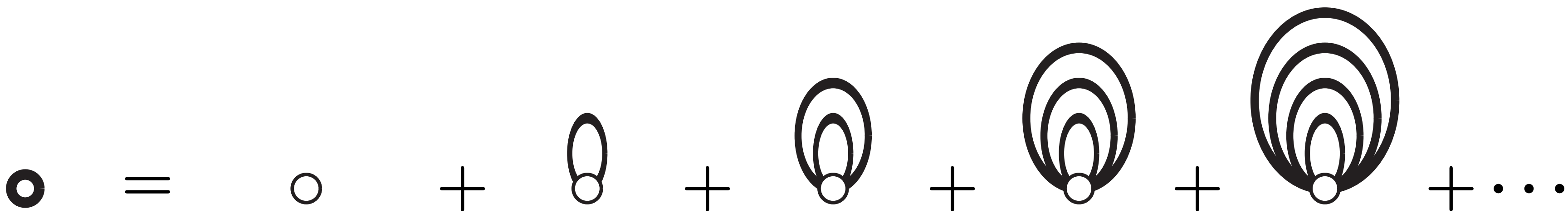
$\Rightarrow$ and include wave-function renormalization

$$F_{\text{phys}} = F\sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N: Vacuum Expectation Value



$$\langle \bar{q}q \rangle_{\text{phys}} = \langle \bar{q}q \rangle_0 \sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

Comments:

- These are the full* leading N results, not just leading log
- But depends on the choice of N-dependence of higher order coefficients
- Assumes higher LECs zero ($< N^{n+1}$ for $\hbar^n$)
- Large N as in $O(N)$ not large N_c

Large N: Checking expansions

$$M^2 = M^2_{\text{phys}}\sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

much smaller expansion coefficients than the table, try

$$M^2 = M^2_{\text{phys}}(1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + \ldots)$$

i	d_i, $N = 3$	d_i for general N
1	$\frac{1}{2}$	$-1 + \frac{1}{2}\ N$
2	$-\frac{13}{8}$	$\frac{1}{4} - \frac{1}{4}\ N - \frac{1}{8}\ N^2$
3	$-\frac{19}{48}$	$\frac{2}{3} - \frac{11}{12}\ N + \frac{1}{16}\ N^3$
4	$-\frac{5773}{1152}$	$-\frac{8}{9} + \frac{107}{144}\ N - \frac{1}{6}\ N^2 - \frac{1}{16}\ N^3 - \frac{5}{128}\ N^4$
5	$-\frac{3343}{768}$	$-\frac{18383}{7200} + \frac{130807}{43200}\ N - \frac{2771}{2160}\ N^2 - \frac{527}{1600}\ N^3 + \frac{23}{640}\ N^4 + \frac{7}{256}\ N^5$

Large N: Checking expansions

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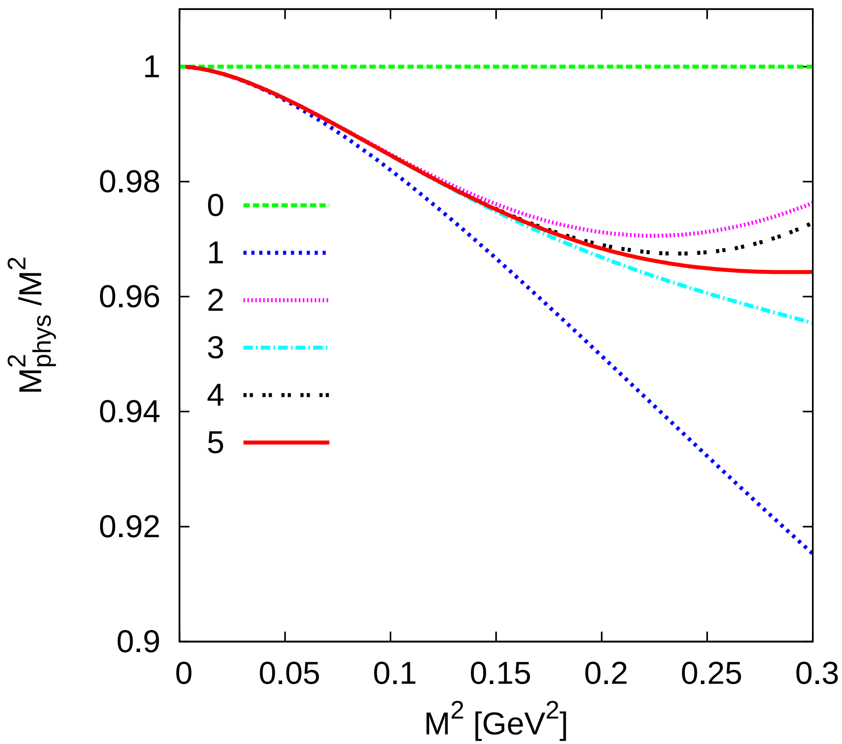
i	a_i, $N = 3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$

