Loops and Volumes in Chiral Perturbation Theory

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Overview

Chiral Perturbation Theory: A selective history

- Goldberger-Treiman relation:
  - M. L. Goldberger and S. B. Treiman, "Decay of the pi meson," Phys. Rev. 110 (1958) 1178.
- PCAC:
  - M. Gell-Mann and M. Levy, "The Axial Vector Current In Beta Decay," Nuovo Cim. 16 (1960) 705.
  - Y. Nambu, "Axial Vector Current Conservation in Weak Interactions," Phys. Rev. Lett. 4 (1960) 380.
- Nambu-Goldstone bosons:
  - Y. Nambu, "Quasi-Particles and Gauge Invariance in the Theory of Superconductivity", Phys. Rev. 117 (1960) 648.
  - J. Goldstone, "Field Theories with Superconductor Solutions", Nuovo Cim. 19 (1961) 154.
  - J. Goldstone, A. Salam, S. Weinberg, "Broken Symmetries", Phys. Rev. 127 (1962) 965.

Chiral Perturbation Theory: A selective history

- Current algebra:
  - M. Gell-Mann, “The Symmetry group of vector and axial vector currents,” Physics **1** (1964) 63.
  - S. Fubini and G. Furlan, “Renormalization effects for partially conserved currents,” Physics **1** (1965) 229.
- Many applications:
  - S. L. Adler, “Sum rules for the axial vector coupling constant renormalization in beta decay,” Phys. Rev. **140** (1965) B736 Erratum: [Phys. Rev. **149** (1966) 1294] Erratum: [Phys. Rev. **175** (1968) 2224].
  - W. I. Weisberger, “Unsubtracted Dispersion Relations and the Renormalization of the Weak Axial Vector Coupling Constants,” Phys. Rev. **143** (1966) 1302.
  - S. Weinberg, “Pion scattering lengths,” Phys. Rev. Lett. **17** (1966) 616.

Chiral Perturbation Theory: A selective history

- Effective Lagrangians:
  - S. Weinberg, "Nonlinear realizations of chiral symmetry," Phys. Rev. **166** (1968) 1568.
  - S. R. Coleman, J. Wess and B. Zumino, "Structure of phenomenological Lagrangians. 1.," Phys. Rev. **177** (1969) 2239.
  - C. G. Callan, Jr., S. R. Coleman, J. Wess and B. Zumino, "Structure of phenomenological Lagrangians. 2.," Phys. Rev. **177** (1969) 2247.
- Chiral logarithms and loops:
  - L. F. Li and H. Pagels, "Perturbation theory about a Goldstone symmetry," Phys. Rev. Lett. **26** (1971) 1204.
  - H. Lehmann, "Chiral invariance and effective range expansion for pion pion scattering," PLB **41** (1972) 529.
  - G. Ecker and J. Honerkamp, "Pion pion scattering from an su(3) x su(3) invariant lagrangian," NPB **62** (1973) 509.
  - P. Langacker and H. Pagels, "Chiral perturbation theory," Phys. Rev. D **8** (1973) 4595.

Chiral Perturbation Theory: A selective history

- S. Weinberg, "Phenomenological Lagrangians," Physica A **96** (1979) 327. (2886 citations on 01 Nov 2016 Inspire)
- J. Gasser and H. Leutwyler, "Chiral Perturbation Theory to One Loop," Annals Phys. **158** (1984) 142. (3645)
- J. Gasser and H. Leutwyler, "Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark," Nucl. Phys. B **250** (1985) 465. (3454)
- 5931 citations to at least one of the above
- 184 due to Ulf (out of more than 700 inspire items)
- Some others: Oset (135), Bijnens (134), Pelaez (104), Pich (101), Scherer (87), Leutwyler (79), Ecker (74), Gasser (66), Donoghue (60), Holstein (59), V.Bernard (54), G.Colangelo (52),...
  (done by checking some names)
- Now sufficiently well known to no longer be cited

Chiral Perturbation Theory: A selective history

Early followers (i.e. p^4 and loop calculations, selection):

- G. D'Ambrosio and D. Espriu, "Rare Decay Modes of the K Mesons in the Chiral Lagrangian," PLB **175** (1986) 237.
- J. L. Goity, "The Decays $K^0(s) \to \gamma\gamma$ and $K^0(\ell) \to \gamma\gamma$ in the Chiral Approach," Z. Phys. C **34** (1987) 341.
- D. B. Kaplan and A. V. Manohar, "Current Mass Ratios of the Light Quarks," Phys. Rev. Lett. **56** (1986) 2004.
- J. Gasser, M. E. Sainio and A. Svarc, "Nucleons with Chiral Loops," Nucl. Phys. B **307** (1988) 779.
- G. Ecker, A. Pich and E. de Rafael, "Radiative Kaon Decays and CP Violation in ChPT," NPB **303** (1988) 665.
- J. Bijnens and F. Cornet, "Two Pion Production in Photon-Photon Collisions," Nucl. Phys. B **296** (1988) 557.
- J. Donoghue, B. Holstein and Y. C. Lin, "The Reaction $\gamma\gamma \to \pi^0\pi^0$ and Chiral Loops," PRD**37**(1988)2423.

Chiral Perturbation Theory: A selective history

And when did Ulf start (first citation and first ChPT papers)

- I. Zahed, A. Wirzba and U. G. Meissner, "Soft pion corrections to the Skyrme soliton," PRD **33** (1986) 830.
- J. Gasser and U. G. Meissner, "Chiral expansion of pion form-factors beyond one loop," NPB **357** (1991) 90.
- V. Bernard, N. Kaiser and U. G. Meissner, "πK scattering in ChPT to one loop," NPB **357** (1991) 129.
- V. Bernard, N. Kaiser and U. G. Meissner, "$\pi\eta$ scattering in QCD," PRD **44** (1991) 3698.
- V. Bernard,N. Kaiser, U. G. Meissner, "Chiral expansion of the nucleon's electromagnetic polarizabilities," PRL**67**(1991)1515
- V. Bernard,N. Kaiser,J. Gasser, U. G. Meissner, "Neutral pion photoproduction at threshold," PLB **268** (1991) 291.
- V. Bernard, N. Kaiser, J. Kambor and U. G. Meissner, "Chiral structure of the nucleon," NPB **388** (1992) 315.

Chiral Perturbation Theory

Exploring the consequences of
the chiral symmetry of QCD
and its spontaneous breaking
using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
On The Foundations Of Chiral Perturbation Theory,
Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

For references to lectures see:
http://www.thep.lu.se/~bijnens/chpt/

Chiral Perturbation Theory

A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about M_ρ.

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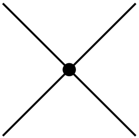
Goldstone Bosons

Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\Longrightarrow$ 8 massless degrees of freedom and interaction vanishes at zero momentum

Power counting in momenta: Meson loops, Weinberg powercounting

rules

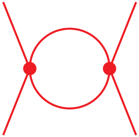


p^2

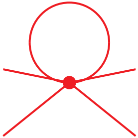
$1/p^2$

$\int d^4p$ $\quad p^4$

one loop example



$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$



$(p^2)\,(1/p^2)\,p^4 = p^4$

Chiral Perturbation Theories

- Which chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$, for $N_f = 2, 3, \ldots$ and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite T
- Which interactions to include beyond the strong one
- Which particles included as non Goldstone Bosons
- Ulf has done a lot in applying it to new areas

Lagrangians: Lagrangian structure (mesons, strong)

	2 flavour		3 flavour		PQChPT/N_f flavour	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	51+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00; Weber 2008

- L_i LEC = Low Energy Constants = ChPT parameters
- H_i: contact terms: value depends on definition of currents/densities
- Finite volume: no new LECs
- Other effects: (many) new LECs

Mesons: which Lagrangians are known ($n_f = 3$)

Loops	$\mathcal{L}_{\text{order}}$	LECs	effects included
$L = 0$	$\mathcal{L}_{p^2}$	2	strong (+ external W, γ)
	$\mathcal{L}_{e^2 p^0}$	1	internal γ
	$\mathcal{L}^{\Delta S=1}_{G_F p^2}$	2	nonleptonic weak
	$\mathcal{L}^{\Delta S=1}_{G_8 e^2 p^0}$	1	nonleptonic weak+internal γ
	$\mathcal{L}^{\text{odd}}_{p^4}$	0	WZW, anomaly
$L \leq 1$	$\mathcal{L}_{p^4}$	10	strong (+ external W, γ)
	$\mathcal{L}_{e^2 p^2}$	13	internal γ
	$\mathcal{L}^{\Delta S=1}_{G_8 F p^4}$	22	nonleptonic weak
	$\mathcal{L}^{\Delta S=1}_{G_{27} p^4}$	28	nonleptonic weak
	$\mathcal{L}^{\Delta S=1}_{G_8 e^2 p^2}$	14	nonleptonic weak+internal γ
	$\mathcal{L}^{\text{odd}}_{p^6}$	23	WZW, anomaly
	$\mathcal{L}^{\text{leptons}}_{e^2 p^2}$	5	leptons, internal γ
$L \leq 2$	$\mathcal{L}_{p^6}$	90	strong (+ external W, γ)

