Loops and Volumes in Chiral Perturbation Theory

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Loops and Volumes in ChPT
Johan Bijnens
Overview

Chiral Perturbation Theory: A selective history

Goldberger-Treiman relation:

PCAC:

Nambu-Goldstone bosons:
Chiral Perturbation Theory: A selective history

- **Current algebra:**

- **Many applications:**
Chiral Perturbation Theory: A selective history

**Effective Lagrangians:**

**Chiral logarithms and loops:**
- G. Ecker and J. Honerkamp, “Pion pion scattering from an su(3) x su(3) invariant lagrangian,” NPB **62** (1973) 509.
Chiral Perturbation Theory: A selective history

- 5931 citations to at least one of the above
- 184 due to Ulf (out of more than 700 inspire items)
- Some others: Oset (135), Bijnens (134), Pelaez (104), Pich (101), Scherer (87), Leutwyler (79), Ecker (74), Gasser (66), Donoghue (60), Holstein (59), V.Bernard (54), G.Colangelo (52),…
  (done by checking some names)
- Now sufficiently well known to no longer be cited
Chiral Perturbation Theory: A selective history

Early followers (i.e. $p^4$ and loop calculations, selection):

- G. D’Ambrosio and D. Espriu, “Rare Decay Modes of the K Mesons in the Chiral Lagrangian,” PLB 175 (1986) 237.
- J. L. Goity, “The Decays $K^0(s) \to \gamma\gamma$ and $K^0(\ell) \to \gamma\gamma$ in the Chiral Approach,” Z. Phys. C 34 (1987) 341.
- J. Donoghue, B. Holstein and Y. C. Lin, “The Reaction $\gamma\gamma \to \pi^0\pi^0$ and Chiral Loops,” PRD 37 (1988) 2423.
Chiral Perturbation Theory: A selective history

And when did Ulf start (first citation and first ChPT papers)

Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
*On The Foundations Of Chiral Perturbation Theory*,

For references to lectures see:
http://www.thep.lu.se/~bijnens/chpt/
Chiral Perturbation Theory

A general Effective Field Theory:
- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:
- **Degrees of freedom**: Goldstone Bosons from spontaneous breaking of chiral symmetry
- **Powercounting**: Dimensional counting in momenta/masses
- **Breakdown scale**: Resonances, so about $M_\rho$. 
Chiral Perturbation Theory

A general Effective Field Theory:
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Goldstone Bosons

Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\implies$ 8 massless degrees of freedom and interaction vanishes at zero momentum
Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg powercounting

\[ \int d^4 p \]

\[ p^2 \]

\[ 1/p^2 \]

\[ p^4 \]

\[ \frac{(p^2)^2 (1/p^2)^2}{p^4} = p^4 \]

\[ (p^2)(1/p^2)p^4 = p^4 \]
Chiral Perturbation Theories

- Which chiral symmetry: \( SU(N_f)_L \times SU(N_f)_R \), for \( N_f = 2, 3, \ldots \) and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite \( T \)
- Which interactions to include beyond the strong one
- Which particles included as non Goldstone Bosons
- Ulf has done a lot in applying it to new areas
Lagrangians: Lagrangian structure (mesons, strong)

<table>
<thead>
<tr>
<th></th>
<th>2 flavour</th>
<th>3 flavour</th>
<th>PQChPT/N_f flavour</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p^2 )</td>
<td>( F, B ) 2</td>
<td>( F_0, B_0 ) 2</td>
<td>( F_0, B_0 ) 2</td>
</tr>
<tr>
<td>( p^4 )</td>
<td>( l_i^r, h_i^r ) 7+3</td>
<td>( L_i^r, H_i^r ) 10+2</td>
<td>( \hat{L}_i^r, \hat{H}_i^r ) 11+2</td>
</tr>
<tr>
<td>( p^6 )</td>
<td>( c_i^r ) 51+4</td>
<td>( C_i^r ) 90+4</td>
<td>( K_i^r ) 112+3</td>
</tr>
</tbody>
</table>

\( p^2 \): Weinberg 1966
\( p^4 \): Gasser, Leutwyler 84,85
\( p^6 \): JB, Colangelo, Ecker 99,00; Weber 2008

\( L_i \): LEC = Low Energy Constants = ChPT parameters
\( H_i \): contact terms: value depends on definition of currents/densities
Finite volume: no new LECs
Other effects: (many) new LECs
Mesons: which Lagrangians are known ($n_f = 3$)

