CHPT RESULTS FOR HVP AND A NEW EVALUATION OF THE PION LOOP CONTRIBUTION

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Towards high precision muon g-2/EDM measurement at J-PARC
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Why do we do this?

The muon $a_{\mu} = \frac{g-2}{2}$ will be measured more precisely.
The blobs are hadronic contributions
I will present some results on both HVP and HLbL
There are higher order contributions of both types
This talk

- HLbL (models)
  - Disconnected contributions are (expected to be) large
  - A new evaluation of the pion loop

- HVP (Chiral perturbation theory (ChPT))
  - An estimate of the disconnected and strange quark contributions
  - Finite volume corrections
    - J. Bijnens and J. Relefors, to be published

- All can be found in the PhD thesis of Johan Relefors (with some smaller changes)
To ChPT or not to ChPT

- ChPT = Effective field theory describing the lowest order pseudo-scalar representation
- or the (pseudo) Goldstone bosons from spontaneous breaking of chiral symmetry.
- Describes pions, kaons and etas at low-energies
- It’s an effective field theory: new parameters or LECs at each new order
- Recent review of LECs:
  

- $a_\mu$ is a very low-energy quantity, why not just calculate it in ChPT?
To ChPT or not to ChPT

- Fill the blobs with pions and kaons
- Lowest order for both HVP and HLbL: pure pion loop (or scalar QED): well defined answer
- NLO: the blob is nicely finite
  but not after the muon/photon integrations
- Needs a counterterm (NLO LEC) that is the muon $g - 2$
To ChPT or not to ChPT

- So need more than ChPT
- Experiment
- Dispersion relations
- lattice QCD
- Models
- ChPT can be used to put constraints, help understanding results and estimate not evaluated parts,...
HLbL: the main object to calculate

- Muon line and photons: well known
- The blob: **fill in with hadrons/QCD**
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks
A separation proposal: a start


- Use ChPT \( p \) counting and large \( N_c \)
  - \( p^4 \), order 1: pion-loop
  - \( p^8 \), order \( N_c \): quark-loop and heavier meson exchanges
  - \( p^6 \), order \( N_c \): pion exchange

Does not fully solve the problem
only short-distance part of quark-loop is really \( p^8 \)
but it’s a start
A separation proposal: a start


- Use ChPT $p$ counting and large $N_c$
  - $p^4$, order 1: pion-loop
  - $p^8$, order $N_c$: quark-loop and heavier meson exchanges
  - $p^6$, order $N_c$: pion exchange

Implemented by two groups in the 1990s:

- Hayakawa, Kinoshita, Sanda: meson models, pion loop using hidden local symmetry, quark-loop with VMD, calculation in Minkowski space (HKS)
- JB, Pallante, Prades: Try using as much as possible a consistent model-approach, ENJL, calculation in Euclidean space (BPP)
ChPT for HVP and a new evaluation of the pion loop contribution

Introduction

HLbL
Disconnected/connected π-loop

HVP

Conclusions

Papers: BPP and HKS

- **JB, E. Pallante and J. Prades**

- **Hayakawa, Kinoshita, (Sanda)**
Some main observations

- The largest contribution is $\pi^0$ (and $\eta, \eta'$) exchange
  - For the pseudo-scalar exchange most evaluations are in reasonable agreement
  - That will be used for an estimate of disconnected/connected
- The pion loop can be sizable but a large difference between the two evaluations
  - For the pure pion loop part, even larger numbers have been proposed by Engel, Ramsey-Musolf
  - Discussed in my second part
  - Another approach is the dispersive by Colangelo et al.
- There are other contributions but the sum is smaller than the leading pseudo-scalar exchange
- I interpret present HLbL lattice results as saying that no major contributions have been missed in the model estimates (but error and actual value might still change)
Disconnected/Connected

- Estimate the full result with pseudo-scalar exchange
- Connected diagrams only:
  - the gluon exchanges responsible for $U(1)_A$ breaking are not included at all
  - $\eta'$ becomes light, mainly $(\bar{u}u + \bar{d}d)/\sqrt{2} (\pi_\eta)$ and has the same mass as the pion
  - Or the two-light states are $\pi_u (\bar{u}u)$ and $\pi_d (\bar{d}d)$
  - $\eta$ becomes mainly $\bar{s}s$ and much heavier than the pion (and thus small contribution)
- Assume that couplings are not affected (not too bad experimentally)
Disconnected/Connected