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars/axial currents
- Chiral logarithms
- includes Isospin and the eightfold way ($SU(3)_V$)
- Unitarity included perturbatively

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$$M^2 = 2B\hat{m}$$

LECs and μ

$l_3^r(\mu)$

$$\bar{l}_i = \frac{32\pi^2}{\gamma_i}\, l_i^r(\mu) - \log\frac{M_\pi^2}{\mu^2}\,.$$

is independent of the scale μ.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

Choice of μ :

- m_π, m_K: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$
  what about large N_c arguments????
- compromise: $\mu = m_\rho = 0.77$ GeV

Program availability

Making the programs more accessible for others to use:

- Two-loop results have very long expressions
- Many not published but available from http://www.thep.lu.se/∼bijnens/chpt/
- Many programs available on request from the authors
- Idea: make a more general framework
- CHIRON:

JB,
"CHIRON: a package for ChPT numerical results at two loops,"
Eur. Phys. J. C **75** (2015) 27 [arXiv:1412.0887]
http://www.thep.lu.se/∼bijnens/chiron/

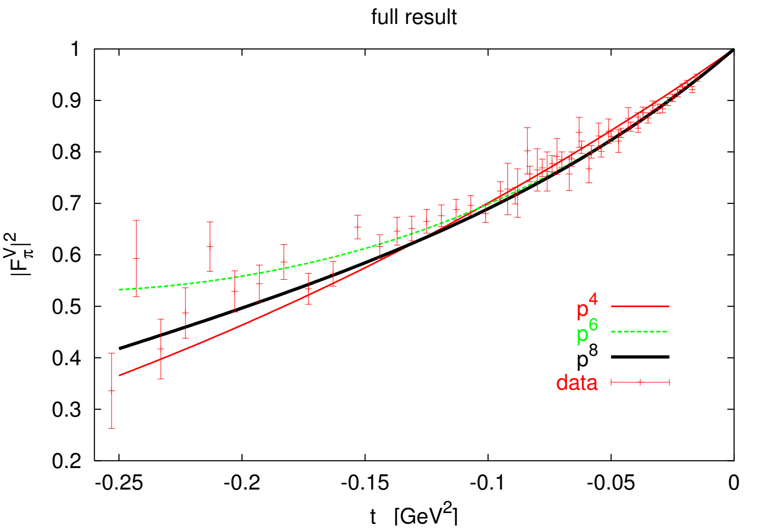
Program availability: CHIRON

- Present version: 0.54
- Classes to deal with $L_i, C_i,\ L_i^{(n)}, K_i$, standardized in/output, changing the scale,...
- Loop integrals: one-loop and sunsetintegrals
- Included so far (at two-loop order):
  - Masses, decay constants and $\langle \bar{q}q \rangle$ for the three flavour case
  - Masses and decay constants at finite volume in the three flavour case
  - Masses and decay constants in the partially quenched case for three sea quarks
  - Masses and decay constants in the partially quenched case for three sea quarks at finite volume
- A large number of example programs is included
- Manual has already reached 94 pages
- I am continually adding results from my earlier work

The first one (at least some parts)

- J. Gasser and U. G. Meissner, "Chiral expansion of pion form-factors beyond one loop," NPB **357** (1991) 90.
- Using dispersive methods to determine the nonanalytic dependence on q^2 in the pion vector and scalar form-factor for two-flavours
- Done to two-loop order fully: JB, G. Colangelo and P. Talavera, "The Vector and scalar form-factors of the pion to two loops," JHEP **9805** (1998) 014 [hep-ph/9805389]
- also known for three-flavour case: JB and P. Talavera, "Pion and kaon electromagnetic form-factors," JHEP **0203** (2002) 046 [hep-ph/0203049]; JB and P. Dhonte, "Scalar form-factors in SU(3) chiral perturbation theory," JHEP **0310** (2003) 061 [hep-ph/0307044]
- Back to dispersive (just one example) B. Ananthanarayan, I. Caprini, D. Das and I. Sentitemsu Imsong, "Parametrisation-free determination of the shape parameters for the pion electromagnetic form factor," Eur. Phys. J. C **73** (2013) 2520 [arXiv:1302.6373 [hep-ph]]

Pion form-factor: quality of fit



Charged pion polarizabilities: experiment

An example where ChPT triumphed

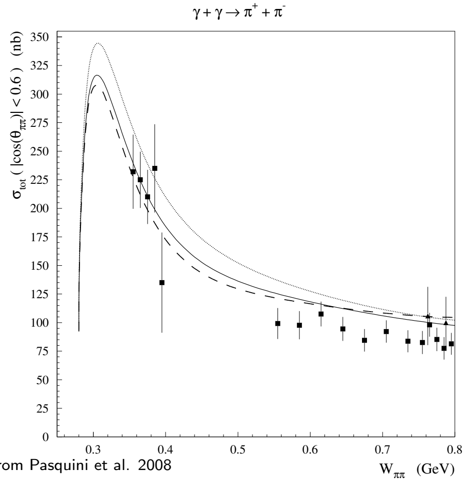
Review: Holstein, Scherer, Ann. Rev. Nucl. Part. Sci. **64** (2014) 51 [1401.0140]

- Expand $\gamma\pi^\pm \to \gamma\pi^\pm$ near threshold: $(z_\pm = 1 \pm \cos\theta_{\rm cm})$

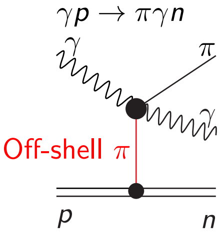
$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\rm Born} - \frac{\alpha m_\pi^3\left((s-m_\pi^2\right)^2}{4s^2\left(sz_+ + m_\pi^2 z_-\right)}\left(z_-^2(\alpha-\beta) + \frac{s^2}{m_\pi^4} z_+^2(\alpha+\beta)\right)$$

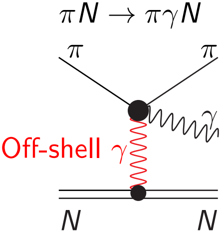
- Three ways to measure: (all assume $\alpha+\beta=0$)
  - $\pi\gamma \to \pi\gamma$ (Primakoff, high energy pion beam)
    Dubna (1985) $\alpha = (6.8 \pm 1.4)\ 10^{-4}\ \text{fm}^3$
    Compass (CERN, 2015) $\alpha = (2.0 \pm 0.6 \pm 0.7)\ 10^{-4}\ \text{fm}^3$
  - $\gamma\pi \to \pi\gamma$ (via one-pion exchange)
    Lebedev (1986) $\alpha = (20 \pm 12)\ 10^{-4}\ \text{fm}^3$
    Mainz (2005) $\alpha = (5.8 \pm 0.75 \pm 1.5 \pm 0.25)\ 10^{-4}\ \text{fm}^3$
  - $\gamma\gamma \to \pi\pi$ (in $e^+e^- \to e^+e^-\pi^+\pi^-$)
    MarkII data analyzed (1992) $\alpha = (2.2 \pm 1.1)\ 10^{-4}\ \text{fm}^3$
- Extrapolation and subtraction: difficult experiments

Polarizabilities: extrapolations needed



from Pasquini et al. 2008





Charged pion polarizabilities: theory

- ChPT:
  - One-loop JB, Cornet, 1986, Donoghue-Holstein 1989
    $\alpha + \beta = 0$, $\alpha = (2.8 \pm 0.2)\ 10^{-4}\ \text{fm}^3$
    input $\pi \to e\nu\gamma$ (error only from this)
  - Two-loop Bürgi, 1996, Gasser, Ivanov, Sainio 2006
    $\alpha + \beta = 0.16\ 10^{-4}\ \text{fm}^3$,$\alpha = (2.8 \pm 0.5)\ 10^{-4}\ \text{fm}^3$
- Dispersive analysis from $\gamma\gamma \to \pi\pi$:
  - Fil'kov-Kashevarov, 2005 $(\alpha_1 - \beta_1) = (13.0^{+2.6}_{-1.9}) \cdot 10^{-4}\text{fm}^3$
  - Critized by Pasquini-Drechsel-Scherer, 2008
    "Large model dependence in their extraction"
    "Our calculations. . . are in reasonable agreement with ChPT for charged pions"
    $(\alpha_1 - \beta_1) = (5.7) \cdot 10^{-4}\text{fm}^3$ perfectly possible