<table>
<thead>
<tr>
<th>Loops</th>
<th>$L_{\text{order}}$</th>
<th>LECs</th>
<th>effects included</th>
</tr>
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<tr>
<td>$L = 0$</td>
<td>$L_p^2$</td>
<td>2</td>
<td>strong (+ external $W, \gamma$)</td>
</tr>
<tr>
<td></td>
<td>$L_{e^2p^0}$</td>
<td>1</td>
<td>internal $\gamma$</td>
</tr>
<tr>
<td></td>
<td>$L_{\Delta S=1}^{G_F p^2}$</td>
<td>2</td>
<td>nonleptonic weak</td>
</tr>
<tr>
<td></td>
<td>$L_{\Delta S=1}^{G_8 e^2 p^0}$</td>
<td>1</td>
<td>nonleptonic weak + internal $\gamma$</td>
</tr>
<tr>
<td></td>
<td>$L_{\text{odd} p^4}$</td>
<td>0</td>
<td>WZW, anomaly</td>
</tr>
<tr>
<td>$L \leq 1$</td>
<td>$L_p^4$</td>
<td>10</td>
<td>strong (+ external $W, \gamma$)</td>
</tr>
<tr>
<td></td>
<td>$L_{e^2p^2}$</td>
<td>13</td>
<td>internal $\gamma$</td>
</tr>
<tr>
<td></td>
<td>$L_{\Delta S=1}^{G_8 F p^4}$</td>
<td>22</td>
<td>nonleptonic weak</td>
</tr>
<tr>
<td></td>
<td>$L_{\Delta S=1}^{G_{27} p^4}$</td>
<td>28</td>
<td>nonleptonic weak</td>
</tr>
<tr>
<td></td>
<td>$L_{\Delta S=1}^{G_8 e^2 p^2}$</td>
<td>14</td>
<td>nonleptonic weak + internal $\gamma$</td>
</tr>
<tr>
<td></td>
<td>$L_{\text{odd} p^6}$</td>
<td>23</td>
<td>WZW, anomaly</td>
</tr>
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<td></td>
<td>$L_{\text{leptons} e^2 p^2}$</td>
<td>5</td>
<td>leptons, internal $\gamma$</td>
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<tr>
<td>$L \leq 2$</td>
<td>$L_p^6$</td>
<td>90</td>
<td>strong (+ external $W, \gamma$)</td>
</tr>
</tbody>
</table>
Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars/axial currents
- Chiral logarithms
- includes Isospin and the eightfold way ($SU(3)_V$)
- Unitarity included perturbatively

$$m_{\pi}^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2} \log \frac{(2B\hat{m})}{\mu^2} + 2I_3^r(\mu)\right] + \cdots$$

$$M^2 = 2B\hat{m}$$
LEC$	ext{s}$ and $\mu$

$$l_3^r(\mu)$$

$$\bar{l}_i = \frac{32 \pi^2}{\gamma_i} l_i^r(\mu) - \log \frac{M_{\pi}^2}{\mu^2}.$$ 

is independent of the scale $\mu$.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

Choice of $\mu$:

- $m_\pi, m_K$: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$
  what about large $N_c$ arguments???
- compromise: $\mu = m_\rho = 0.77$ GeV
Program availability

Making the programs more accessible for others to use:

- Two-loop results have very long expressions
- Many not published but available from http://www.thep.lu.se/~bijnens/chpt/
- Many programs available on request from the authors
- Idea: make a more general framework
- CHIRON:

  JB,
  “CHIRON: a package for ChPT numerical results at two loops,”
  http://www.thep.lu.se/~bijnens/chiron/
Loops and Volumes in ChPT

Overview

ChPT
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Three flavour: more masses
Conclusions

Program availability: CHIRON

- Present version: 0.54
- Classes to deal with $L_i$, $C_i$, $L_i^{(n)}$, $K_i$, standardized in/output, changing the scale,
- Loop integrals: one-loop and sunset integrals
- Included so far (at two-loop order):
  - Masses, decay constants and $\langle \bar{q}q \rangle$ for the three flavour case
  - Masses and decay constants at finite volume in the three flavour case
  - Masses and decay constants in the partially quenched case for three sea quarks
  - Masses and decay constants in the partially quenched case for three sea quarks at finite volume
- A large number of example programs is included
- Manual has already reached 94 pages
- I am continually adding results from my earlier work

Using dispersive methods to determine the nonanalytic dependence on $q^2$ in the pion vector and scalar form-factor for two-flavours


Pion form-factor: quality of fit

```plaintext
|t [GeV^2]| |F_{\pi}V|^2 |
|---|---|---|
|p^4 | |  |
|p^6 | |  |
|p^8 | |  |
|data | |  |
```

The graph shows the fit quality of the pion form-factor $|F_{\pi}V|^2$ as a function of $t [GeV^2]$. The full result is compared to data points, with four-loop (p^4) and six-loop (p^6) calculations also included. The graph indicates a good fit quality, with the data points closely following the curve.

**References:**

- *ChPT*
- **Overview**
- Two or more loops
- pion form-factor
- pi polarizability
- Beyond QCD or BSM
- Leading logarithms
- Three flavour: more masses
- Conclusions
An example where ChPT triumphed


- Expand $\gamma \pi^\pm \rightarrow \gamma \pi^\pm$ near threshold: $(z_\pm = 1 \pm \cos \theta_{cm})$

$$
\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega_{Born}} - \frac{\alpha m_\pi^3 ((s - m_\pi^2)^2}{4s^2 (sz_+ + m_\pi^2 z_-)} \left(z_+^2 (\alpha - \beta) + \frac{s^2}{m_\pi^4} z_-^2 (\alpha + \beta) \right)
$$

- Three ways to measure: (all assume $\alpha + \beta = 0$)
  - $\pi\gamma \rightarrow \pi\gamma$ (Primakoff, high energy pion beam)
    Dubna (1985) $\alpha = (6.8 \pm 1.4) \times 10^{-4}$ fm$^3$
    Compass (CERN, 2015) $\alpha = (2.0 \pm 0.6 \pm 0.7) \times 10^{-4}$ fm$^3$
  - $\gamma\pi \rightarrow \pi\gamma$ (via one-pion exchange)
    Lebedev (1986) $\alpha = (20 \pm 12) \times 10^{-4}$ fm$^3$
    Mainz (2005) $\alpha = (5.8 \pm 0.75 \pm 1.5 \pm 0.25) \times 10^{-4}$ fm$^3$
  - $\gamma\gamma \rightarrow \pi\pi$ (in $e^+e^- \rightarrow e^+e^-\pi^+\pi^-$)
    MarkII data analyzed (1992) $\alpha = (2.2 \pm 1.1) \times 10^{-4}$ fm$^3$