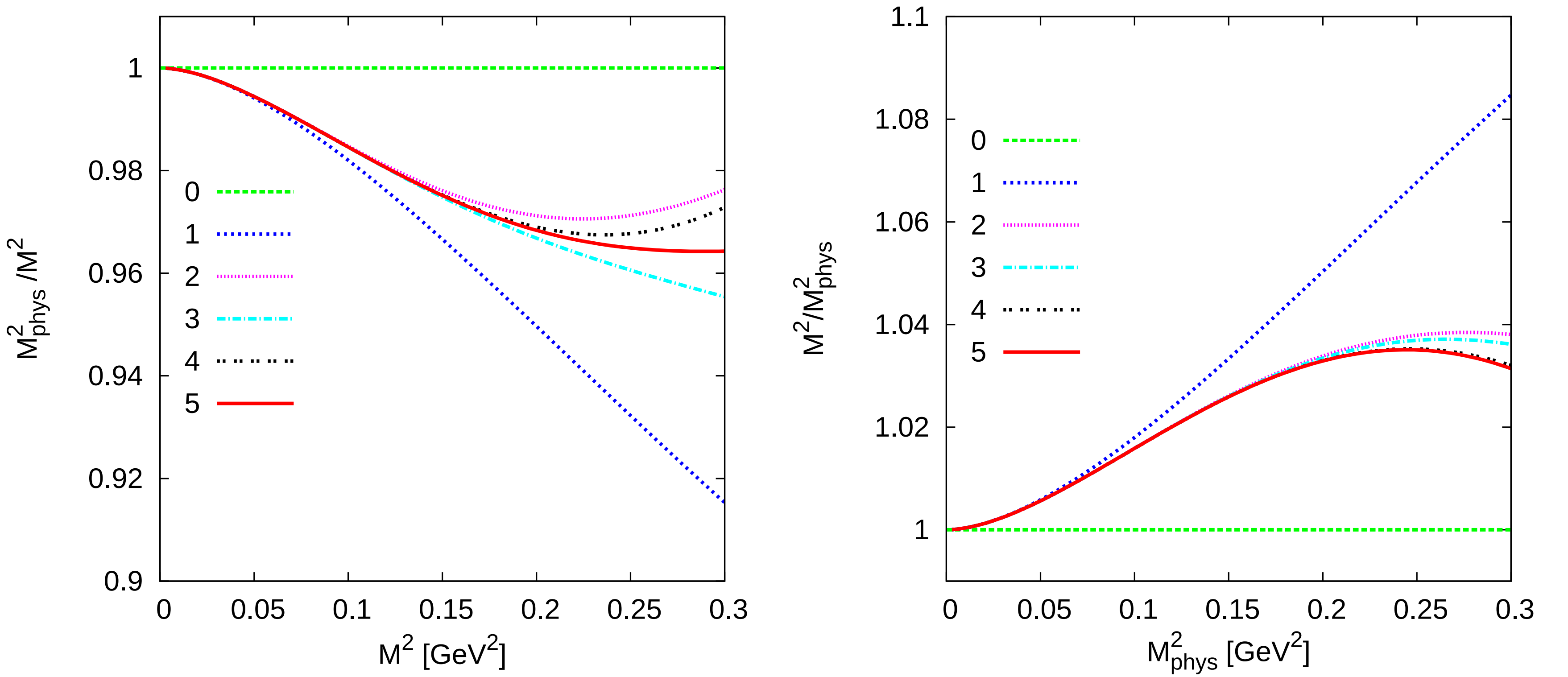
Numerical results



Left: $\frac{M^2_{\rm phys}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

F = 90 MeV, $\mu = 1$ GeV

Numerical results

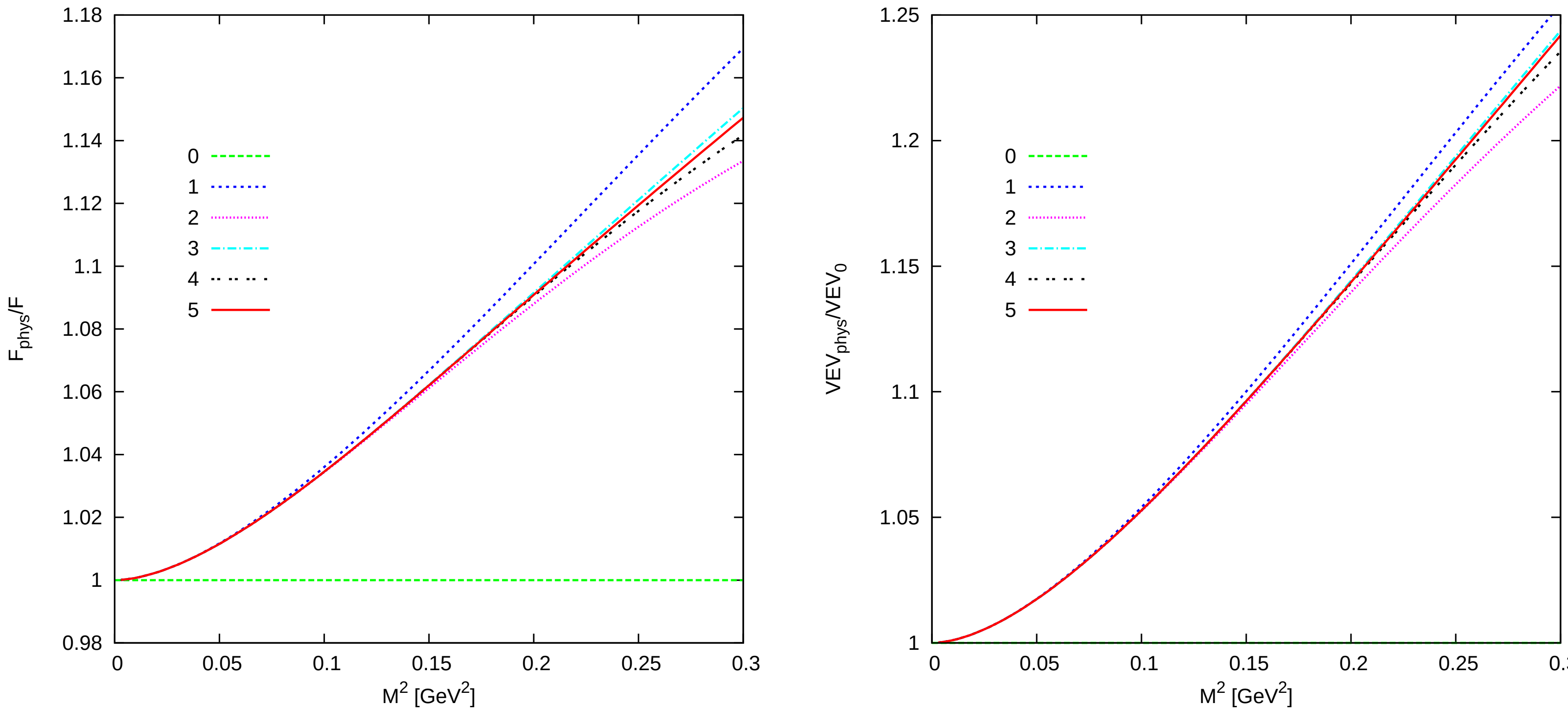


Left: $\frac{M^2_{\rm phys}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

Right: $\frac{M^2}{M^2_{\rm phys}} = 1 + d_1 L_{M_{\rm phys}} + d_2 L^2_{M_{\rm phys}} + d_3 L^3_{M_{\rm phys}} + \cdots$