QCDlike and/or technicolor theories

- One can also have different symmetry breaking patterns from underlying fermions
- Three generic cases
  - $SU(N) \times SU(N)/SU(N)$
  - $SU(2N)/SO(2N)$ (Dirac) or $SU(N)/SO(N)$ (Majorana)
  - $SU(2N)/Sp(2N)$
- Many one-loop results existed especially for the first case (several discovered only after we published our work)
- Equal mass case pushed to two loops JB, Lu, 2009-11
- Majorana, Finite Volume and partially quenched added JB, Rössler, arXiv:1509.04082

N_F fermions in a representation of the gauge group

- complex (QCD):
  - $q^T = (q_1\ q_2 \dots\ q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$
    $q_L \to g_L q_L$ and $g_R \to g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$
  - $g_L = g_R$ then $\Sigma_{ij} \to \Sigma_{ij} \Longrightarrow$ conserved $H = SU(N_F)_V$:
- Real (e.g. adjoint): $\hat{q}^T = (q_{R1}\ \dots\ q_{RN_F}\ \tilde{q}_{R1}\ \dots\ \tilde{q}_{RN_F})$
  - $\tilde{q}_{Ri} \equiv C\overline{q}_{Li}^T$ goes under gauge group as q_{Ri}
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - $\langle \overline{q}_j q_i \rangle$ is really $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto J_{Sij}$ $J_S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$
  - Conserved if $g J_S g^T = J_S \Longrightarrow H = SO(2N_F)$
- Real with N_F Majorana fermions
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - Majorana condensate is $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto \delta_{ij} = I_{ij}$
  - Conserved $g I g^T = I$

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N_F fermions in a representation of the gauge group

- complex (QCD): $q^T = (q_1 \ q_2 \ldots \ q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$ $\quad q_L \to g_L q_L$ and $g_R \to g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$
  - Conserved $H = SU(N_F)_V$: $g_L = g_R$ then $\Sigma_{ij} \to \Sigma_{ij}$
- Pseudoreal (e.g. two-colours):
  $\hat{q}^T = (q_{R1} \ \ldots \ q_{RN_F} \ \tilde{q}_{R1} \ \ldots \ \tilde{q}_{RN_F})$
  - $\tilde{q}_{R\alpha i} \equiv \epsilon_{\alpha\beta} C \bar{q}^T_{L\beta i}$ goes under gauge group as $q_{R\alpha i}$
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - $\langle \overline{q}_j q_i \rangle$ is really $\epsilon_{\alpha\beta}\langle (\hat{q}_{\alpha j})^T C \hat{q}_{\beta i}\rangle \propto J_{Aij}$ $J_A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$
  - Conserved if $g J_A g^T = J_A \Longrightarrow H = Sp(2N_F)$

Lagrangians

JB, Lu, arXiv:0910.5424, JB Rössler 1509.04082:

4 cases similar with $u = \exp\left(\frac{i}{\sqrt{2}F}\phi^a X^a\right)$

But the matrices X^a are:

- Complex or $SU(N) \times SU(N)/SU(N)$: all $SU(N)$ generators
- Real or $SU(2N)/SO(2N)$: $SU(2N)$ generators with $X^a J_S = J_S X^{aT}$
- Pseudoreal or $SU(2N)/Sp(2N)$: $SU(2N)$ generators with $X^a J_A = J_A X^{aT}$
- Real Majorana or $SU(N)/SO(N)$: $SU(N)$ generators with $X^a = X^{aT}$
- $SO(2N)$: not usual way of parametrizing $SO(2N)$ matrices
  - the two are related by a $U(2N)$ transformation:
  - same ChPT except for anomalous sector

Lagrangians

- $u \to h u g_L^\dagger \equiv g_R u h^\dagger$ for complex
- $u \to h u g^\dagger$ for real, pseudoreal
- h is in the conserved part of the group for all cases
- $u_\mu = i\left(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger\right) \to h u_\mu h^\dagger$
- external fields can also be included.
- a generalized mass term $\chi_\pm \to h \chi_\pm h^\dagger$ can be defined with $\chi_\pm = u^\dagger \tilde{\chi} u^\dagger \pm u \tilde{\chi} u$
- $\mathcal{L}_{LO} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+ \rangle$
- $$\begin{aligned} \mathcal{L}_4 \quad = \quad & L_0 \langle u^\mu u^\nu u_\mu u_\nu \rangle + L_1 \langle u^\mu u_\mu \rangle \langle u^\nu u_\nu \rangle + L_2 \langle u^\mu u^\nu \rangle \langle u_\mu u_\nu \rangle + L_3 \langle u^\mu u_\mu u^\nu u_\nu \rangle \\ & + L_4 \langle u^\mu u_\mu \rangle \langle \chi_+ \rangle + L_5 \langle u^\mu u_\mu \chi_+ \rangle + L_6 \langle \chi_+ \rangle^2 + L_7 \langle \chi_- \rangle^2 + \frac{1}{2} L_8 \langle \chi_+^2 + \chi_-^2 \rangle \\ & - i L_9 \langle f_{+\mu\nu} u^\mu u^\nu \rangle + \frac{1}{4} L_{10} \langle f_+^2 - f_-^2 \rangle + H_1 \langle l_{\mu\nu} l^{\mu\nu} + r_{\mu\nu} r^{\mu\nu} \rangle + H_2 \langle \chi \chi^\dagger \rangle . \end{aligned}$$

The main useful formulae

Calculating for equal mass case goes through using:

$$
\begin{aligned}
\text{Complex}: \quad & \langle X^a A X^a B \rangle = \langle A \rangle \langle B \rangle - \frac{1}{N_F} \langle AB \rangle \,, \\
& \langle X^a A \rangle \langle X^a B \rangle = \langle AB \rangle - \frac{1}{N_F} \langle A \rangle \langle B \rangle \,. \\
\text{Real}: \quad & \langle X^a A X^a B \rangle = \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \left\langle A J_S B^T J_S \right\rangle - \frac{1}{2N_F} \langle AB \rangle \,, \\
& \langle X^a A \rangle \langle X^a B \rangle = \frac{1}{2} \langle AB \rangle + \frac{1}{2} \left\langle A J_S B^T J_S \right\rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle \,. \\
\text{Pseudoreal}: \quad & \langle X^a A X^a B \rangle = \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \left\langle A J_A B^T J_A \right\rangle - \frac{1}{2N_F} \langle AB \rangle \,, \\
& \langle X^a A \rangle \langle X^a B \rangle = \frac{1}{2} \langle AB \rangle - \frac{1}{2} \left\langle A J_A B^T J_A \right\rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle
\end{aligned}
$$

So can do the calculations for all cases

For the partially quenched case: extension needed

JB, Rössler, arXiv:1509.04082 (done with quark-flow method)

$\phi\phi \to \phi\phi$

- $\pi\pi$ scattering
  - Amplitude in terms of $A(s,t,u)$

    $M_{\pi\pi}(s,t,u) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t)\,.$
  - Three intermediate states $I = 0, 1, 2$
- Our three cases
  - Two amplitudes needed $B(s,t,u)$ and $C(s,t,u)$

$$
\begin{aligned}
M(s,t,u) \quad = \quad & \left[\left\langle X^aX^bX^cX^d\right\rangle + \left\langle X^aX^dX^cX^b\right\rangle\right] B(s,t,u) \\
& + \left[\left\langle X^aX^cX^dX^b\right\rangle + \left\langle X^aX^bX^dX^c\right\rangle\right] B(t,u,s) \\
& + \left[\left\langle X^aX^dX^bX^c\right\rangle + \left\langle X^aX^cX^bX^d\right\rangle\right] B(u,s,t) \\
& + \delta^{ab}\delta^{cd}C(s,t,u) + \delta^{ac}\delta^{bd}C(t,u,s) + \delta^{ad}\delta^{bc}C(u,s,t)\,.
\end{aligned}
$$

$B(s,t,u) = B(u,t,s) \qquad C(s,t,u) = C(s,u,t)\,.$

  - 7, 6 and 6 possible intermediate states
- All formulas similar length to $\pi\pi$ cases but there are so many of them