- Extrapolation and subtraction: difficult experiments
Polarizabilities: extrapolations needed

\[ \gamma + \gamma \rightarrow \pi^+ + \pi^- \]

\[ \sigma_{tot}(|\cos(\theta_{\pi\pi})| < 0.6) \text{ (nb)} \]

from Pasquini et al. 2008

\[ W_{\pi\pi} \text{ (GeV)} \]

\[ \gamma p \rightarrow \pi\gamma n \]

Off-shell \( \pi \)

\[ \pi N \rightarrow \pi\gamma N \]

Off-shell \( \gamma \)

Conclusions
Charged pion polarizabilities: theory

- **ChPT:**
    \[ \alpha + \beta = 0, \quad \alpha = (2.8 \pm 0.2) \times 10^{-4} \text{ fm}^3 \]
    Input \( \pi \to e\nu\gamma \) (error only from this)
  - Two-loop: Bürgi, 1996, Gasser, Ivanov, Sainio 2006
    \[ \alpha + \beta = 0.16 \times 10^{-4} \text{ fm}^3, \quad \alpha = (2.8 \pm 0.5) \times 10^{-4} \text{ fm}^3 \]

- **Dispersive analysis from \( \gamma\gamma \to \pi\pi \):**
  - Fil’kov-Kashevarov, 2005
    \[ (\alpha_1 - \beta_1) = (13.0^{+2.6}_{-1.9}) \times 10^{-4} \text{ fm}^3 \]
  - Critized by Pasquini-Drechsel-Scherer, 2008
    “Large model dependence in their extraction”
    “Our calculations... are in reasonable agreement with ChPT for charged pions”
    \[ (\alpha_1 - \beta_1) = (5.7) \times 10^{-4} \text{ fm}^3 \] perfectly possible
QCD-like and/or technicolor theories

- One can also have different symmetry breaking patterns from underlying fermions
- Three generic cases
  - $SU(N) \times SU(N)/SU(N)$
  - $SU(2N)/SO(2N)$ (Dirac) or $SU(N)/SO(N)$ (Majorana)
  - $SU(2N)/Sp(2N)$
- Many one-loop results existed especially for the first case (several discovered only after we published our work)
- Equal mass case pushed to two loops JB, Lu, 2009-11
- Majorana, Finite Volume and partially quenched added
  
  JB, Rössler, arXiv:1509.04082
$N_F$ fermions in a representation of the gauge group

- complex (QCD):
  - $q^T = (q_1 \ q_2 \ldots\ q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$
  - $q_L \rightarrow g_L q_L$ and $g_R \rightarrow g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \bar{q}_j q_i \rangle \propto \delta_{ij}$
  - $g_L = g_R$ then $\Sigma_{ij} \rightarrow \Sigma_{ij} \implies$ conserved $H = SU(N_F)_V$:
- Real (e.g. adjoint): $\hat{q}^T = (q_{R1} \ldots\ q_{RN_F} \ \tilde{q}_{R1} \ldots\ \tilde{q}_{RN_F})$
  - $\tilde{q}_{Ri} \equiv C\tilde{q}_{Li}$ goes under gauge group as $q_{Ri}$
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \rightarrow g\hat{q}$
  - $\langle \bar{q}_j q_i \rangle$ is really $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto J_{Sij}$ $J_S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$
  - Conserved if $gJ_S g^T = J_S \implies H = SO(2N_F)$
- Real with $N_F$ Majorana fermions
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \rightarrow g\hat{q}$
  - Majorana condensate is $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto \delta_{ij} = I_{ij}$
  - Conserved $gI g^T = I$
$N_F$ fermions in a representation of the gauge group

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  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \rightarrow g \hat{q}$
  - Majorana condensate is $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto \delta_{ij} = l_{ij}$
  - Conserved $glg^T = I$
$N_F$ fermions in a representation of the gauge group

- **complex (QCD):** $q^T = (q_1 \; q_2 \ldots \; q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$ \ $q_L \rightarrow g_L q_L$ and $g_R \rightarrow g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \bar{q}_j q_i \rangle \propto \delta_{ij}$
  - Conserved $H = SU(N_F)_V$: $g_L = g_R$ then $\Sigma_{ij} \rightarrow \Sigma_{ij}$

- **Pseudoreal (e.g. two-colours):**
  $\hat{q}^T = (q_{R1} \; \ldots \; q_{RN_F} \; \tilde{q}_{R1} \; \ldots \; \tilde{q}_{RN_F})$
  - $\tilde{q}_{R\alpha i} \equiv \epsilon_{\alpha\beta} C \tilde{q}_{L\beta i}$ goes under gauge group as $q_{R\alpha i}$
  - Some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \rightarrow g \hat{q}$
  - $\langle \bar{q}_j q_i \rangle$ is really $\epsilon_{\alpha\beta} \langle (\hat{q}_{\alpha j})^T C \hat{q}_{\beta i} \rangle \propto J_{Aij}$ \ $J_A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$
  - Conserved if $g J_A g^T = J_A \implies H = Sp(2N_F)$
Lagrangians