- Two flavour case only: up and down quarks (three flavour not more difficult, just more numbers)
- Meson couplings to two-photons is via quark-loop
- Look at charge factors for Connected
  - As “quark-loop”: $q_u^4 + q_d^4 = \frac{17}{81}$
  - As $\pi_u, \pi_d$: $q_u^2 q_u^2 + q_d^2 q_d^2 = \frac{17}{81}$
  - As $\pi^0, \pi^\eta$: $\left(\frac{q_u^2 - q_d^2}{\sqrt{2}}\right)^2 + \left(\frac{q_u^2 + q_d^2}{\sqrt{2}}\right)^2 = \frac{9}{162} + \frac{25}{162} = \frac{17}{81}$
- Include $U(1)_A$ breaking: $\pi^\eta$ heavy
  - $\pi^0$: $\left(\frac{q_u^2 - q_d^2}{\sqrt{2}}\right)^2 = \frac{9}{162}$
Disconnected/Connected

- So in this limit:
  - Two-flavour case
  - $U(1)_A$ breaking makes $\pi\eta$ infinitely heavy
  - Full result dominated by pseudo-scalar exchange
  - $U(1)_A$ breaking does not affect couplings

  \[
  \begin{align*}
  \text{Connected:} & \quad \frac{34}{162} \\
  \text{Disconnected:} & \quad -\frac{25}{162} \\
  \text{Sum:} & \quad \frac{9}{162}
  \end{align*}
  \]

- All assumptions get corrections but final conclusion stays

  The disconnected contribution is expected to be large and of opposite sign with significant cancellations

- Argument used to go from large-$N_c$ to $\pi^0, \eta, \eta'$ in
  

Disconnected/Connected

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General properties

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) = \]

Actually we really need

\[ \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_3\lambda} \bigg|_{p_3=0} \]
General properties

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3): \]

- In general 138 Lorentz structures (but only 28 contribute to \(g-2\))
- Using \(q_\rho \Pi^{\rho\nu\alpha\beta} = p_{1\nu} \Pi^{\rho\nu\alpha\beta} = p_{2\alpha} \Pi^{\rho\nu\alpha\beta} = p_{3\beta} \Pi^{\rho\nu\alpha\beta} = 0\)
- 43 gauge invariant structures
- Bose symmetry relates some of them
- All depend on \(p_1^2, p_2^2\) and \(q^2\), but before derivative and \(p_3 \to 0\) also \(p_3^2, p_1 \cdot p_2, p_1 \cdot p_3\)
- Actually 2 less but singular basis Fischer et al.
- Compare HVP: one function, one variable
- General calculation from experiment: how difficult: Colangelo
- In four photon measurement: lepton contribution
General properties

\[ \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \] plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
- 5 remain: \( p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu \)
- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.
- More recent: can do two more using Gegenbauer techniques Knecht-Nyffeler, Jegerlehner-Nyffeler, JB–Zahiri-Abyaneh–Relefors
- \( P_1^2, P_2^2 \) and \( Q^2 \) remain
- study \( a^X_\mu = \int dl_{P_1} dl_{P_2} a^{XLL}_{\mu} = \int dl_{P_1} dl_{P_2} dl_{Q} a^{XLLQ}_{\mu} \)

\[ l_P = \ln \left( \frac{P}{\text{GeV}} \right) \], to see where the contributions are
- Study the dependence on the cut-off for the photons
General properties

\[ \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \]

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- \( P_1^2, P_2^2 \) and \( Q^2 \) remain
- study \( a_{\mu}^{X} = \int dl_{P_1} dl_{P_2} a_{\mu}^{XLL} = \int dl_{P_1} dl_{P_2} dl_{Q} a_{\mu}^{XLLQ} \)
  \( l_P = \ln \left( \frac{P}{\text{GeV}} \right) \), to see where the contributions are
- Study the dependence on the cut-off for the photons
A bare $\pi$-loop (sQED) give about $-4 \cdot 10^{-10}$