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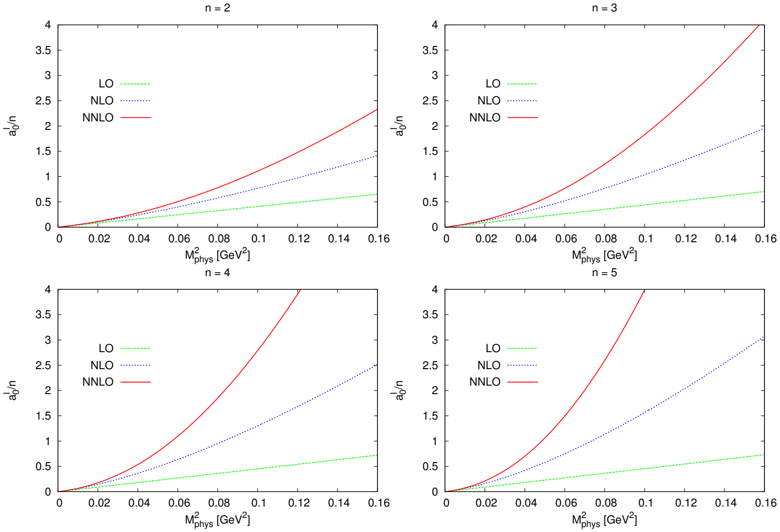
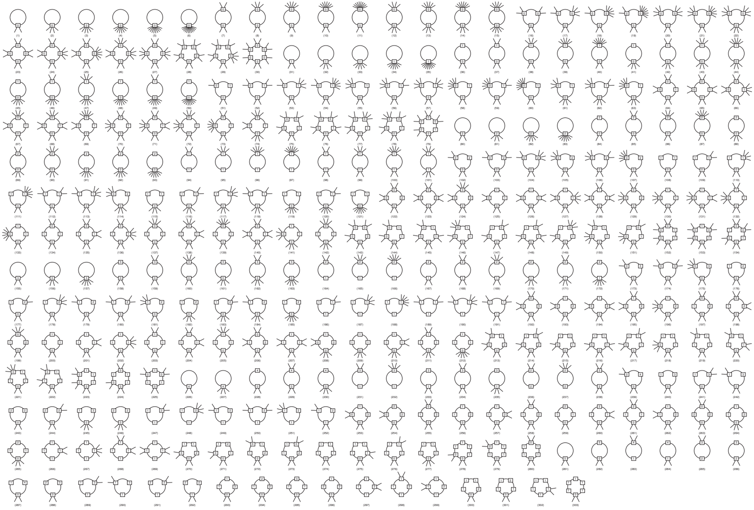
$\phi\phi \to \phi\phi$: a_0^I/n



Weinberg's argument for leading logarithms

- The recursive argument of renromalizable theories does not work
- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop: Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- General at 2 loop: JB, Colangelo, Ecker, hep-ph/9808421
- Proof at all orders
  - using β-functions: Büchler, Colangelo, hep-ph/0309049
  - with diagrams: JB, Carloni, arXiv:0909.5086
  - Extension to nucleons: JB, Vladimirov, arXiv:1409.6127
- Underlying reason: nonlocal divergences must cancel
- JB, Vladimirov, Polyakov, Carloni, Lanz, Kampf

Mass to order $\hbar^6$



General

- Calculate the divergence
- rewrite it in terms of a local Lagrangian
  - Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
  - Luckily: we do not need to go to a minimal Lagrangian
  - So everything can be computerized
  - Thank Jos Vermaseren for FORM
- We keep all terms to have all 1PI (one particle irreducible) diagrams finite

Result Mass $O(N+1)/O(N)$

$$M^2_{\text{phys}} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

i	a_i, $N=3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$
6	$\frac{1922964667}{6220800}$	$158393809/3888000 - 182792131/2592000\ N$ $+1046805817/7776000\ N^2 - 17241967/103680\ N^3$ $+70046633/576000\ N^4 - 23775/512\ N^5 + 7293/1024\ N^6$

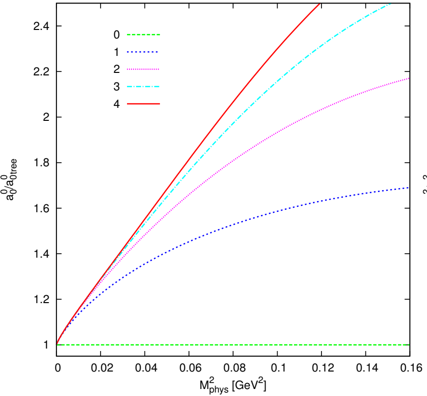
Result Mass $SU(n) \times SU(N)/SU(N)$

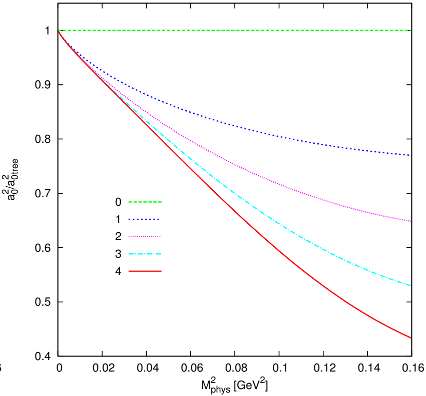
$$M^2_{\rm phys} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

i	a_i for $N=2$	a_i for $N=3$	a_i for general N
1	$-1/2$	$-1/3$	$-N^{-1}$
2	$17/8$	$27/8$	$9/2\,N^{-2} - 1/2 + 3/8\,N^2$
3	$-103/24$	$-3799/648$	$-89/3\,N^{-3} + 19/3\,N^{-1} - 37/24\,N - 1/12\,N^3$
4	$24367/1152$	$146657/2592$	$2015/8\,N^{-4} - 773/12\,N^{-2} + 193/18 + 121/288\,N^2$ $+41/72\,N^4$
5	$-8821/144$	$-\frac{27470059}{186624}$	$-38684/15\,N^{-5} + 6633/10\,N^{-3} - 59303/1080\,N^{-1}$ $-5077/1440\,N - 11327/4320\,N^3 - 8743/34560\,N^5$
6*	$\frac{1922964667}{6220800}$	$\frac{12902773163}{9331200}$	$7329919/240\,N^{-6} - 1652293/240\,N^{-4}$ $-4910303/15552\,N^{-2} + 205365409/972000$ $-69368761/7776000\,N^2 + 14222209/2592000\,N^4$ $+3778133/3110400\,N^6$

Numerics: π-π scattering



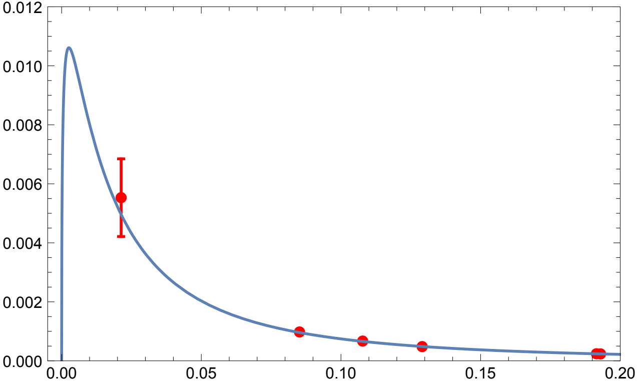


Reviews and general comments

- General loop review: JB, Prog. Part. Nucl. Phys. **58** (2007) 521 [hep-ph/0604043]
- Recent review of LECs:
  JB, Ecker,Ann.Rev.Nucl.Part.Sci. 64 (2014) 149 [arXiv:1405.6488]
- Highlights of my recent work:
  - $g-2$ HVP: estimates of connected vs disconnected vs strange contributions
  - Finite volume and twisting
    - Masses to two-loop
    - Form-factors one-loop
    - $g-2$ HVP: two-loop

Two-point: Why

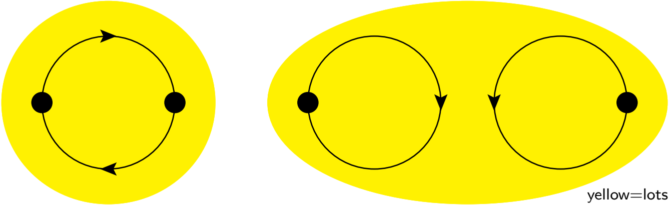
Muon: $a_\mu = (g-2)/2$ and $a_\mu^{\mathrm{LO,HVP}} = \int_0^\infty dQ^2 f\left(Q^2\right)\hat{\Pi}\left(Q^2\right)$



plot: $f\left(Q^2\right)\hat{\Pi}\left(Q^2\right)$ with $Q^2 = -q^2$ in GeV^2

Figure and data: Aubin, Blum, Chau, Golterman, Peris, Tu, Phys. Rev. D93 (2016) 054508 [arXiv:1512.07555]