JB, Lu, arXiv:0910.5424, JB Rössler 1509.04082:
4 cases similar with \( u = \exp \left( \frac{i}{\sqrt{2F}} \phi^a X^a \right) \)

But the matrices \( X^a \) are:
- Complex or \( SU(N) \times SU(N)/SU(N) \):
  all \( SU(N) \) generators
- Real or \( SU(2N)/SO(2N) \):
  \( SU(2N) \) generators with \( X^a J_S = J_S X^a T \)
- Pseudoreal or \( SU(2N)/Sp(2N) \):
  \( SU(2N) \) generators with \( X^a J_A = J_A X^a T \)
- Real Majorana or \( SU(N)/SO(N) \):
  \( SU(N) \) generators with \( X^a = X^a T \)
- \( SO(2N) \): not usual way of parametrizing \( SO(2N) \) matrices
  the two are related by a \( U(2N) \) transformation:
  same ChPT except for anomalous sector
Lagrangians

- $u \rightarrow hug_L^\dagger \equiv g_R u h^\dagger$ for complex
- $u \rightarrow hug^\dagger$ for real, pseudoreal
- $h$ is in the conserved part of the group for all cases
- $u_\mu = i (u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \rightarrow h u_\mu h^\dagger$
- external fields can also be included.
- a generalized mass term $\chi_\pm \rightarrow h \chi_\pm h^\dagger$ can be defined with $\chi_\pm = u^{\dagger} \tilde{\chi} u^\dagger \pm u \tilde{\chi} u$
- $\mathcal{L}_{LO} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle$
- $\mathcal{L}_4 = L_0 \langle u_\mu^\nu u_\mu u_\nu \rangle + L_1 \langle u_\mu^\nu u_\mu \rangle \langle u_\nu^\mu u_\nu \rangle + L_2 \langle u_\mu^\nu u_\nu \rangle \langle u_\mu u_\nu \rangle + L_3 \langle u_\mu u_\mu u_\nu u_\nu \rangle$
  + $L_4 \langle u_\mu u_\mu \rangle \langle \chi_+ \rangle + L_5 \langle u_\mu u_\mu \chi_+ \rangle + L_6 \langle \chi_+ \rangle^2 + L_7 \langle \chi_- \rangle^2 + \frac{1}{2} L_8 \langle \chi_+^2 + \chi_-^2 \rangle$
  - $-iL_9 \langle f_+ u_\mu^\nu u^\mu u^\nu \rangle + \frac{1}{4} L_{10} \langle f_+^2 - f_-^2 \rangle + H_1 \langle l_\mu^\nu m^\mu^\nu + r_\mu^\nu r^\mu^\nu \rangle + H_2 \langle \chi \chi^+ \rangle$. 
The main useful formulae

Calculating for equal mass case goes through using:

**Complex:**
\[
\langle X^a A X^a B \rangle = \langle A \rangle \langle B \rangle - \frac{1}{N_F} \langle AB \rangle,
\]
\[
\langle X^a A \rangle \langle X^a B \rangle = \langle AB \rangle - \frac{1}{N_F} \langle A \rangle \langle B \rangle.
\]

**Real:**
\[
\langle X^a A X^a B \rangle = \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \langle A J_S B^T J_S \rangle - \frac{1}{2N_F} \langle AB \rangle,
\]
\[
\langle X^a A \rangle \langle X^a B \rangle = \frac{1}{2} \langle AB \rangle + \frac{1}{2} \langle A J_S B^T J_S \rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle.
\]

**Pseudoreal:**
\[
\langle X^a A X^a B \rangle = \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \langle A J_A B^T J_A \rangle - \frac{1}{2N_F} \langle AB \rangle,
\]
\[
\langle X^a A \rangle \langle X^a B \rangle = \frac{1}{2} \langle AB \rangle - \frac{1}{2} \langle A J_A B^T J_A \rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle.
\]

So can do the calculations for all cases

For the partially quenched case: extension needed

\[ \phi \phi \rightarrow \phi \phi \]

- **\(\pi \pi\) scattering**
  - Amplitude in terms of \(A(s, t, u)\)
    \[ M_{\pi \pi}(s, t, u) = \delta^{ab}\delta^{cd} A(s, t, u) + \delta^{ac}\delta^{bd} A(t, u, s) + \delta^{ad}\delta^{bc} A(u, s, t) . \]
  - Three intermediate states \(I = 0, 1, 2\)

- **Our three cases**
  - Two amplitudes needed \(B(s, t, u)\) and \(C(s, t, u)\)
    \[ M(s, t, u) = \left[ \left\langle X^a X^b X^c X^d \right\rangle + \left\langle X^a X^d X^c X^b \right\rangle \right] B(s, t, u) \]
    \[ + \left[ \left\langle X^a X^c X^d X^b \right\rangle + \left\langle X^a X^b X^d X^c \right\rangle \right] B(t, u, s) \]
    \[ + \left[ \left\langle X^a X^d X^b X^c \right\rangle + \left\langle X^a X^c X^b X^d \right\rangle \right] B(u, s, t) \]
    \[ + \delta^{ab}\delta^{cd} C(s, t, u) + \delta^{ac}\delta^{bd} C(t, u, s) + \delta^{ad}\delta^{bc} C(u, s, t) . \]

- \(B(s, t, u) = B(u, t, s)\) \(C(s, t, u) = C(s, u, t)\).