The $\pi\pi\gamma^*$ vertex is always done using VMD

$\pi\pi\gamma^*\gamma^*$ vertex two choices:
- Hidden local symmetry model: only one $\gamma$ has VMD
- Full VMD
- Both are chirally symmetric
- The HLS model used has problems with $\pi^+ - \pi^0$ mass difference (due to not having an $a_1$)

Final numbers quite different: $-0.45$ and $-1.9 \times 10^{-10}$

For BPP stopped at 1 GeV but within 10% of higher $\Lambda$
\( \pi \) loop: Bare vs VMD

- Plotted \( a^{LLQ}_\mu \) for \( P_1 = P_2 \)
- \( a_\mu = \int dl_{P_1} dl_{P_2} dl_Q a^{LLQ}_\mu \)
- \( l_Q = \log(Q/1 \text{ GeV}) \)
ChPT for HVP and a new evaluation of the pion loop contribution

Johan Bijnens

Introduction

HLbL

Disconnected/connected \(\pi\)-loop

HVP

Conclusions

\[ \pi \text{ loop: VMD vs HLS} \]

Usual HLS, \( a = 2 \)
π loop: VMD vs HLS

HLS with \( a = 1 \), satisfies more short-distance constraints
\( \pi \) loop

- \( \pi \pi \gamma^* \gamma^* \) for \( q_1^2 = q_2^2 \) has a short-distance constraint from the OPE as well.
- HLS does not satisfy it
- full VMD does: so probably better estimate
- Ramsey-Musolf suggested to do pure ChPT for the \( \pi \) loop
- Polarizability (\( L_9 + L_{10} \)) up to 10\%, charge radius 30\% at low energies, more at higher
- Both HLS and VMD have charge radius effect but not polarizability
\( \pi \) loop

- \( \pi\pi\gamma^*\gamma^* \) for \( q_1^2 = q_2^2 \) has a short-distance constraint from the OPE as well.
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- Ramsey-Musolf suggested to do pure ChPT for the \( \pi \) loop
- Polarizability \( (L_9 + L_{10}) \) up to 10\%, charge radius 30\% at low energies, more at higher
- Both HLS and VMD have charge radius effect but not polarizability
ChPT for muon $g - 2$ at order $p^6$ is not powercounting finite so no prediction for $a_\mu$ exists.

But can be used to study the low momentum end of the integral over $P_1, P_2, Q$

The four-photon amplitude is finite still at two-loop order (counterterms start at order $p^8$)

Add $L_9$ and $L_{10}$ vertices to the bare pion loop:

JB, Relefors, Zahiri-Abyaneh, 1208.3548, 1208.2554, 1308.2575, 1510.05796

π loop: VMD vs charge radius

low scale, charge radius effect well reproduced
\( \pi \) loop: VMD vs \( L_9 \) and \( L_{10} \)

- \( L_9 + L_{10} \neq 0 \) gives an enhancement of 10-15%
- To do it fully need to get a model: include \( a_1 \)
Include $a_1$

- $L_9 + L_{10}$ effect is from

- But to get gauge invariance correctly need
Include $a_1$

- Consistency problem: full $a_1$-loop?
- Treat $a_1$ and $\rho$ classical and $\pi$ quantum: there must be a $\pi$ that closes the loop
  - Argument: integrate out $\rho$ and $a_1$ classically, then do pion loops with the resulting Lagrangian
- To avoid problems: representation without $a_1-\pi$ mixing
- Check for curiosity what happens if we add $a_1$-loop
Include $a_1$

- Use antisymmetric vector representation for $a_1$ and $\rho$
- Fields $A_{\mu\nu}$, $V_{\mu\nu}$ (nonets)
- Kinetic terms:
  \[
  -\frac{1}{2} \left\langle \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{M^2_V}{2} V_{\mu\nu} V^{\mu\nu} \right\rangle 
  \]
  \[
  -\frac{1}{2} \left\langle \nabla^\lambda A_{\lambda\mu} \nabla_\nu A^{\nu\mu} - \frac{M^2_A}{2} A_{\mu\nu} A^{\mu\nu} \right\rangle 
  \]
- Terms that give contributions to the $L^r_i$:
  \[
  \frac{F_V}{2\sqrt{2}} \left\langle f_{+\mu\nu} V^{\mu\nu} \right\rangle + \frac{iG_V}{\sqrt{2}} \left\langle V^{\mu\nu} u_\mu u_\nu \right\rangle + \frac{F_A}{2\sqrt{2}} \left\langle f_{-\mu\nu} A^{\mu\nu} \right\rangle 
  \]
- \[
  L_9 = \frac{F_V G_V}{2M^2_V}, \quad L_{10} = -\frac{F_V^2}{4M^2_V} + \frac{F_A^2}{4M^2_A} 
  \]
- Weinberg sum rules: (Chiral limit)
  \[
  F_V^2 = F_A^2 + F^2_\pi \quad F_V M^2_V = F_A^2 M^2_A 
  \]
- VMD for $\pi\pi\gamma$:
  \[
  F_V G_V = F^2_\pi 
  \]
\( V_{\mu \nu} \) only