Two-point: Connected versus disconnected



- $\Pi^{\mu\nu}_{ab}(q) \equiv i \int d^4x e^{iq\cdot x} \langle T(j^\mu_a(x) j^{\nu\dagger}_a(0)) \rangle$
- $j^\mu_{\pi^+} = \bar{d}\gamma^\mu u$
- $j^\mu_u = \bar{u}\gamma^\mu u, \quad j^\mu_d = \bar{d}\gamma^\mu d, \quad j^\mu_s = \bar{s}\gamma^\mu s$
- $j^\mu_e = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s$
- Study in ChPT at one-loop:

  Della Morte, Jüttner, JHEP 1011 (2010) 154 [arXiv:1009.3783]
- Two-loop JB, Relefors [arXiv:1609.01573]

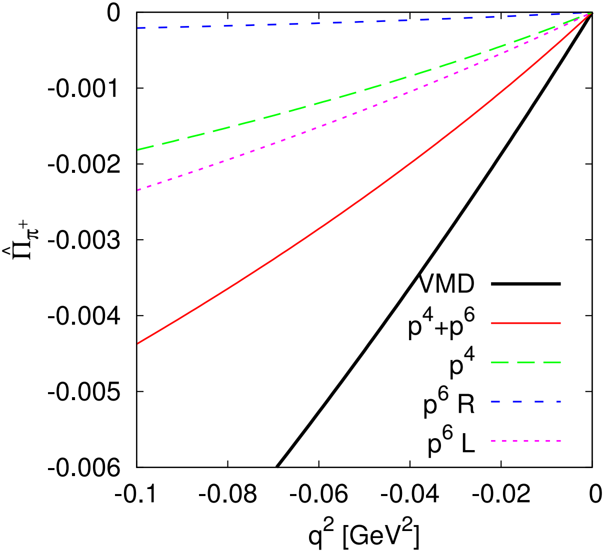
Two-point: Connected versus disconnected

- Include also singlet part of the vector current
- There are new terms in the Lagrangian
- p^4 only one more: $\langle L_{\mu\nu}\rangle \langle L^{\mu\nu}\rangle + \langle R_{\mu\nu}\rangle \langle R^{\mu\nu}\rangle$
- $\Longrightarrow$ The pure singlet vector current does not couple to mesons until p^6
- $\Longrightarrow$ Loop diagrams involving the pure singlet vector current only appear at p^8 (implies relations)
- p^6 (no full classification, just some examples)
  $\langle D_\rho L_{\mu\nu}\rangle \langle D^\rho L^{\mu\nu}\rangle + \langle D_\rho R_{\mu\nu}\rangle \langle D^\rho R^{\mu\nu}\rangle$,
  $\langle L_{\mu\nu}\rangle \left\langle L^{\mu\nu}\chi^\dagger U\right\rangle + \langle R_{\mu\nu}\rangle \left\langle R^{\mu\nu}\chi U^\dagger\right\rangle, \ldots$
- Results at two-loop order, unquenched isospin limit

Two-point: Connected versus disconnected

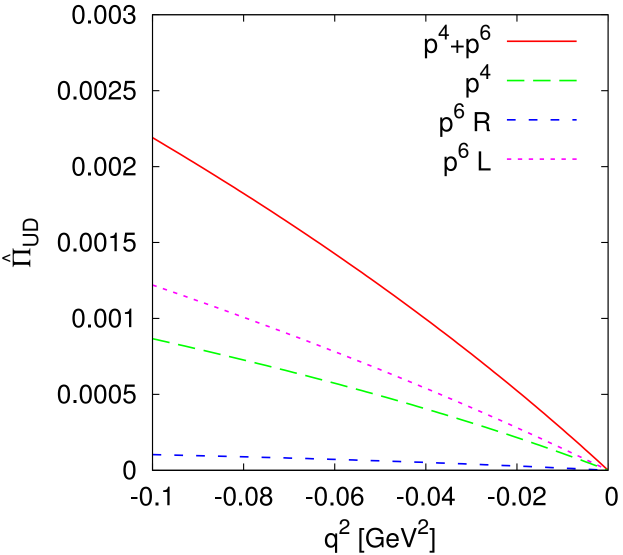
- $\Pi^{\mu\nu}_{\pi^+\pi^+}$: only connected
- $\Pi^{\mu\nu}_{ud}$: only disconnected
- $\Pi^{\mu\nu}_{uu} = \Pi^{\mu\nu}_{\pi^+\pi^+} + \Pi^{\mu\nu}_{ud}$
- $\Pi^{\mu\nu}_{ee} = \frac{5}{9}\Pi^{\mu\nu}_{\pi^+\pi^+} + \frac{1}{9}\Pi^{\mu\nu}_{ud}$
- Infinite volume (and the *ab* considered here): $\Pi^{\mu\nu}_{ab} = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\Pi^{(1)}_{ab}$
- Large N_c + VMD estimate: $\Pi^{(1)}_{\pi^+\pi^+} = \dfrac{4F_\pi^2}{M_V^2 - q^2}$
- Plots on next pages are for $\Pi^{(1)}_{ab0}(q^2) = \Pi^{(1)}_{ab}(q^2) - \Pi^{(1)}_{ab}(0)$
- At p^4 the extra LEC cancels, at p^6 there are new LEC contributions, but no new ones in the loop parts

Two-point: Connected versus disconnected



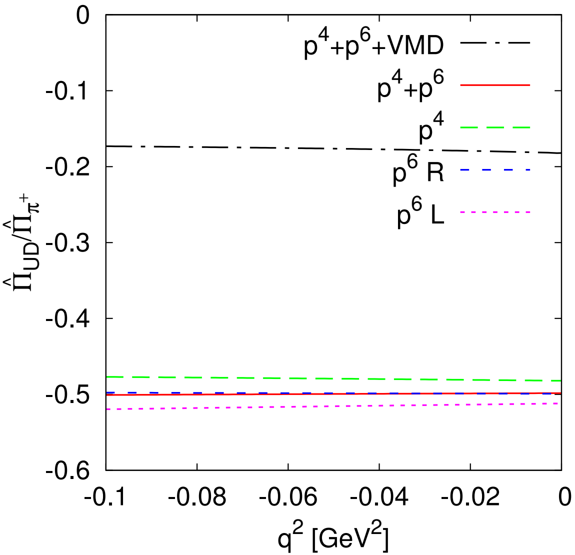
- Connected
- p^6 is large
- Due to the L_i^r loops

Two-point: Connected versus disconnected



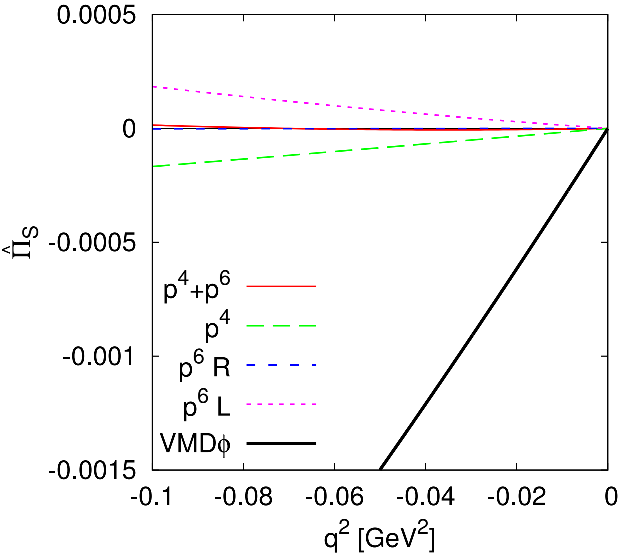
- Disconnected
- p^6 is large
- Due to the L_i^r loops
- about $-\frac{1}{2}$ connected
- $-\frac{1}{10}$ is from $\Pi^{(1)}_{ee} = \frac{5}{9}\Pi^{(1)}_{\pi^+\pi^+} + \frac{1}{9}\Pi^{(1)}_{ud}$

Two-point: Connected versus disconnected



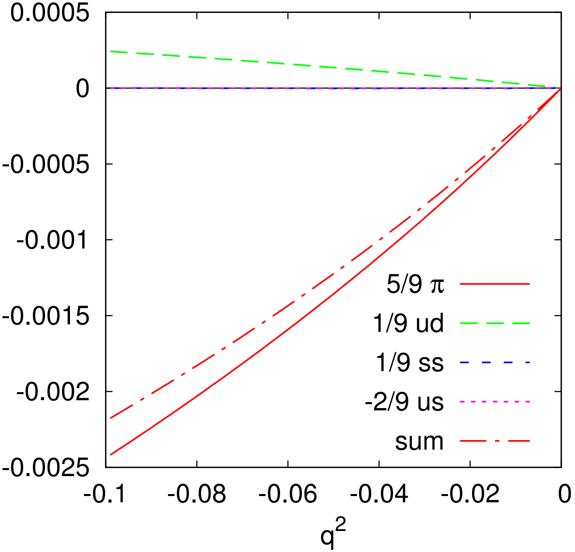
- p^4 and p^6 pion part exactly $-\frac{1}{2}$
- not true for unsubtracted at p^4(LEC)
- not true for pure LEC at p^6