- 7, 6 and 6 possible intermediate states

- All formulas similar length to \(\pi \pi\) cases but there are so many of them
\( \phi \phi \rightarrow \phi \phi: \frac{a_0^l}{n} \)
Weinberg’s argument for leading logarithms

The recursive argument of renormalizable theories does not work

Weinberg, Physica A96 (1979) 327

Two-loop leading logarithms can be calculated using only one-loop: Weinberg consistency conditions

\( \pi \pi \) at 2-loop: Colangelo, hep-ph/9502285


Proof at all orders

- using \( \beta \)-functions: Büchler, Colangelo, hep-ph/0309049

Underlying reason: nonlocal divergences must cancel

JB, Vladimirov, Polyakov, Carloni, Lanz, Kampf
Mass to order $\hbar^6$
Calculate the divergence
- rewrite it in terms of a local Lagrangian
  - Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
  - Luckily: we do not need to go to a minimal Lagrangian
  - So everything can be computerized
  - Thank Jos Vermaseren for FORM

- We keep all terms to have all 1PI (one particle irreducible) diagrams finite
Result Mass $O(N + 1)/O(N)$

\[ M_{\text{phys}}^2 = M^2 (1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots) \]

\[ L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \]

<table>
<thead>
<tr>
<th>i</th>
<th>$a_i$, $N = 3$</th>
<th>$a_i$ for general $N$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>$1 - \frac{N}{2}$</td>
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<tr>
<td>2</td>
<td>$\frac{17}{8}$</td>
<td>$\frac{7}{4} - \frac{7N}{4} + \frac{5}{8} N^2$</td>
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<tr>
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<td>6</td>
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Result Mass $SU(n) \times SU(N)/SU(N)$

$$M_{\text{phys}}^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

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<td>$7329919/240 N^{-6} - 1652293/240 N^{-4}$</td>
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</table>

$\text{pi form-factor}$

$\text{pi polarizability}$

Beyond QCD or BSM

Leading logarithms

Three flavour: more masses

Conclusions
Numerics: $\pi-\pi$ scattering

Graphs showing $a_0/a_{0\text{tree}}$ and $a_2/a_{2\text{tree}}$ as functions of $M^2_{\text{phys}}$ for different numbers of loops.
Reviews and general comments

- Recent review of LECs:
- Highlights of my recent work:
  - $g - 2$ HVP: estimates of connected vs disconnected vs strange contributions
  - Finite volume and twisting
    - Masses to two-loop
    - Form-factors one-loop
    - $g - 2$ HVP: two-loop
Two-point: Why

Muon: $a_\mu = (g - 2)/2$ and $a_\mu^{\text{LO,HVP}} = \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2)$

plot: $f(Q^2) \hat{\Pi}(Q^2)$ with $Q^2 = -q^2$ in GeV$^2$

Figure and data: Aubin, Blum, Chau, Golterman, Peris, Tu, Phys. Rev. D93 (2016) 054508 [arXiv:1512.07555]
Two-point: Connected versus disconnected

\[ \Pi_{ab}^{\mu\nu}(q) \equiv i \int d^4 x e^{i q \cdot x} \langle T(j_{a}^{\mu}(x)j_{a}^{\nu}\dagger(0)) \rangle \]

- Connected:
  - \( j_{u}^{\mu} = \bar{u}\gamma^{\mu}u \)
  - \( j_{d}^{\mu} = \bar{d}\gamma^{\mu}d, \quad j_{s}^{\mu} = \bar{s}\gamma^{\mu}s \)

- Disconnected:
  - \( j_{e}^{\mu} = \frac{2}{3} \bar{u}\gamma^{\mu}u - \frac{1}{3} \bar{d}\gamma^{\mu}d - \frac{1}{3} \bar{s}\gamma^{\mu}s \)

Study in ChPT at one-loop:


Two-loop JB, Relefors [arXiv:1609.01573]
Two-point: Connected versus disconnected

- Include also singlet part of the vector current
- There are new terms in the Lagrangian
- $p^4$ only one more: $\langle L_{\mu\nu} \rangle \langle L^{\mu\nu} \rangle + \langle R_{\mu\nu} \rangle \langle R^{\mu\nu} \rangle$
- $\implies$ The pure singlet vector current does not couple to mesons until $p^6$
- $\implies$ Loop diagrams involving the pure singlet vector current only appear at $p^8$ (implies relations)
- $p^6$ (no full classification, just some examples)
- $\langle D_\rho L_{\mu\nu} \rangle \langle D^\rho L^{\mu\nu} \rangle + \langle D_\rho R_{\mu\nu} \rangle \langle D^\rho R^{\mu\nu} \rangle,$
- $\langle L_{\mu\nu} \rangle \langle L^{\mu\nu} \chi^\dagger U \rangle + \langle R_{\mu\nu} \rangle \langle R^{\mu\nu} \chi U^\dagger \rangle,$
- Results at two-loop order, unquenched isospin limit
Two-point: Connected versus disconnected

