ChPT loops for the lattice: pion mass and decay constant; HVP at finite volume and $n\bar{n}$-oscillations

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Overview

1. Introduction
2. Vector two-point functions for $a_\mu$ LO-HVP
   - Connected and disconnected in infinite volume
   - Finite volume
   - Twisting
   - Results
3. Pion mass and decay constant
4. $n\bar{n}$ oscillations
5. Conclusions
Chiral Perturbation Theory

- ChPT = Effective field theory describing the lowest order pseudo-scalar representation
- or the (pseudo) Goldstone bosons from spontaneous breaking of chiral symmetry.
- The number of degrees of freedom depend on the case we look at
- Treat $\pi, \eta, K$ as light and pointlike with a derivative and quark-mass expansion
- Recent review of LECs:
  
Why

Muon: $a_\mu = (g - 2)/2$ and $a_\mu^{\text{LO,HVP}} = \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2)$

plot: $f(Q^2) \hat{\Pi}(Q^2)$ with $Q^2 = -q^2$ in GeV$^2$

Aubin, Blum, Chau, Golterman, Peris, Tu,


Low energy quantity so ChPT should be useful
Two-point: Connected versus disconnected

\[ \Pi_{ab}^{\mu\nu}(q) \equiv i \int d^4x e^{iq \cdot x} \langle T(j_\alpha^\mu(x)j_\alpha^{\nu\dagger}(0)) \rangle \]

\[ j_{\pi^+}^\mu = \bar{d}\gamma^\mu u \]

\[ j_u^\mu = \bar{u}\gamma^\mu u, \quad j_d^\mu = \bar{d}\gamma^\mu d, \quad j_s^\mu = \bar{s}\gamma^\mu s \]

\[ j_e^\mu = \frac{2}{3} \bar{u}\gamma^\mu u - \frac{1}{3} \bar{d}\gamma^\mu d - \frac{1}{3} \bar{s}\gamma^\mu s \]


Two-point: Connected versus disconnected

Include also singlet part of the vector current
There are new terms in the Lagrangian
\( p^4 \) only one more:
\[
\left\langle L_{\mu\nu} \right\rangle \left\langle L^{\mu\nu} \right\rangle + \left\langle R_{\mu\nu} \right\rangle \left\langle R^{\mu\nu} \right\rangle
\]
(drops out when subtracting \( \Pi(0) \))

\[ \Rightarrow \] The pure singlet vector current does not couple to mesons until \( p^6 \)

\[ \Rightarrow \] Loop diagrams involving the pure singlet vector current only appear at \( p^8 \) (implies relations)

\( p^6 \) (no full classification, just some examples)
\[
\left\langle D_\rho L_{\mu\nu} \right\rangle \left\langle D_\rho L^{\mu\nu} \right\rangle + \left\langle D_\rho R_{\mu\nu} \right\rangle \left\langle D_\rho R^{\mu\nu} \right\rangle,
\]
\[
\left\langle L_{\mu\nu} \right\rangle \left\langle L^{\mu\nu} \chi^\dagger U \right\rangle + \left\langle R_{\mu\nu} \right\rangle \left\langle R^{\mu\nu} \chi U^\dagger \right\rangle, \ldots
\]

Results at two-loop order, unquenched isospin limit
Two-point: Connected versus disconnected

- $\Pi_{\pi^+\pi^+}^{\mu\nu}$: only connected
- $\Pi_{ud}^{\mu\nu}$: only disconnected
- $\Pi_{uu}^{\mu\nu} = \Pi_{\pi^+\pi^+}^{\mu\nu} + \Pi_{ud}^{\mu\nu}$
- $\Pi_{ee}^{\mu\nu} = \frac{5}{9}\Pi_{\pi^+\pi^+}^{\mu\nu} + \frac{1}{9}\Pi_{ud}^{\mu\nu}$

Infinite volume (and the $ab$ considered here):

$$\Pi_{ab}^{\mu\nu} = (q^{\mu}q^{\nu} - q^2 g^{\mu\nu}) \Pi_{ab}^{(1)}$$

Large $N_c +$ VMD estimate:

$$\Pi_{\pi^+\pi^+}^{(1)} = \frac{4F_\pi^2}{M_{V}^2 - q^2}$$

Plots on next pages are for $\Pi_{ab0}^{(1)}(q^2) = \Pi_{ab}^{(1)}(q^2) - \Pi_{ab}^{(1)}(0)$