Two-point: Including strange



- π connected u,d
- ud disconnected u,d
- ss strange current
- us mixed strange-u,d
- strange part is very small: $q^2 = 0$ subtraction (only kaon loops) p^4 and p^6 cancel largely

Two-point: with strange, electromagnetic current



- π connected u,d
- ud disconnected u,d
- ss strange current
- us mixed s–u,d
- new p^6 LEC cancels
- VMD, VMDϕ not included

Finite volume

- Lattice QCD calculates at different quark masses, volumes boundary conditions,. . .
- A general result by Lüscher: relate finite volume effects to scattering (1986)
- Chiral Perturbation Theory is also useful for this
- Start: Gasser and Leutwyler, Phys. Lett. B184 (1987) 83, Nucl. Phys. B 307 (1988) 763
  M_π, F_π, $\langle\bar{q}q\rangle$ one-loop equal mass case
- I will stay with ChPT and the p regime ($M_\pi L >> 1$)
- $1/m_\pi = 1.4$ fm
  may need to (and I will) go beyond leading $e^{-m_\pi L}$ terms
  "around the world as often as you like"
- Convergence of ChPT is given by $1/m_\rho \approx 0.25$ fm

Finite volume: selection of earlier ChPT results

- masses and decay constants for π, K, η one-loop

  Becirevic, Villadoro, Phys. Rev. D 69 (2004) 054010
- M_π at 2-loops (2-flavour)

  Colangelo, Haefeli, Nucl.Phys. B744 (2006) 14 [hep-lat/0602017]
- $\langle \bar{q}q \rangle$ at 2 loops (3-flavour)

  JB, Ghorbani, Phys. Lett. B636 (2006) 51 [hep-lat/0602019]
- Twisted mass at one-loop

  Colangelo, Wenger, Wu, Phys.Rev. D82 (2010) 034502 [arXiv:1003.0847]
- Twisted boundary conditions

  Sachrajda, Villadoro, Phys. Lett. B 609 (2005) 73 [hep-lat/0411033]

Papers

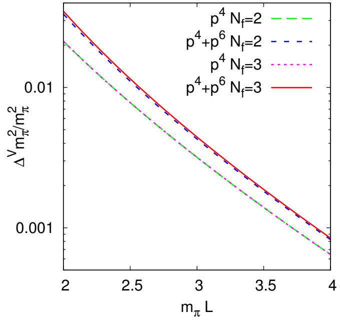
- Finite volume at two-loops (periodic)
  - Two-loop sunset integrals at finite volume,
    JB, Boström, Lähde, JHEP 1401(2014)019 [arXiv:1311.3531]
  - Finite Volume at two-loops in Chiral Perturbation Theory,
    JB, Rössler, JHEP 1501 (2015) 034 [arXiv:1411.6384]
  - Finite Volume for Three-Flavour Partially Quenched Chiral Perturbation Theory through NNLO in the Meson Sector,
    JB, Rössler, JHEP 1511 (2015) 097 [arXiv:1508.07238]
  - Finite Volume and Partially Quenched QCD-like Effective Field Theories, JB, Rössler, JHEP 1511 (2015) 017 [arXiv:1509.04082]
- Twisted boundary conditions
  - Masses, Decay Constants and Electromagnetic Form-factors with Twisted Boundary Conditions,
    JB, Relefors, JHEP 1405 (2014) 015 [arXiv:1402.1385]
  - The vector two-point function with twisted boundary conditions,
    JB, Relefors, to be published
  - $K_{\ell 3}$ wth staggered, finite volume and twisting,
    Bernard, JB, Gamiz, Relefors, to be published

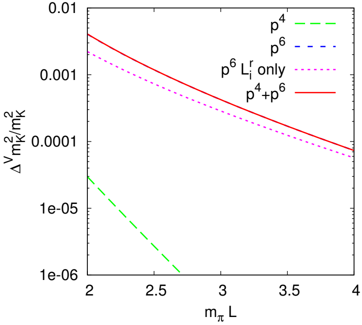
Masses at two-loop order

- Sunset integrals at finite volume done

  JB, Boström and Lähde, JHEP 01 (2014) 019 [arXiv:1311.3531]
- Loop calculations:

  JB, Rössler, JHEP **1501** (2015) 034 [arXiv:1411.6384]





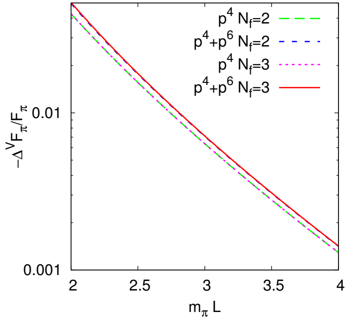
- Agreement for $N_f = 2, 3$ for pion
- K has no pion loop at LO

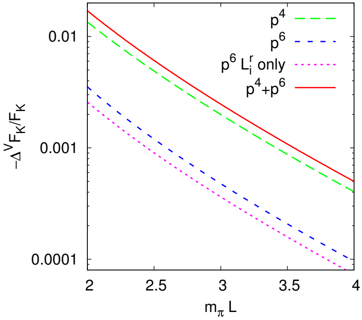
Decay constants at two-loop order

- Sunset integrals at finite volume done

  JB, Boström and Lähde, JHEP 01 (2014) 019 [arXiv:1311.3531]
- Loop calculations:

  JB, Rössler, JHEP **1501** (2015) 034 [arXiv:1411.6384]





- Agreement for $N_f = 2, 3$ for pion
- K now has a pion loop at LO

Other p^6

Masses and decay constants at finite volume:

- Finite volume for PQ three flavour (all cases) JB, Rössler, JHEP **1511** (2015) 097, [arXiv:1508.07238]
- QCD-like theories, normal and PQ (one valence mass, one sea mass) JB, Rössler, JHEP **1511** (2015) 017, [arXiv:1509.04082]
  - $SU(N) \times SU(N)/SU(N)$
  - $SU(N)/SO(N)$ (including Majorana case)
  - $SU(2N)/Sp(2N)$
- If you want more graphs: look at the papers or play with the programs in CHIRON

Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i/L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta^i_q} q(x^i)$
- Then momenta are $p^i = \theta^i/L + 2\pi n^i/L$. Allows to map out momentum space on the lattice much better

  Bedaque,...
- Small note:
  - Beware what people call momentum: is θ^i/L included or not?
  - Reason: a colour singlet gauge transformation $G^S_\mu \to G^S_\mu - \partial_\mu \epsilon(x)$, $q(x) \to e^{i\epsilon(x)} q(x)$, $\epsilon(x) = -\theta^i_q x^i/L$
  - Boundary condition Twisted $\Leftrightarrow$ constant background field+periodic

Twisted boundary conditions: Drawbacks

Drawbacks:

- Box: Rotation invariance $\rightarrow$ cubic invariance
- Twisting: reduces symmetry further

Consequences:

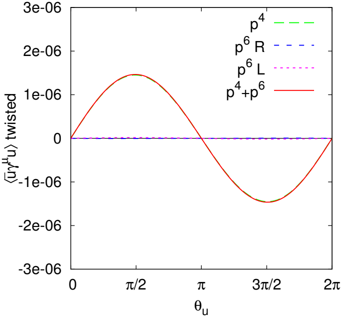
- $m^2(\vec{p}^2) = E^2 - \vec{p}^2$ is not constant
  - dispersion relation so what is a "mass" not clear
  - already present in "moving frame" without twist
  - Renormalized momentum is quantity-dependent
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum

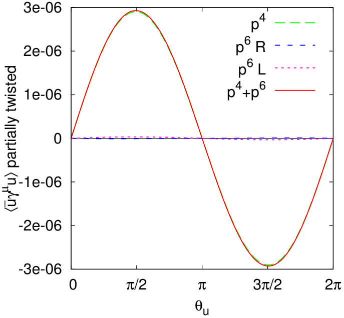
Two-point function: twisted boundary conditions

JB, Relefors, JHEP 05 (201)4 015 [arXiv:1402.1385]