- $\Pi_{\pi^+\pi^+}^{\mu\nu}$: only connected
- $\Pi_{ud}^{\mu\nu}$: only disconnected
- $\Pi_{uu}^{\mu\nu} = \Pi_{\pi^+\pi^+}^{\mu\nu} + \Pi_{ud}^{\mu\nu}$
- $\Pi_{ee}^{\mu\nu} = \frac{5}{9}\Pi_{\pi^+\pi^+}^{\mu\nu} + \frac{1}{9}\Pi_{ud}^{\mu\nu}$
- Infinite volume (and the $ab$ considered here):
  $\Pi_{ab}^{\mu\nu} = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{ab}^{(1)}$
- Large $N_c$ + VMD estimate: $\Pi_{\pi^+\pi^+}^{(1)} = \frac{4 F_{\pi}^2}{M_V^2 - q^2}$
- Plots on next pages are for $\Pi_{ab0}^{(1)}(q^2) = \Pi_{ab}^{(1)}(q^2) - \Pi_{ab}^{(1)}(0)$
- At $p^4$ the extra LEC cancels, at $p^6$ there are new LEC contributions, but no new ones in the loop parts
Two-point: Connected versus disconnected

- Connected
- $p^6$ is large
- Due to the $L_i^r$ loops

$\hat{\Pi}_\pi^+$ vs. $q^2$ [GeV$^2$]

VMD, $p^4 + p^6$, $p^4$, $p^6 R$, $p^6 L$
Two-point: Connected versus disconnected

- Disconnected
- $p^6$ is large
- Due to the $L^r_i$ loops
- About $-\frac{1}{2}$ connected
- $-\frac{1}{10}$ is from $\Pi^{(1)}_{ee} = \frac{5}{9} \Pi^{(1)}_{\pi^+\pi^+} + \frac{1}{9} \Pi^{(1)}_{ud}$

**Diagram:**
- $p^4 + p^6$
- $p^4$
- $p^6 R$
- $p^6 L$

**Graph:**
- $q^2$ vs. $\hat{\Pi}_{UD}$
- $q^2$ range: $-0.1$ to $0$
- $\hat{\Pi}_{UD}$ range: $0$ to $0.003$
Two-point: Connected versus disconnected

\[ p^4 + p^6 + \text{VMD} \]

- \( p^4 + p^6 \)
- \( p^4 \)
- \( p^6 \) R
- \( p^6 \) L

\[ \hat{\Pi}_{UD}/\Pi_{\pi}^+ \]

- \( p^4 \) and \( p^6 \) pion part exactly \(-\frac{1}{2}\)
- Not true for unsubtracted at \( p^4 \) (LEC)
- Not true for pure LEC at \( p^6 \)
Two-point: Including strange

- \( \pi \) connected \( u,d \)
- \( ud \) disconnected \( u,d \)
- \( ss \) strange current
- \( us \) mixed strange-\( u,d \)

- strange part is very small:
  - \( q^2 = 0 \) subtraction
  - \( p^4 \) and \( p^6 \) cancel largely

\[ \frac{\tilde{\Pi}}{s_q^2} \]
Two-point: with strange, electromagnetic current

- $\pi$ connected $u,d$
- $ud$ disconnected $u,d$
- $ss$ strange current
- $us$ mixed $s-u,d$
- new $p^6$ LEC cancels
- VMD, VMD used

not included
Finite volume

- Lattice QCD calculates at different quark masses, volumes boundary conditions, . . .
- A general result by Lüscher: relate finite volume effects to scattering (1986)
- Chiral Perturbation Theory is also useful for this
  \( M_\pi, F_\pi, \langle \bar{q}q \rangle \) one-loop equal mass case
- I will stay with ChPT and the \( p \) regime (\( M_\pi L \gg 1 \))
- \( 1/m_\pi = 1.4 \text{ fm} \)
  - may need to (and I will) go beyond leading \( e^{-m_\pi L} \) terms “around the world as often as you like”
- Convergence of ChPT is given by \( 1/m_\rho \approx 0.25 \text{ fm} \)
Finite volume: selection of earlier ChPT results

- masses and decay constants for $\pi, K, \eta$ one-loop

- $M_\pi$ at 2-loops (2-flavour)

- $\langle \bar{q}q \rangle$ at 2 loops (3-flavour)

- Twisted mass at one-loop

- Twisted boundary conditions
Papers

Finite volume at two-loops (periodic)
- Two-loop sunset integrals at finite volume,
- Finite Volume at two-loops in Chiral Perturbation Theory,
- Finite Volume for Three-Flavour Partially Quenched Chiral Perturbation Theory through NNLO in the Meson Sector,

Twisted boundary conditions
- Masses, Decay Constants and Electromagnetic Form-factors with Twisted Boundary Conditions,
- The vector two-point function with twisted boundary conditions,
  JB, Relefors, to be published
- $K_{\ell 3}$ with staggered, finite volume and twisting,
  Bernard, JB, Gamiz, Relefors, to be published
Masses at two-loop order

- Sunset integrals at finite volume done
- Loop calculations:

Agreement for $N_f = 2, 3$ for pion
- $K$ has no pion loop at LO
Decay constants at two-loop order

- Sunset integrals at finite volume done
  

- Loop calculations:
  

![Graphs showing decay constants at two-loop order](attachment:image.png)

- Agreement for \( N_f = 2, 3 \) for pion
- \( K \) now has a pion loop at LO
Other $p^6$

Masses and decay constants at finite volume:

  - $SU(N) \times SU(N)/SU(N)$
  - $SU(N)/SO(N)$ (including Majorana case)
  - $SU(2N)/Sp(2N)$
- If you want more graphs: look at the papers or play with the programs in CHIRON
Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i / L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta^i} q(x^i)$
- Then momenta are $p^i = \theta^i / L + 2\pi n^i / L$. Allows to map out momentum space on the lattice much better
  Sandor Bedaque, ...
- Small note:
  - Beware what people call momentum: is $\theta^i / L$ included or not?
  - Reason: a colour singlet gauge transformation $G^S_\mu \rightarrow G^S_\mu - \partial_\mu \epsilon(x)$, $q(x) \rightarrow e^{i\epsilon(x)} q(x)$, $\epsilon(x) = -\theta^i x^i / L$
  - Boundary condition
    Twisted $\iff$ constant background field + periodic
Twisted boundary conditions: Drawbacks

Drawbacks:
- Box: Rotation invariance $\rightarrow$ cubic invariance
- Twisting: reduces symmetry further

Consequences:
- $m^2(p^2) = E^2 - \vec{p}^2$ is not constant
  - dispersion relation so what is a “mass” not clear
  - already present in “moving frame” without twist
  - Renormalized momentum is quantity-dependent
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum
Two-point function: twisted boundary conditions


\[ \int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0 \]

\[ \langle \bar{u} \gamma^\mu u \rangle \neq 0 \]

\[ j_\pi^\mu = \bar{d} \gamma^\mu u \]

satisfies \( \partial_\mu \langle T(j_\pi^\mu + (x)j_\pi^\mu (0)\rangle = \delta^{(4)}(x)\langle \bar{d} \gamma^\nu d - \bar{u} \gamma^\nu u \rangle \)

\[ \Pi^{\mu\nu}_a(q) \equiv i \int d^4 x e^{i q \cdot x} \langle T(j_a^\mu (x)j_a^\nu \dagger (0)\rangle \]

Satisfies WT identity. \( q_\mu \Pi^{\mu\nu}_{\pi^+} = \langle \bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d \rangle \)

ChPT at one-loop satisfies this


two-loop in partially quenched: JB, Relefors, in preparation

satisfies the WT identity (as it should)
\[ \langle \bar{u} \gamma^\mu u \rangle \]

\[ \langle \bar{u} u \rangle \]

\[ \langle \bar{u} u \rangle^V \approx -2.4 \times 10^{-5} \text{ GeV}^3 \]

\[ \langle \bar{u} u \rangle \approx -1.2 \times 10^{-2} \text{ GeV}^3 \]

**Fully twisted**

\[ \theta_u = (0, \theta_u, 0, 0), \text{ all others untwisted} \]

\[ m_\pi L = 4 \]

**Partially twisted**

(ratio at \( p^4 = 2 \) up to kaon loops)

For comparison:
Two-point: partially twisted, one-loop

\[ q = \left(0, \sqrt{-q^2}, 0, 0\right) \]

\[ \Pi^{22} = \Pi^{33} \]

\[ \tilde{\theta}_u = L q \]

\[ m_{\pi 0} L = 4 \]

\[ m_{\pi 0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi_{\text{VMD}}^{(1)} = \frac{-4q^2 F_{\pi}^2}{M_V^2 - q^2} \]

\[ \approx 5e-3 \cdot \frac{q^2}{0.1} \]

Diamond: periodic
Note: \( \Pi^{\mu \nu}(0) \neq 0 \)

Correction is at the % level
Two-point: partially twisted, with two-loop

\[ q = \left(0, \sqrt{-q^2}, 0, 0\right) \]

\[ \Pi_{22} = \Pi_{33} \]

\[ \vec{\theta}_u = L q \]

\[ m_{\pi_0} L = 4 \]

\[ m_{\pi_0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi_{VMD}^{(1)} = \frac{-4q^2F_\pi^2}{M_V^2 - q^2} \approx 5 \cdot 10^{-3} \cdot \frac{q^2}{0.1} \]

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Correction from two loop is reasonable (thin lines are \( p^4 \))
Two-point: partially twisted, one-loop

\[ q = \left( 0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0 \right) \]

\[ \Pi_{\mu\nu}^{(1)}_{\text{VMD}} = \frac{-4q^2F_\pi^2}{M_V^2-q^2} \]

\[ \approx 5e^{-3}\cdot \frac{q^2}{0.1} \]

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Correction is at the % level
Two-point: partially twisted, one-loop

\[ q = \left( 0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0 \right) \]

\[ \Pi^{11} = \Pi^{22} \]

\[ \vec{\theta}_u = L q \]

\[ m_{\pi 0} L = 4 \]

\[ m_{\pi 0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi^{(1)}_{\text{VMD}} = \frac{-4q^2 F^2_\pi}{M^2_V - q^2} \approx 5 \cdot 10^{-3} \cdot \frac{q^2}{0.1} \]

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Two loop correction again reasonable (thin lines are \( p^4 \))
Partial twisting: masses

Bernard, JB, Gamiz, Relefors, in preparation

\[ m_\pi L = 3, \quad \vec{\theta}_u = (\theta, 0, 0), \quad \vec{\theta}_d = \vec{\theta}_s = \vec{\theta}_{d\text{sea}} = \vec{\theta}_{s\text{sea}} = 0 \]

\[
\begin{align*}
\vec{\theta}_{\text{usea}} &= 0 \\
\vec{p}_1 &= (\theta, 0, 0) / L, \\
\vec{p}_2 &= (\theta + 2\pi, 0, 0) / L, \\
\vec{p}_3 &= (\theta - 2\pi, 0, 0) / L,
\end{align*}
\]
$K_{\ell 3}$: Twisting and finite volume