At $p^4$ the extra LEC cancels, at $p^6$ there are new LEC contributions, but no new ones in the loop parts
Two-point: Connected versus disconnected

- Connected
- $p^6$ is large
- Due to the $L_i^r$ loops

\[ \Pi_{\pi^+\pi^0}^{(1)} \]
Two-point: Connected versus disconnected

- **Disconnected**
- $p^6$ is large
- Due to the $L_i^r$ loops
- about $-\frac{1}{2}$ connected
- $-\frac{1}{10}$ is from

\[
\Pi_{ee}^{(1)} = \frac{5}{9} \Pi_{\pi^+\pi^+}^{(1)} + \frac{1}{9} \Pi_{ud}^{(1)}
\]
Two-point: with strange, electromagnetic current

\begin{itemize}
  \item $\pi$
  \item connected $u,d$
  \item $ud$
  \item disconnected $u,d$
  \item $ss$
  \item strange current
  \item $us$
  \item mixed $s-u,d$
  \item new $p^6$ LEC cancels
  \item Disconnected strange $\approx -15\%$
  \item of total strange
\end{itemize}

JB, Relefors, LU TP 16-51 to appear
One-loop calculation in finite volume done by
and found to fit lattice data well

Two-loop in partially quenched
JB, Relefors, LU TP 16-51 to appear

I will stay with ChPT and the $p$ regime ($M_\pi L >> 1$)

$1/m_\pi = 1.4$ fm
may need to (and I will) go beyond leading $e^{-m_\pi L}$ terms
“around the world as often as you like”

Convergence of ChPT is given by $1/m_\rho \approx 0.25$ fm
Finite volume and Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i / L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta^i_q} q(x^i)$
- Then momenta are $p^i = \theta^i / L + 2\pi n^i / L$. Allows to map out momentum space on the lattice much better
  Bedaque, …
- Small note:
  - Beware what people call momentum: is $\theta^i / L$ included or not?
  - Reason: a colour singlet gauge transformation $G^S_{\mu} \rightarrow G^S_{\mu} - \partial_{\mu} \epsilon(x)$, $q(x) \rightarrow e^{i\epsilon(x)} q(x)$, $\epsilon(x) = -\theta_q x^i / L$
  - Boundary condition
    Twisted $\Leftrightarrow$ constant background field + periodic
Finite volume and twisting: Drawbacks

Drawbacks:
- Box: Rotation invariance → cubic invariance
- Twisting: reduces symmetry further

Consequences:
- $m^2(\bar{p}^2) = E^2 - \bar{p}^2$ is not constant
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum
Two-point function: twisted boundary conditions

\begin{itemize}
  \item \[ \int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0 \]
  \item \[ \langle \bar{u} \gamma^\mu u \rangle \neq 0 \]
  \item \[ j^\mu_{\pi^+} = \bar{d} \gamma^\mu u \]
    satisfies \[ \partial_\mu \langle T(j^\mu_{\pi^+}(x) j^{\nu \dagger}_{\pi^+}(0)) \rangle = \delta^{(4)}(x) \langle \bar{d} \gamma^\nu d - \bar{u} \gamma^\nu u \rangle \]
  \item \[ \Pi^{\mu\nu}_a(q) \equiv i \int d^4 x e^{i q \cdot x} \langle T(j^\mu_a(x) j^{\nu \dagger}_a(0)) \rangle \]
    Satisfies WT identity. \[ q_\mu \Pi^\mu\nu_{\pi^+} = \langle \bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d \rangle \]
  \item ChPT at one-loop satisfies this
  \item two-loop in partially quenched
    JB, Relefors, LU TP 16-51, to appear
    satisfies the WT identity (as it should)
\end{itemize}
\[ \langle \bar{u} \gamma^\mu u \rangle \]

- Fully twisted
  \( \theta_u = (0, \theta_u, 0, 0) \), all others untwisted
  \( m_\pi L = 4 \)

- Partially twisted
  (ratio at \( p^4 \equiv 2 \) up to kaon loops)

For comparison:
\[ \langle \bar{u}u \rangle^V \approx -2.4 \times 10^{-5} \text{ GeV}^3 \]
\[ \langle \bar{u}u \rangle \approx -1.2 \times 10^{-2} \text{ GeV}^3 \]
Two-point partially twisted: components