- $\int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0$
- $\langle \bar{u}\gamma^\mu u \rangle \neq 0$
- $j^\mu_{\pi^+} = \bar{d}\gamma^\mu u$
  satisfies $\partial_\mu \left\langle T(j^\mu_{\pi^+}(x) j^{\nu\dagger}_{\pi^+}(0)) \right\rangle = \delta^{(4)}(x) \left\langle \bar{d}\gamma^\nu d - \bar{u}\gamma^\nu u \right\rangle$
- $\Pi^{\mu\nu}_a(q) \equiv i \int d^4x e^{iq\cdot x} \left\langle T(j^\mu_a(x) j^{\nu\dagger}_a(0)) \right\rangle$
  Satisfies WT identity. $q_\mu \Pi^{\mu\nu}_{\pi^+} = \left\langle \bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d \right\rangle$
- ChPT at one-loop satisfies this
  see also Aubin et al, Phys.Rev. D88 (2013) 7, 074505 [arXiv:1307.4701]
- two-loop in partially quenched: JB, Relefors, in preparation
  satisfies the WT identity (as it should)

$\langle \bar{u}\gamma^\mu u \rangle$





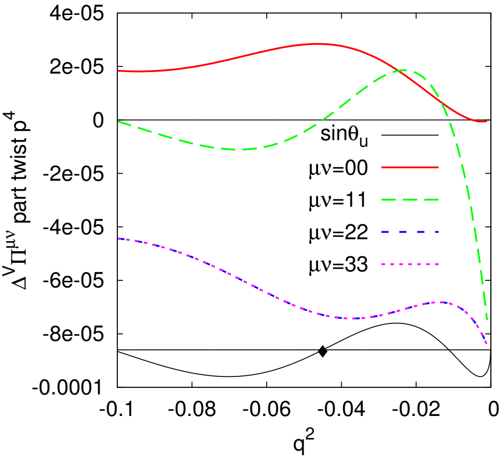
Fully twisted | Partially twisted

$\theta_u = (0, \theta_u, 0, 0)$, all others untwisted

$m_\pi L = 4$ (ratio at p^4=2 up to kaon loops)

For comparison: $\langle \bar{u}u \rangle^V \approx -2.4\ 10^{-5}\ \mathrm{GeV}^3$
$\langle \bar{u}u \rangle \approx -1.2\ 10^{-2}\ \mathrm{GeV}^3$

Two-point: partially twisted, one-loop



$$q = \left(0, \sqrt{-q^2}, 0, 0\right)$$

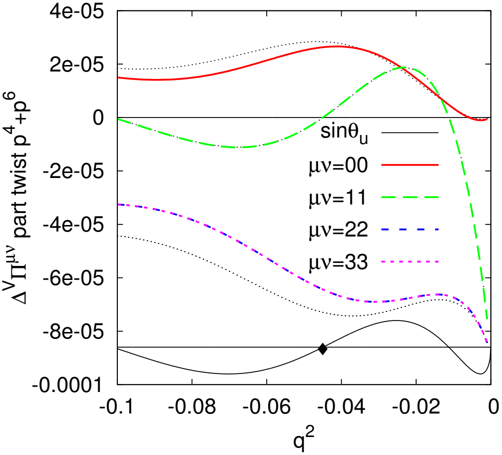
$\Pi^{22} = \Pi^{33}$
$\vec{\theta}_u = L\, q$
$m_{\pi 0} L = 4$
$m_{\pi 0} = 0.135$ GeV

$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic
Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction is at the % level

Two-point: partially twisted, with two-loop



$$q = \left(0, \sqrt{-q^2}, 0, 0\right)$$

$\Pi^{22} = \Pi^{33}$

$\vec{\theta}_u = L\, q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2}$$

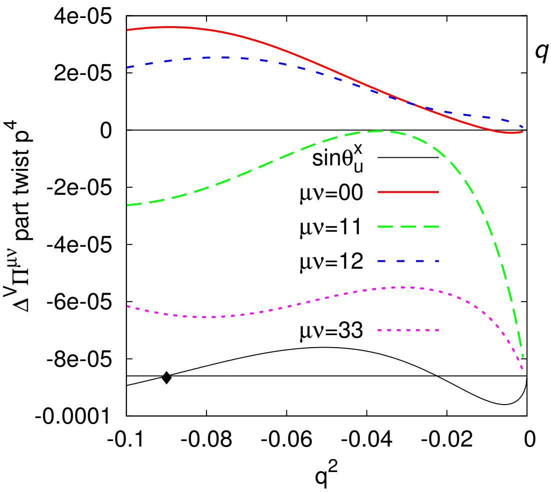
$$\approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction from two loop is reasonable (thin lines are p^4)

Two-point: partially twisted, one-loop



$$q = \left(0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0\right)$$

$\Pi^{11} = \Pi^{22}$

$\vec{\theta}_u = L\, q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

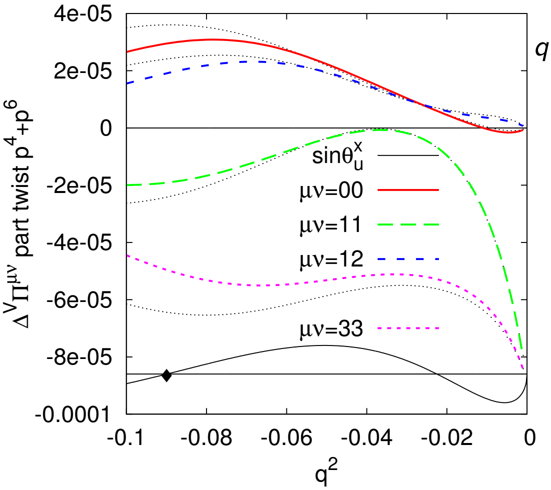
$$-q^2 \Pi^{(1)}_{\rm VMD} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction is at the % level

Two-point: partially twisted, one-loop



$$q = \left(0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0\right)$$

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$\vec{\theta}_u = L\, q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4 q^2 F_\pi^2}{M_V^2 - q^2} \approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

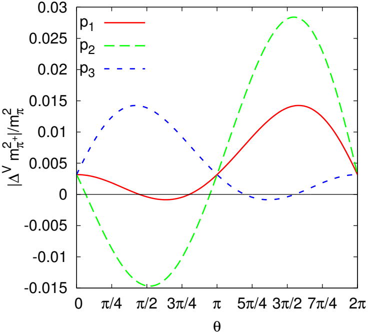
Note: $\Pi^{\mu\nu}(0) \neq 0$

Two loop correction again reasonable (thin lines are p^4)

Partial twisting: masses

Bernard, JB, Gamiz, Relefors, in preparation

$m_\pi L = 3$, $\vec{\theta}_u = (\theta, 0, 0)$, $\vec{\theta}_d = \vec{\theta}_s = \vec{\theta}_{d\mathrm{sea}} = \vec{\theta}_{s\mathrm{sea}} = 0$



$\vec{\theta}_{u\mathrm{sea}} = 0$ $\qquad$ $\vec{\theta}_{u\mathrm{sea}} = (\pi/3, 0, 0)$

$\vec{p}_1 = (\theta, 0, 0)/L$, $\vec{p}_2 = (\theta + 2\pi, 0, 0)/L$, $\vec{p}_3 = (\theta - 2\pi, 0, 0)/L$,

$K_{\ell 3}$: Twisting and finite volume

- There are more form-factors since Lorentz-invariance and even cubic symmetry is broken
- Masses become twist and volume dependent
- All these need to be remembered in the Ward identities
- Masses needed when checking Ward identities
- For unquenched twisted masses, decay constants and electromagnetic form-factor (see there for earlier work):
  JB, Relefors, JHEP 05 (2014) 015 [arXiv:1402.1385]
- Partial twisting and quenching, staggered: masses and $K_{\ell 3}$
  Bernard, JB, Gamiz, Relefors, in preparation