- There are more form-factors since Lorentz-invariance and even cubic symmetry is broken
- Masses become twist and volume dependent
- All these need to be remembered in the Ward identities
- Masses needed when checking Ward identities
- For unquenched twisted masses, decay constants and electromagnetic form-factor (see there for earlier work):
  

- Partial twisting and quenching, staggered: masses and $K_{\ell 3}$
  
  Bernard, JB, Gamiz, Relefors, in preparation
\[ q = p - p' \]
\[
\langle \pi^- (p') | \bar{s} \gamma_\mu u(0) | K^0(p) \rangle = f_+ (p_\mu + p'_\mu) + f_- q_\mu + h_\mu .
\]
\[
\langle \pi^- (p') | (m_s - m_u) \bar{s} u(0) | K^0(p) \rangle = \rho .
\]
Ward identity: \((p^2 - p'^2) f_+ + q^2 f_- + q^\mu h_\mu = \rho\)

ChPT:
- \(p^4\) Isopin conserving and breaking Gasser, Leutwyler, 1985
- \(p^6\) Isopin conserving JB, Talavera, 2003
- \(p^6\) Isopin breaking JB, Ghorbani, 2007
- \(p^4\) partially quenched, staggered Bernard, JB, Gamiz, 2013
- \(p^4\) Finite volume Ghorbani, Ghorbani, 2013 \((q^2 = 0)\)
- \(p^4\) Finite volume, twisted, partially quenched, staggered Bernard, JB, Gamiz, Relefors, in preparation
- Rare decays: \(p^4\) Mescia, Smith 2007, \(p^6\) JB, Ghorbani, 2007

Split in \(f_+\), \(f_-\) and \(h_\mu\) not unique
Masses: finite volume masses with twist effect included.

\[ p = \left( \sqrt{m_K^2(\bar{p}) + \bar{p}^2}, \bar{p} \right) \]

\[ p' = \left( \sqrt{m_\pi^2(\bar{p}')} + \bar{p}'^2, \bar{p}' \right) \]

\( q^2 \) calculated with \( m_K^2 \) and \( m_\pi^2 \) at \( V = \infty \) will also have volume corrections (small effect)

First: Twisting and partially quenched

Second: Staggered as well
$K_{\ell 3}$: infinite volume, pure loop part

- The components are the well defined ones at finite volume plots: $\rho^4$ (neglecting the $L_9' q^2$ term)
- Valence masses with $m_\pi = 135$ GeV and $m_K = 0.495$ GeV
- PQ case with $\hat{m}_{\text{sea}} = 1.1\hat{m}$, $m_{s\text{sea}} = 1.1m_s$.
- case A: $\vec{p} = 0$, case B: $\vec{p}' = 0$
$K_{\ell 3}$

\[ \rho \approx 0.23 \text{ GeV}^2 \]

$m_\pi L = 3$

\[ \rho \approx 0.23 \text{ GeV}^2 \]

\[ m_\pi L = 3 \]
Calculate the volume corrections for exactly what you did
What do you calculate on the lattice?

- Want \( f_+(0) \) at infinite volume and physical masses
- WT identity: \((p^2 - p'^2)f_+ + q^2 f_- + q_\mu h^{\mu} = \rho\)
- Assume calculation at physical masses
- All parts in the WTI at fixed \( \vec{p}, \vec{p}' \) have finite volume corrections: \( p^2, p'^2, q^2, f_-, q_\mu h^{\mu} \) and \( \rho \)
- Can use WTI at finite volume and then extrapolate \( f_+ \) or extrapolate \( \rho \) and then use WTI
### MILC lattices and numbers

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<th>a(fm)</th>
<th>$m_l/m_s$</th>
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Results: $\vec{\theta}_u = (0, \theta, \theta, \theta)$ (staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\Delta V \frac{m_K^2 - \Delta V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \Delta V \rho$$

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<th>“mass”</th>
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Results: \( \vec{\theta}_u = (0, \theta, 0, 0) \) (staggered)

Finite volume part of WI divided by \( m_K^2 - m_\pi^2 \):

\[
\frac{\Delta V m_K^2 - \Delta V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta V \rho}{m_K^2 - m_\pi^2}
\]

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Results: $\vec{\theta}_u = (0, \theta, 0, 0)$ (not staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\Delta^V \frac{m_K^2 - m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h_\mu}{m_K^2 - m_\pi^2} = \Delta^V \rho$$

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Comments FV

- Many results available
- Range from very relevant to really small at present lattices
- Partial twisting can be used to check our FV corrections on the same underlying lattice
- Be careful: ChPT must exactly correspond to your lattice calculation
- Programs available (for published ones) via CHIRON
  Those for this talk: sometime later this year (I hope)
Conclusions

- A very short (and biased) history
- A few older results
- Finite volume, twisting, staggered, . . .
- Lots of stuff left from many people (especially Ulf)
- One thing left:
CONGRATULATIONS ULF

“for his developments and applications of effective field theories in hadron and nuclear physics, that allowed for systematic and precise investigations of the structure and dynamics of nucleons and nuclei based on Quantum Chromodynamics.”