- \( \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) \) is not finite
  (but was also not finite for HLS)

- But \( \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \) also not finite
  (but was finite for HLS)

- Derivative one finite for \( G_V = F_V / 2 \)

- Surprise: \( g - 2 \) identical to HLS with \( a = \frac{F_V^2}{F_\pi^2} \)

- Yes I know, different representations are identical BUT
  they do differ in higher order terms and even in what is
  higher order

- Same comments as for HLS numerics
$V_{\mu\nu}$ only

- $\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)$ is not finite (but was also not finite for HLS)
- But $\frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}}|_{p_3=0}$ also not finite (but was finite for HLS)
- Derivative one finite for $G_V = F_V/2$
- Surprise: $g - 2$ identical to HLS with $a = \frac{F_V^2}{F_\pi^2}$
- Yes I know, different representations are identical BUT they do differ in higher order terms and even in what is higher order
- Same comments as for HLS numerics
$V_{\mu\nu}$ and $A_{\mu\nu}$

- Add $a_1$
- Calculate a lot

\[ \frac{\delta \Pi^{\mu\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \]

finite for:

- $G_V = F_V = 0$ and $F_A^2 = -2F_\pi^2$
- If adding full $a_1$-loop $G_V = F_V = 0$ and $F_A^2 = -F_\pi^2$
$V_{\mu\nu}$ and $A_{\mu\nu}$

- Start by adding $\rho a_1 \pi$ vertices
  
  $\lambda_1 \langle [V_{\mu\nu}, A_{\mu\nu}] \chi_- \rangle + \lambda_2 \langle [V_{\mu\nu}, A_{\nu\alpha}] h_{\mu} \rangle$
  
  $\lambda_3 \langle i [\nabla^\mu V_{\mu\nu}, A_{\nu\alpha}] u_\alpha \rangle + \lambda_4 \langle i [\nabla_\alpha V_{\mu\nu}, A_{\alpha\nu}] u^\mu \rangle$
  
  $\lambda_5 \langle i [\nabla^\alpha V_{\mu\nu}, A_{\mu\nu}] u_\alpha \rangle + \lambda_6 \langle i [V_{\mu\nu}, A_{\mu\nu}] f^-_{\alpha} \rangle$
  
  $\lambda_7 \langle iV_{\mu\nu} A^{\mu\rho} A^{\nu}_\rho \rangle$

- All lowest dimensional vertices of their respective type
- Not all independent, there are three relations
- Follow from the constraints on $V_{\mu\nu}$ and $A_{\mu\nu}$ (thanks to Stefan Leupold)
$V_{\mu\nu}$ and $A_{\mu\nu}$: big disappointment

- Work a whole lot
  \[
  \frac{\delta \Pi^{\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_3 \lambda} \bigg|_{p_3=0}
  \]
  not obviously finite

- Work a lot more
  \[
  \frac{\delta \Pi^{\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_3 \lambda} \bigg|_{p_3=0}
  \]
  finite, only same solutions as before

- Try the combination that show up in $g - 2$ only

- Work a lot

- Again, only same solutions as before

- Small loophole left: after the integration for $g - 2$ could be finite but many funny functions of $m_\pi, m_\mu, M_V$ and $M_A$ show up.
$a_1$-loop: cases with good $L_9$ and $L_{10}$

- Add $F_V$, $G_V$ and $F_A$
- Fix values by Weinberg sum rules and VMD in $\gamma^*\pi\pi$
- no $a_1$-loop
$a_1$-loop: cases with good $L_9$ and $L_{10}$