F = 90 MeV, $\mu = 1$ GeV

Numerical results

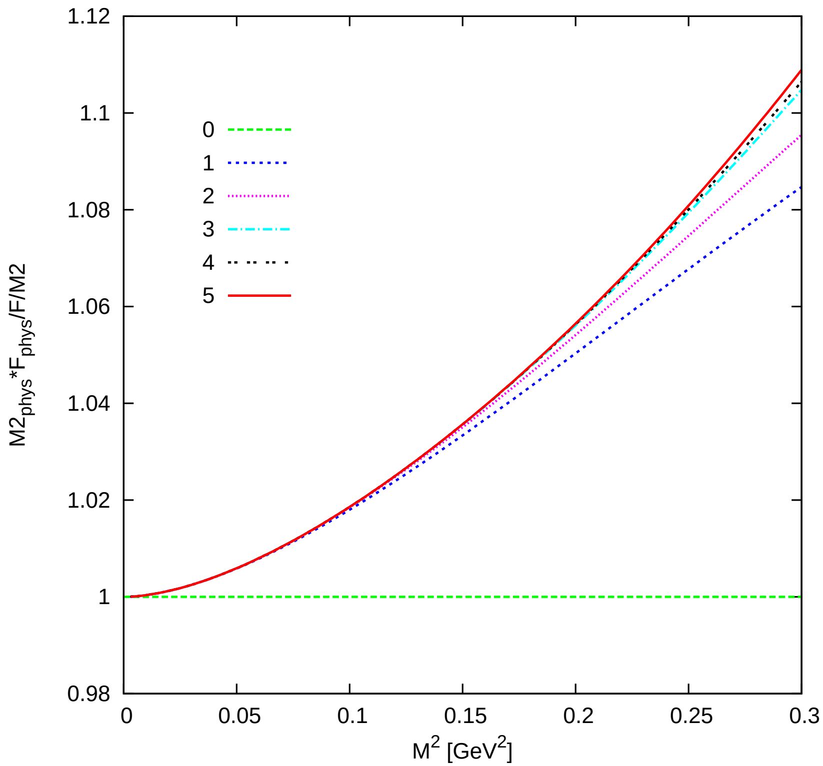


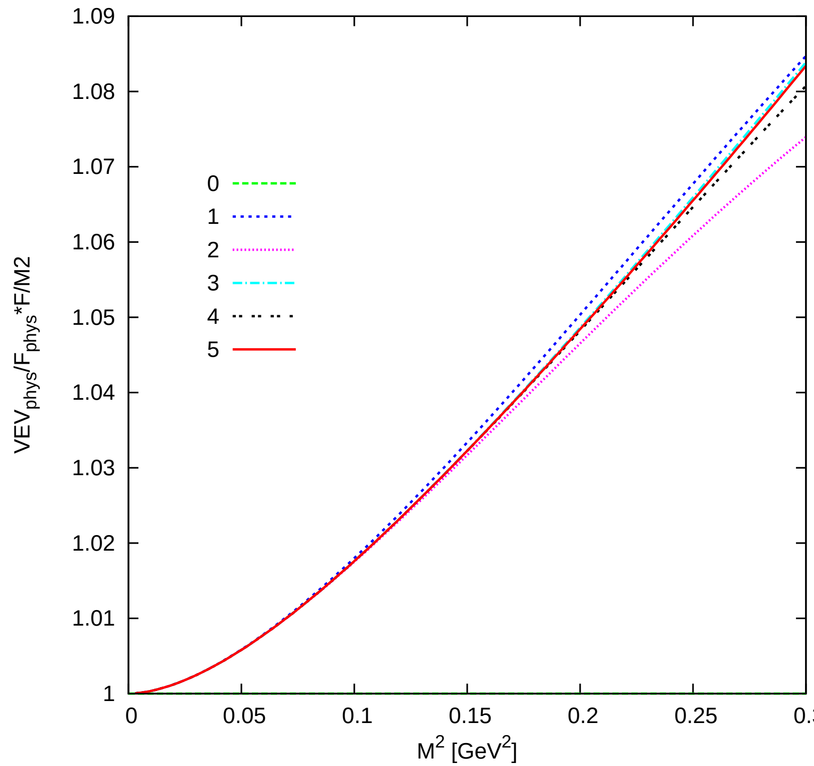
Left: $\frac{F_{\rm phys}}{F} = 1 + b_1 L_M + b_2 L_M^2 + b_3 L_M^3 + \cdots$

Right: $\frac{\langle \bar{q}q\rangle_{\rm phys}}{\langle \bar{q}q\rangle} = 1 + c_1 L_M + c_2 L_M^2 + c_3 L_M^3 + \cdots$

F = 90 MeV, $\mu = 1$ GeV

Numerical results





Left: $\frac{M^2_{\text{phys}}F_{\text{phys}}}{FM^2} = 1 + b_1 L_M + b_2 L_M^2 + b_3 L_M^3 + \cdots$

Right: $\frac{\langle\bar{q}q\rangle_{\text{phys}}/F_{\text{phys}}}{\langle\bar{q}q\rangle/F} = 1 + c_1 L_M + c_2 L_M^2 + c_3 L_M^3 + \cdots$

F = 90 MeV, $\mu = 1$ GeV,

large N cancels

Other results

- Bissegger, Fuhrer, hep-ph/0612096 Dispersive methods, massless Π_S to five loops