Twisting spatially symmetric

Twisting in x-direction

- Small $p^6$ corrections (thin lines: $p^4$ only)
- $m_{\pi 0} L = 4 \ m_{\pi 0} = 0.135 \ \text{GeV}$
- $-q^2 \Pi^{(1)}_{\text{VMD}} = \frac{-4q^2F^2_\pi}{M^2_V - q^2} \approx 5 e^{-3} \cdot \frac{q^2}{0.1} \implies \text{Correction at } \% \text{ level}$
- Can use the difference between different twists with same $q^2$ to check the finite volume corrections
Two-point partially twisted: spatial average

- Plotted: volume correction to $\Pi = \frac{1}{3} \sum_{i=x,y,z} \Pi_{ii}$
- Small $p^6$ corrections: compare left and right
- Can use the difference between different twists with same $q^2$ to check the finite volume corrections
Two flavour ChPT: mass and decay constant

- First step towards finding out why hard-pion ChPT does not work at three-loops
- Lowest order: Gell-Mann, Oakes, Renner (1968)
- Chiral logarithm Langacker, Pagels (1973)
- Full NLO (and properly starting ChPT) Gasser-Leutwyler (1984)
- NNNLO JB, Hermansson-Truedsson (2017)
Methods used

- LO and Chiral logs: current algebra
- NLO: Feynman diagrams (by hand) and direct expansion of functional integral (with REDUCE)
- NNLO: Feynman diagrams (with a little help from FORM)
- NNLO: Feynman diagrams purely with FORM
- Main stumbling block: integrals
  - Reduction to master integrals with REDUCE Studerus (2009)
  - Master Integrals known
    Laporta-Remiddi (1996); Melnikov, van Ritbergen (2001)
- Lots of book-keeping: FORM
- Checks:
  - All nonlocal divergences must cancel
  - Use different parametrizations of the Lagrangian
  - Agree with known leading log result JB, Carloni (2009)
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Diagrams

\( p^2; \quad \bullet p^4; \quad \Box p^6; \quad \bigcirc p^8 \)
Results: LO or $x$-expansion/physical or $\xi$-expansion

- $x = \frac{M^2}{16\pi^2 F^2}$, \quad $L_x = \log \frac{M^2}{\mu^2}$, \quad $M^2 = 2B\hat{m}$

- $\frac{M^2_\pi}{M^2} = 1 + x \left( a_{11}^M L_x + a_{10}^M \right) + x^2 \left( a_{22}^M L_x^2 + a_{21}^M L_x + a_{20}^M \right) + x^3 \left( a_{33}^M L_x^3 + a_{32}^M L_x^2 + a_{31}^M L_x + a_{30}^M \right) + \cdots$

- $\frac{F_\pi}{F} = 1 + x \left( a_{11}^F L_x + a_{10}^F \right) + x^2 \left( a_{22}^F L_x^2 + a_{21}^F L_x + a_{20}^F \right) + x^3 \left( a_{33}^F L_x^3 + a_{32}^F L_x^2 + a_{31}^F L_x + a_{30}^F \right) + \cdots$

- $\xi = \frac{M^2_\pi}{16\pi^2 F^2_\pi}$, \quad $L_\pi = \log \frac{M^2_\pi}{\mu^2}$

- $\frac{M^2}{M^2_\pi} = 1 + \xi \left( b_{11}^M L_\pi + b_{10}^M \right) + \xi^2 \left( b_{22}^M L_\pi^2 + b_{21}^M L_\pi + b_{20}^M \right) + \xi^3 \left( b_{33}^M L_\pi^3 + b_{32}^M L_\pi^2 + b_{31}^M L_\pi + b_{30}^M \right) + \cdots$