- Add $F_V$, $G_V$ and $F_A$
- Fix values by Weinberg sum rules and VMD in $\gamma^*\pi\pi$
- With $a_1$-loop (is different plot!!)
$a_1$-loop: cases with good $L_9$ and $L_{10}$

- Add $a_1$ with $F_A^2 = +F_\pi^2$
- Add the full VMD as done earlier for the bare pion loop
Integration results

\[ a_1 F_A^2 = -2F^2 \]

\[ a_1 F_A^2 = -1 \text{ a}_1\text{-loop} \]

HLS

HLS a=1

VMD

a_1 VMD

a_1 Weinberg

\[ P_1, P_2, Q \leq \Lambda \]
Integration results with $a_1$

- Problem: get high energy behaviour good enough
- But all models with reasonable $L_9$ and $L_{10}$ fall way inside the error quoted earlier ($-1.9 \pm 1.3 \times 10^{-10}$)
- Tentative conclusion: Use hadrons only below about 1 GeV: $a_\mu^{\pi-\text{loop}} = (-2.0 \pm 0.5) \times 10^{-10}$
- Note that Engel and Ramsey-Musolf, arXiv:1309.2225 is a bit more pessimistic quoting numbers from ($-1.1$ to $-7.1$) $10^{-10}$
Two-point: Why

Muon: \( a_\mu = (g - 2)/2 \) and \( a_{\mu,\text{LO},\text{HVP}} = \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2) \)

plot: \( f(Q^2) \hat{\Pi}(Q^2) \) with \( Q^2 = -q^2 \) in GeV^2

Figure and data: Aubin, Blum, Chau, Golterman, Peris, Tu, Phys. Rev. D93 (2016) 054508 [arXiv:1512.07555]
Two-point: Connected versus disconnected

- $\Pi_{a b}^{\mu \nu}(q) \equiv i \int d^4 x e^{i q \cdot x} \langle T(j^\mu_a(x) j^{\nu \dagger}_a(0)) \rangle$
- $j_{\pi^+}^\mu = \bar{d} \gamma^\mu u$
- $j_u^\mu = \bar{u} \gamma^\mu u$, $j_d^\mu = \bar{d} \gamma^\mu d$, $j_s^\mu = \bar{s} \gamma^\mu s$
- $j_e^\mu = \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d - \frac{1}{3} \bar{s} \gamma^\mu s$
- Study in ChPT at one-loop:
- Extend to two-loops and some more:
Two-point: Connected versus disconnected

- Include also singlet part of the vector current
- There are new terms in the Lagrangian
- \( p^4 \) only one more: \( \langle L_{\mu\nu} \rangle \langle L^{\mu\nu} \rangle + \langle R_{\mu\nu} \rangle \langle R^{\mu\nu} \rangle \)
- \( \Rightarrow \) The pure singlet vector current does not couple to mesons until \( p^6 \)
- \( \Rightarrow \) Loop diagrams involving the pure singlet vector current only appear at \( p^8 \) (implies relations)
- \( \Rightarrow \) Loop diagrams with singlet vector and WZW only at \( p^{10} \)
- \( p^6 \) (no full classification, just some examples)
  \( \langle D_\rho L_{\mu\nu} \rangle \langle D^\rho L^{\mu\nu} \rangle + \langle D_\rho R_{\mu\nu} \rangle \langle D^\rho R^{\mu\nu} \rangle, \)
  \( \langle L_{\mu\nu} \rangle \langle L^{\mu\nu} \chi^\dagger U \rangle + \langle R_{\mu\nu} \rangle \langle R^{\mu\nu} \chi U^\dagger \rangle, \ldots \)
- Results at two-loop order, unquenched isospin limit
Two-point: Connected versus disconnected

- $\Pi^{\mu\nu}_{\pi^+\pi^+}$: only connected
- $\Pi^{\mu\nu}_{ud}$: only disconnected
- $\Pi^{\mu\nu}_{uu} = \Pi^{\mu\nu}_{\pi^+\pi^+} + \Pi^{\mu\nu}_{ud}$
- $\Pi^{\mu\nu}_{ee} = \frac{5}{9}\Pi^{\mu\nu}_{\pi^+\pi^+} + \frac{1}{9}\Pi^{\mu\nu}_{ud}$
- Infinite volume (and the $ab$ considered here):
  $\Pi^{\mu\nu}_{ab} = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}_{ab}$