- Kivel, Polyakov, Vladimirov, 0809.3236, 0904.3008
  - In the massless case tadpoles vanish
  - hence the number of external legs needed does not grow
  - All 4-meson vertices via Legendre polynomials
  - can do divergence of all one-loop diagrams analytically
  - algebraic (but quadratic) recursion relations
  - massless $\pi\pi$, F_V and F_S to arbitrarily high order
  - large N agrees with Coleman, Wess, Zumino
  - large N is not a good approximation

Other results

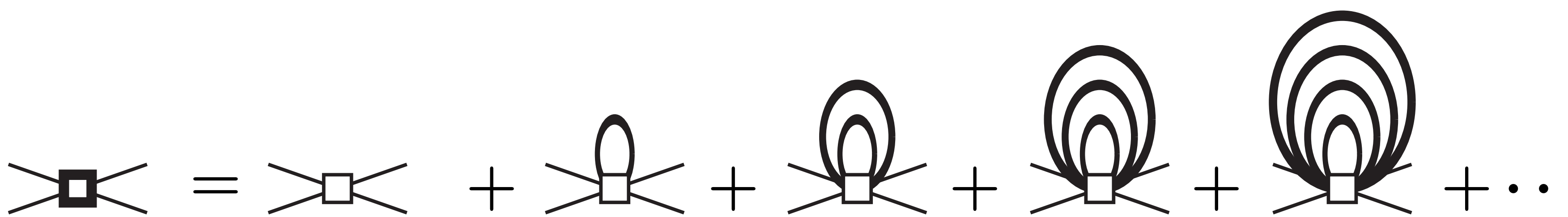
- JB,Carloni to be published
  - massive case: $\pi\pi$, F_V and F_S to 4-loop order
  - large N for these cases also for massive $O(N)$.
  - done using bubble resummations or recursion eqation which can be solved analytically (extension similar to gap equation)