- $\frac{F}{F_\pi} = 1 + \xi \left( b_{11}^F L_\pi + b_{10}^F \right) + \xi^2 \left( b_{22}^F L_\pi^2 + b_{21}^F L_\pi + b_{20}^F \right) + \xi^3 \left( b_{33}^F L_\pi^3 + b_{32}^F L_\pi^2 + b_{31}^F L_\pi + b_{30}^F \right) + \cdots$
Results \[ \tilde{l}_i = 16\pi^2 l'_i, \tilde{c}_i = (16\pi^2)^2 c'_i \]

| \(a^M_{11}\) | \(\frac{1}{2}\) |
| \(a^M_{10}\) | \(2\tilde{l}_3\) |
| \(a^M_{22}\) | \(\frac{17}{8}\) |
| \(a^M_{21}\) | \(-3\tilde{l}_3 - 8\tilde{l}_2 - 14\tilde{l}_1 - \frac{49}{12}\) |
| \(a^M_{20}\) | \(64\tilde{c}_{18} + 32\tilde{c}_{17} + 96\tilde{c}_{11} + 48\tilde{c}_{10} - 16\tilde{c}_9 - 32\tilde{c}_8 - 16\tilde{c}_7 - 32\tilde{c}_6 + \tilde{l}_3 + 2\tilde{l}_2 + \tilde{l}_1 + \frac{193}{96}\) |
| \(a^M_{33}\) | \(\frac{103}{24}\) |
| \(a^M_{32}\) | \(\frac{23}{2} \tilde{l}_3 - 11\tilde{l}_2 - 38\tilde{l}_1 - \frac{91}{24}\) |
| \(a^M_{31}\) | \(-416\tilde{c}_{18} - 208\tilde{c}_{17} - 32\tilde{c}_{16} + 96\tilde{c}_{14} + 8\tilde{c}_{13} - 48\tilde{c}_{12} - 384\tilde{c}_{11} - 192\tilde{c}_{10} + 72\tilde{c}_9 + 144\tilde{c}_8 + 72\tilde{c}_7 + 64\tilde{c}_6 - 8\tilde{c}_5 - 56\tilde{c}_4 + 16\tilde{c}_3 + 32\tilde{c}_2 - 96\tilde{c}_1 - 8\tilde{l}_3^2 - 48\tilde{l}_3\tilde{l}_2 - 84\tilde{l}_3\tilde{l}_1 - \frac{88}{3} \tilde{l}_3 - \frac{231}{10} \tilde{l}_2 - \frac{69}{5} \tilde{l}_1 - \frac{74971}{8640}\) |
| \(a^M_{30}\) | contains free \(p^8\) LECs (and a lot more terms) |
Results: comments

- Similar tables for $a_i^F$, $b_i^M$, $b_i^F$
- Coefficients depend on scale $\mu$, but whole expression is $\mu$-independent
- Can be rewritten in terms of scales in the logarithm rather than in terms of LECs à la FLAG
- Leading log: a number
- NLL: depends on $l'_i$
- NNLL: depends on $c'_i$
- For the mass all needed $c'_i$ can be had from mass, decay-constant and $\pi\pi$ parameters fitted to two-loop or $p^6$ (i.e. $r_M$, $r_F$, $r_1$, ..., $r_6$).
- For decay need one more (busy checking if it can be had)
Results: numerics preliminary

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<tr>
<th>$ij$</th>
<th>$a^M_{ij}$</th>
<th>$b^M_{ij}$</th>
<th>$a^F_{ij}$</th>
<th>$b^F_{ij}$</th>
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<td>−0.00282</td>
<td>+1.09436</td>
<td>−1.09436</td>
</tr>
<tr>
<td>11</td>
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<td>−1</td>
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<td>−0.41666</td>
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</tbody>
</table>

Note the large coefficients in the decay constant
Pion mass

\[ F_\pi = 92.2 \text{ MeV}, \quad F = F_\pi / 1.037, \quad \tilde{t}_1 = -0.4, \quad \tilde{t}_2 = 4.3, \quad \tilde{t}_3 = 3.41, \quad \tilde{t}_4 = 4.51, \]

\[ r_i \text{ from JB et al 1997, other } c_i^r = 0, \mu = 0.77 \text{ GeV} \]

\[ F_\pi = 92.2 \text{ MeV}, \quad F = F_\pi / 1.037, \quad \tilde{t}_1 = -0.4, \quad \tilde{t}_2 = 4.3, \quad \tilde{t}_3 = 3.41, \quad \tilde{t}_4 = 4.51, \]

\[ r_i \text{ from JB et al 1997, other } c_i^r = 0, \mu = 0.77 \text{ GeV} \]

\[ x\text{-expansion (}F\text{-fixed)} \]

\[ \xi\text{-expansion (}F_\pi \text{ fixed)} \]

\[ \xi\text{-expansion converges notably better} \]
Pion decay constant

x-expansion ($F$-fixed)

- $\xi$-expansion converges better
- Large $p^6$ due to the “240” in $a_{30}^F$ and $b_{30}^F$
Some GUT models have neutron-anti-neutron oscillations but no proton decay.