Large $N_c$ + VMD estimate: $\Pi^{(1)}_{\pi^+\pi^+} = \frac{4F^2_{\pi}}{M^2_V - q^2}$

Plots on next pages are for $\Pi^{(1)}_{ab0}(q^2) = \Pi^{(1)}_{ab}(q^2) - \Pi^{(1)}_{ab}(0)$

At $\rho^4$ the extra LEC cancels, at $\rho^6$ there are new LEC contributions, but no new ones in the loop parts
Two-point: Connected versus disconnected

- Connected
- $p^6$ is large
- Due to the $L_i^r$ loops

\[ \Pi_{\pi^+}^\pm \]

$q^2 [\text{GeV}^2]$
Two-point: Connected versus disconnected

-Disconnected
- $p^6$ is large
- Due to the $L^r_i$ loops
- About $-\frac{1}{2}$ connected
- $-\frac{1}{10}$ is from

\[
\Pi_{ee}^{(1)} = \frac{5}{9} \Pi^{(1)}_{\pi^+\pi^+} + \frac{1}{9} \Pi^{(1)}_{ud}
\]
Two-point: Connected versus disconnected

- $p^4$ and $p^6$ pion part exactly $-\frac{1}{2}$
- not true for unsubtracted at $p^4$(LEC)
- not true for pure LEC at $p^6$
Two-point: Including strange

- $\pi$ connected $u,d$
- $ud$ disconnected $u,d$
- $ss$ strange current
- $us$ mixed strange-$u,d$

- Strange part is very small:
  - $q^2 = 0$ subtraction (only kaon loops)
  - $p^4$ and $p^6$ cancel largely
Two-point: with strange, electromagnetic current

- $\pi$
  - connected $u,d$
- $ud$
  - disconnected $u,d$
- $ss$
  - strange current
- $us$
  - mixed $s–u,d$
- new $p^6$ LEC cancels
Comparison with other results

- Time moments, Taylor expansion
- \( \hat{\Pi}(q)^2 = \Pi_1 q^2 - \Pi_2 q^4 + \cdots \)
- Lattice data from Miura presentation Lattice2106 and HPQCD
- Phenomenological HLS fit (Benayoun et al)
Comparing with other results

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\Pi_A$</th>
<th>$\Pi_1$ (GeV$^{-2}$)</th>
<th>$\Pi_2$ (GeV$^{-4}$)</th>
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<tbody>
<tr>
<td>$\Pi^{VMD}$</td>
<td>$\hat{\Pi}_{\pi^+}$</td>
<td>0.0967</td>
<td>-0.163</td>
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<td>$p^4$</td>
<td>$\hat{\Pi}_{\pi^+}$</td>
<td>0.0240</td>
<td>-0.091</td>
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<td>$p^6 R$</td>
<td>$\hat{\Pi}_{\pi^+}$</td>
<td>0.0031</td>
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<td>$\hat{\Pi}_{\pi^+}$</td>
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<tr>
<td>sum</td>
<td>$\hat{\Pi}_{\pi^+}$</td>
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<td>-0.336</td>
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<td>Miura Lattice2016</td>
<td>$\hat{\Pi}_{\pi^+}$</td>
<td>0.1657(16)(18)</td>
<td>-0.297(10)(05)</td>
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<tr>
<td>HPQCD</td>
<td>$\hat{\Pi}_{\pi^+}$</td>
<td>0.1460(22)</td>
<td>-0.2228(65)</td>
</tr>
<tr>
<td>$p^4$</td>
<td>$\hat{\Pi}_{UD}$</td>
<td>-0.0116</td>
<td>0.045</td>
</tr>
<tr>
<td>$p^6 R$</td>
<td>$\hat{\Pi}_{UD}$</td>
<td>-0.0015</td>
<td>0.007</td>
</tr>
<tr>
<td>$p^6 L$</td>
<td>$\hat{\Pi}_{UD}$</td>
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<td>0.032</td>
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<tr>
<td>sum</td>
<td>$\hat{\Pi}_{UD}$</td>
<td>-0.0278</td>
<td>0.085</td>
</tr>
<tr>
<td>Miura Lattice2016</td>
<td>$\hat{\Pi}_{UD}$</td>
<td>-0.015(2)(1)</td>
<td>0.046(10)(04)</td>
</tr>
</tbody>
</table>