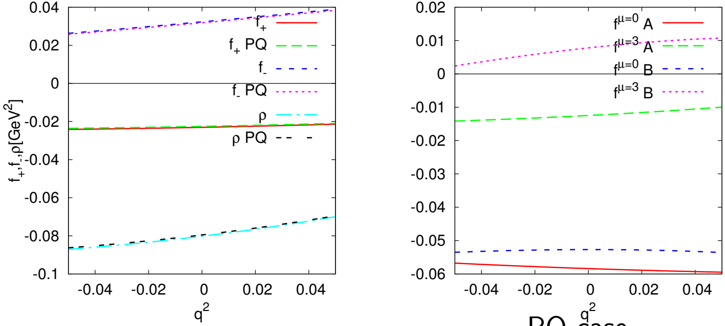
$K_{\ell 3}$

- $q = p - p'$
  $\langle \pi^-(p') | \bar{s}\gamma_\mu u(0) | K^0(p) \rangle = f_+(p_\mu + p'_\mu) + f_- q_\mu + h_\mu \,.$
- $\langle \pi^-(p') | (m_s - m_u)\bar{s}u(0) | K^0(p) \rangle = \rho \,.$
- Ward identity: $(p^2 - p'^2) f_+ + q^2 f_- + q^\mu h_\mu = \rho$
- ChPT:
  - p^4 Isopin conserving and breaking Gasser, Leutwyler, 1985
  - p^6 Isospin conserving JB, Talavera, 2003
  - p^6 Isospin breaking JB, Ghorbani, 2007
  - p^4 partially quenched, staggered Bernard, JB, Gamiz, 2013
  - p^4 Finite volume Ghorbani, Ghorbani, 2013 ($q^2 = 0$)
  - p^4 Finite volume, twisted, partially quenched, staggered Bernard, JB, Gamiz, Relefors, in preparation
  - Rare decays: p^4 Mescia, Smith 2007, p^6 JB, Ghorbani, 2007
- Split in f_+, f_- and h_μ not unique

$K_{\ell 3}$

- Masses: finite volume masses with twist effect included.
- $p = \left(\sqrt{m_K^2(\vec{p}) + \vec{p}^2}, \vec{p}\right)$
- $p' = \left(\sqrt{m_\pi^2(\vec{p}') + \vec{p}'^2}, \vec{p}'\right)$
- q^2 calculated with m_K^2 and m_π^2 at $V = \infty$ will also have volume corrections (small effect)
- First: Twisting and partially quenched
- Second: Staggered as well

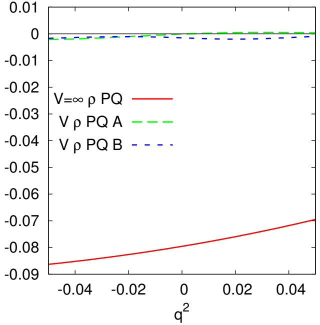
$K_{\ell 3}$: infinite volume, pure loop part



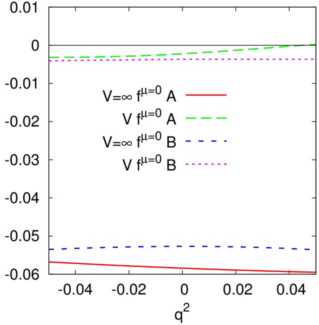
PQ case

- The components are the well defined ones at finite volume
- plots: p^4 (neglecting the $L_9^r q^2$ term)
- Valence masses with $m_\pi = 135$ GeV and $m_K = 0.495$ GeV
- PQ case with $\hat{m}_{\rm sea} = 1.1\hat{m}$, $m_{s{\rm sea}} = 1.1 m_s$.
- case A: $\vec{p} = 0$, case B: $\vec{p}\,' = 0$

$K_{\ell 3}$

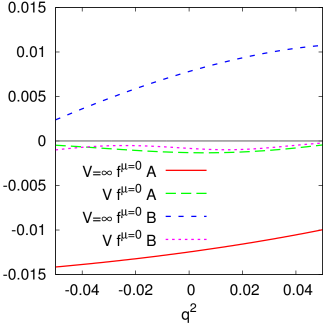


ρ



$\mu = 0$

$\rho_\infty \approx 0.23\ \text{GeV}^2$
$m_\pi L = 3$



$\mu = 3$

Calculate the volume corrections for exactly what you did

What do you calculate on the lattice?

- Want $f_+(0)$ at infinite volume and physical masses
- WT identity: $(p^2 - p'^2)f_+ + q^2 f_- + q_\mu h^\mu = \rho$
- Assume calculation at physical masses
- All parts in the WTI at fixed $\vec{p}, \vec{p}'$ have finite volume corrections: p^2, p'^2, q^2, f_-, $q^\mu h_\mu$ and ρ
- Can use WTI at finite volume and then extrapolate f_+ or extrapolate ρ and then use WTI

MILC lattices and numbers Preliminary

a(fm)	m_l/m_s	L(fm)	m_π(MeV)	m_K(MeV)	$m_\pi L$
0.15	0.035	4.8	134	505	3.25
0.12	0.2	2.9	309	539	4.5
	0.1	2.9	220	516	3.2
	0.1	3.8	220	516	4.3
	0.1	4.8	220	516	5.4
	0.035	5.7	135	504	3.9
0.09	0.2	2.9	312	539	4.5
	0.1	4.2	222	523	4.7
	0.035	5.6	129	495	3.7
0.06	0.2	2.8	319	547	4.5
	0.035	5.5	134	491	3.7

Results: $\vec{\theta}_u = (0, \theta, \theta, \theta)$ (staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\frac{\Delta^V m_K^2 - \Delta^V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta^V \rho}{m_K^2 - m_\pi^2}$$

m_π	$m_\pi L$	"mass"	"f_+"	"h_μ"	"ρ"
134	3.25	0.00000	−0.00042	0.00007	−0.00036
309	4.5	0.00013	−0.00003	−0.00041	−0.00031
220	3.2	0.00054	−0.00048	−0.00084	−0.00077
220	4.3	−0.00007	−0.00009	−0.00005	−0.00021
220	5.4	−0.00005	−0.00003	0.00001	−0.00006
135	3.9	−0.00006	−0.00020	0.00005	−0.00021
312	4.5	0.00047	0.00023	−0.00068	−0.00001
222	4.7	−0.00000	0.00018	−0.00003	0.00014
129	3.7	−0.00013	−0.00004	0.00009	−0.00007
319	4.5	0.00052	0.00037	−0.00081	0.00008
134	3.7	−0.00016	0.00045	0.00013	0.00043

Results: $\vec{\theta}_u = (0, \theta, 0, 0)$ (staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\frac{\Delta^V m_K^2 - \Delta^V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta^V \rho}{m_K^2 - m_\pi^2}$$

m_π	$m_\pi L$	"mass"	"f_+"	"h_μ"	"ρ"
134	3.25	−0.00003	−0.00066	0.00008	−0.00061
309	4.5	−0.00030	−0.00017	−0.00002	−0.00049
220	3.2	−0.00078	−0.00105	0.00036	−0.00148
220	4.3	−0.00033	−0.00034	0.00018	−0.00049
220	5.4	−0.00008	−0.00010	0.00003	−0.00015
135	3.9	−0.00002	−0.00032	0.00001	−0.00033
312	4.5	−0.00019	0.00002	−0.00009	−0.00026
222	4.7	−0.00024	−0.00018	0.00017	−0.00025
129	3.7	−0.00003	−0.00050	−0.00001	−0.00054
319	4.5	−0.00026	0.00013	−0.00012	−0.00025
134	3.7	−0.00005	−0.00058	0.00001	−0.00062

Results: $\vec{\theta}_u = (0, \theta, 0, 0)$ (not staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\frac{\Delta^V m_K^2 - \Delta^V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta^V \rho}{m_K^2 - m_\pi^2}$$

m_π	$m_\pi L$	"mass"	"f_+"	"h_μ"	"ρ"
134	3.25	−0.00049	−0.00124	0.00037	−0.00137
309	4.5	−0.00033	0.00014	−0.00004	0.00022
220	3.2	−0.00113	0.00077	0.00067	0.00031
220	4.3	−0.00062	−0.00011	0.00046	−0.00027
220	5.4	−0.00014	−0.00011	0.00010	−0.00016
135	3.9	0.00004	−0.00045	−0.00008	−0.00049
312	4.5	0.00031	0.00015	−0.00009	−0.00025
222	4.7	−0.00037	−0.00015	0.00027	−0.00025
129	3.7	−0.00000	−0.00066	−0.00005	−0.00071
319	4.5	−0.00031	0.00015	−0.00011	−0.00027
134	3.7	−0.00007	−0.00064	0.00001	−0.00070

Comments FV

- Many results available
- Range from very relevant to really small at present lattices
- Partial twisting can be used to check our FV corrections on the same underlying lattice
- Be careful: ChPT must exactly correspond to your lattice calculation
- Programs available (for published ones) via CHIRON
  Those for this talk: sometime later this year (I hope)

Conclusions

- A very short (and biased) history
- A few older results
- Finite volume, twisting, staggered,...
- Lots of stuff left from many people (especially Ulf)
- One thing left:

CONGRATULATIONS ULF

"for his developments and applications of effective field theories in hadron and nuclear physics, that allowed for systematic and precise investigations of the structure and dynamics of nucleons and nuclei based on Quantum Chromodynamics."