Large N: $\pi\pi$-scattering

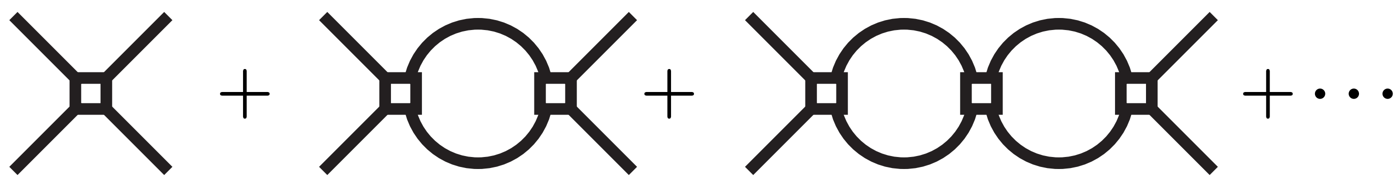
- Semiclassical methods Coleman, Jackiw, Politzer 1974
- Diagram resummation Dobado, Pelaez 1992
- $A(\phi^i\phi^j \to \phi^k\phi^l) =$
  $A(s,t,u)\delta^{ij}\delta^{kl} + A(t,u,s)\delta^{ik}\delta^{jl} + A(u,s,t)\delta^{il}\delta^{jk}$
- $A(s,t,u) = A(s,u,t)$
- Proof same as Weinberg's for $O(4)/O(3)$, group theory and crossing

Large N: $\pi\pi$-scattering

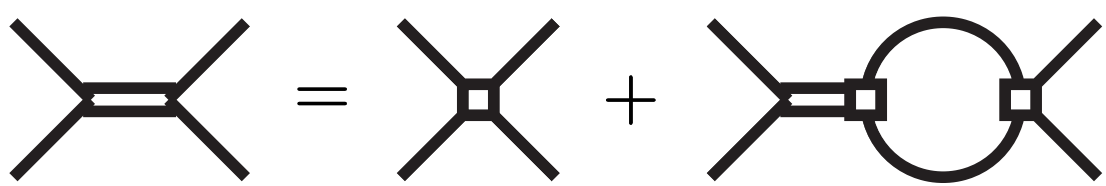
- Cactus diagrams for $A(s,t,u)$
- Branch with no momentum: resummed by —
- Branch starting at vertex: resum by



- The full result is then



- Can be summarized by a recursive equation



Large N: $\pi\pi$ scattering

$$y = \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})$$

$$A(s,t,u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{1}{2}\left(\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}\right)\overline{B}(M^2_{\text{phys}}, M^2_{\text{phys}}, s)}$$

or

$$A(s,t,u) = \frac{\frac{s-M^2_{\text{phys}}}{F_{\text{phys}}}}{1 - \frac{1}{2}\frac{s-M^2_{\text{phys}}}{F^2_{\text{phys}}}\overline{B}(M^2_{\text{phys}}, M^2_{\text{phys}}, s)}$$

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results

Conclusions

- Several quantities in massive $O(N)$ LL known to high loop order
- Large N in massive $O(N)$ model solved
- Had hoped: recognize the series also for general N
- Limited essentially by CPU time and size of intermediate files
- Some first studies on convergence etc.
- In progress: $\pi\pi$, F_V and F_S to four-loop order
- The technique can be generalized to other models/theories
  - $SU(N) \times SU(N)/SU(N)$
  - One nucleon sector