Connected and Disconnected:
size difference (-few%) understood from ChPT and VMD
Comparing with other results

<table>
<thead>
<tr>
<th>Reference</th>
<th>(\hat{\Pi}_S)</th>
<th>(\Pi_1) (GeV(^{-2}))</th>
<th>(\Pi_2) (GeV(^{-4}))</th>
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<tbody>
<tr>
<td>(\Pi^{VMD}_S\phi)</td>
<td>(0.0017)</td>
<td>(-0.005)</td>
<td></td>
</tr>
<tr>
<td>(\Pi^{p^4}_S)</td>
<td>(0.0000)</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td>(\Pi^{p^6R}_S)</td>
<td>(-0.0013)</td>
<td>(-0.005)</td>
<td></td>
</tr>
<tr>
<td>(\Pi^{p^6L}_S)</td>
<td>(0.0318)</td>
<td>(-0.035)</td>
<td></td>
</tr>
<tr>
<td>Miura Lattice2016</td>
<td>(0.0657(1)(2))</td>
<td>(-0.0532(1)(3))</td>
<td></td>
</tr>
<tr>
<td>HPQCD</td>
<td>(0.06625(74))</td>
<td>(-0.0526(11))</td>
<td></td>
</tr>
<tr>
<td>our result</td>
<td>(0.0852)</td>
<td>(-0.182)</td>
<td></td>
</tr>
<tr>
<td>Benayoun et al</td>
<td>(0.0990(7))</td>
<td>(-0.206(2))</td>
<td></td>
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<tr>
<td>Miura Lattice2016</td>
<td>(0.0972(2)(1))</td>
<td>(-0.166(6)(3))</td>
<td></td>
</tr>
</tbody>
</table>

Strange and Total electromagnetic
Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i / L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta^i_q}q(x^i)$
- Then momenta are $p^i = \theta^i / L + 2\pi n^i / L$. Allows to map out momentum space on the lattice much better

Bedaque, ...

- Small note:
  - Beware what people call momentum: is $\theta^i / L$ included or not?
  - Reason: a colour singlet gauge transformation $G^S_{\mu} \to G^S_{\mu} - \partial_\mu \epsilon(x)$, $q(x) \to e^{i\epsilon(x)}q(x)$, $\epsilon(x) = -\theta^i_q x^i / L$
  - Boundary condition
    Twisted $\Leftrightarrow$ constant background field + periodic
Twisted boundary conditions: Drawbacks

Drawbacks:
- Box: Rotation invariance $\rightarrow$ cubic invariance
- Twisting: reduces symmetry further
- Can only be done for the connected part

Consequences:
- $m^2(\bar{p}^2) = E^2 - \bar{p}^2$ is not constant
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum
Two-point function: twisted boundary conditions


\[ \int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0 \]

\[ \langle \bar{u} \gamma^\mu u \rangle \neq 0 \]

\[ j_\pi^\mu = \bar{d} \gamma^\mu u \]

satisfies \( \partial_\mu \langle T(j_\pi^\mu (x)j_\pi^{\nu \dagger} (0)) \rangle = \delta^4(x) \langle \bar{d} \gamma^\nu d - \bar{u} \gamma^\nu u \rangle \)

\[ \Pi_{\sigma}^{\mu \nu}(q) \equiv i \int d^4x e^{iq \cdot x} \langle T(j_\sigma^\mu (x)j_\sigma^{\nu \dagger} (0)) \rangle \]

Satisfies WT identity. \( q_\mu \Pi_{\sigma}^{\mu \nu} = \langle \bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d \rangle \)

- ChPT at one-loop satisfies this

- two-loop in partially quenched: JB, Relefors, in preparation
  satisfies the WT identity (as it should)
\[ \langle \bar{u} \gamma^\mu u \rangle \]

\[ \langle -u \gamma^\mu u \rangle \]

\[ \langle -u \gamma^\mu u \rangle \text{ twisted} \]

\[ \langle -u \gamma^\mu u \rangle \text{ partially twisted} \]

\[ \theta_u = (0, \theta_u, 0, 0), \text{ all others untwisted} \]

\[ m_\pi L = 4 \]

\[ (\text{ratio at } p^4 = 2 \text{ up to kaon loops}) \]

For comparison:

\[ \langle \bar{u}u \rangle^V \approx -2.4 \times 10^{-5} \text{ GeV}^3 \]

\[ \langle \bar{u}u \rangle \approx -1.2 \times 10^{-2} \text{ GeV}^3 \]
Two-point: partially twisted, one-loop

\[ q = \left(0, \sqrt{-q^2}, 0, 0\right) \]

\[ \Pi^{22} = \Pi^{33} \]

\[ \vec{\theta}_u = L \vec{q} \]

\[ m_{\pi 0} L = 4 \]

\[ m_{\pi 0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi^{(1)}_{\text{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5e-3 \cdot \frac{q^2}{0.1} \]

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Correction is at the % level
Two-point: partially twisted, with two-loop

\[ q = \left( 0, \sqrt{-q^2}, 0, 0 \right) \]

\[ \Pi^{22} = \Pi^{33} \]

\[ \vec{\theta}_u = L q \]

\[ m_{\pi 0} L = 4 \]

\[ m_{\pi 0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi_{VMD}^{(1)} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5 \times 10^{-3} \cdot \frac{q^2}{0.1} \]

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Correction from two loop is reasonable (thin lines are \( p^4 \))
Two-point: partially twisted, one-loop

\[ q = \left( 0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0 \right) \]

\[ \Pi^{11} = \Pi^{22} \]

\[ \vec{\theta}_u = L q \]

\[ m_{\pi^0}L = 4 \]

\[ m_{\pi^0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi^{(1)}_{\text{VMD}} = \frac{-4q^2 F^2_\pi}{M^2_V - q^2} \approx 5 \times 10^{-3} \cdot \frac{q^2}{0.1} \]

Correction is at the % level

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)
Two-point: partially twisted, two-loop

\[ q = \left( 0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0 \right) \]

\[ \Pi_{11} = \Pi_{22} \]
\[ \vec{\theta}_u = L q \]
\[ m_{\pi_0} L = 4 \]
\[ m_{\pi_0} = 0.135 \text{ GeV} \]
\[ -q^2 \Pi_{VMD}^{(1)} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5e-3 \cdot \frac{q^2}{0.1} \]

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Two loop correction again reasonable (thin lines are \( p^4 \))
Two-point: untwisted, connected part two-loop

<table>
<thead>
<tr>
<th>(q/(2\pi/L))</th>
<th>(\Pi_{U}^{00}) [10(^{-5}) GeV(^2)]</th>
<th>(\Pi_{U}^{11}) [10(^{-5}) GeV(^2)]</th>
<th>(\Pi_{U}^{33}) [10(^{-5}) GeV(^2)]</th>
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</thead>
<tbody>
<tr>
<td>(0,0,0,0)</td>
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<td></td>
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<tr>
<td>(p^4)</td>
<td>0.000</td>
<td>-8.785</td>
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<tr>
<td>(p^6 R)</td>
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<tr>
<td>(p^6 L)</td>
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<td>(0,1,0,0)</td>
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<tr>
<td>(p^4)</td>
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<td>(p^6 R)</td>
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<td>(p^4)</td>
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<td>(p^6 L)</td>
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<tr>
<td>sum</td>
<td>2.951</td>
<td>-1.911</td>
<td>-4.960</td>
</tr>
</tbody>
</table>

- \(q^2 = 0, 0.045, 0.09\) GeV\(^2\)
- Disconnected = \(-1/2\) shown (with very small differences)
Conclusions

Showed you results for:

- **Hadronic Light by Light**
  - Expect large and opposite sign disconnected contributions
  - A new evaluation of the pion loop including effect of polarizability

- **Hadronic vacuum polarization: vector two-point function**
  - Connected versus disconnected at two-loops
  - Connected: twisting and finite volume at two-loops
    - The WI are satisfied exactly
    - The corrections are sizable for present lattices (few%) but two-loop corrections were normal size
    - **Be careful: ChPT must exactly correspond to your lattice calculation**

- Our results can help with chiral extrapolation
- Programs will be available via CHIRON
  http://www.thep.lu.se/~bijnens/chiron/
  sometime later this year (manual writing takes time)