Limits:
- $8.6 \times 10^7$ s from free neutrons (ILL)
- $2.7 \times 10^8$ s from oxygen nuclei (super-K)
  - (n̄ mass inside nuclei very different from n-mass)
- Possible ESS experiment improvement by up to $10^3$

Effective dimension 9 operator: "uududd" 

Classification of quark operators and RGE to two loops:
- and earlier papers in there

Operators

- 14 operators
- Chiral representations under $SU(2)_L \times SU(2)_R$:

<table>
<thead>
<tr>
<th>Chiral</th>
<th>#operators</th>
<th>Chiral</th>
<th>#operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1_L, 3_R)$</td>
<td>3: $Q_1, Q_2, Q_3$</td>
<td>$(3_L, 1_R)$</td>
<td>3</td>
</tr>
<tr>
<td>$(5_L, 3_R)$</td>
<td>3: $Q_5, Q_6, Q_7$</td>
<td>$(3_L, 5_R)$</td>
<td>3</td>
</tr>
<tr>
<td>$(1_L, 7_R)$</td>
<td>1: $Q_4$</td>
<td>$(7_L, 1_R)$</td>
<td>1</td>
</tr>
</tbody>
</table>

- $n\bar{n}$ is $\Delta l = 1$ so bottom line needs isospin breaking
- For the others the different operators are different elements within the same representation
- Use heavy baryon formalism with a baryon ($\mathcal{N}$) and anti-baryon doublet ($\mathcal{N}^c$) with same velocity $v$
- These correspond two widely separated areas ($v$ and $-v$) in the relativistic field: so no double counting
ChPT terms

\[ \mathcal{L} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle + \overline{N} (iv \cdot D + g_A u \cdot S) N + \overline{N}^c \tau^2 (iv \cdot D + g_A u \cdot S) \tau^2 N^c + \text{higher orders} \]

\[ \mathcal{N} = \begin{pmatrix} p \\ n \end{pmatrix} \rightarrow h \mathcal{N}, \quad \overline{N}^c = \begin{pmatrix} p^c \\ n^c \end{pmatrix} \rightarrow \overline{N}^c h^T \]

\[ h(g_L, g_R, u): SU(2)_V \text{ compensator chiral transformation} \]

Spurions for each quark \( n\bar{n} \) operator:

- \( (1_L, 3_R) \): two \( SU(2)_R \) doublet indices
- \( (5_L, 3_R) \): four \( SU(2)_L \) and two \( SU(2)_L \) doublet indices
- \( (1_L, 7_R) \): six \( SU(2)_R \) double indices
- plus parity conjugates
ChPT terms and diagrams

- **$p^0$:**
  - $(1_L, 3_R)$: $(u\bar{N}c)_{il} (uN)_{jl}$
  - $(5_L, 3_R)$: $(u\bar{N}c)_{il} (u\bar{N})_{jl} (U_T^{-2})_{kr}lL (U_T^{-2})_{mn}lL$
  - $(7_L, 1_R)$: none (first one at $p^2$)

- **$p^1$:** none that directly contribute (but in loops from $p^3$)

- **$p^2$:** many (at least 20 each for $(3_L, 1_R)$ and $(3_L, 5_R)$)
Results (Preliminary)

- $Q_1, Q_2, Q_3$ same factor from loops (isospin)
- $Q_5, Q_6, Q_7$ same factor
  (Conjecture: due to projection on $I = 1$ subspace)
- $Q_1, Q_2, Q_3$:
  \[ 1 + \frac{M_{\pi}^2}{16\pi^2 F_{\pi}^2} \left( \left( -\frac{3}{2} g_A^2 - 1 \right) \log \frac{M_{\pi}^2}{\mu^2} - g_A^2 \right) + \text{order } p^2 \text{ LECs} \]
- $Q_5, Q_6, Q_7$:
  \[ 1 + \frac{M_{\pi}^2}{16\pi^2 F_{\pi}^2} \left( \left( -\frac{3}{2} g_A^2 - 7 \right) \log \frac{M_{\pi}^2}{\mu^2} - g_A^2 \right) + \text{order } p^2 \text{ LECs} \]
- Also done at finite volume
Results (Preliminary)

Relative correction from loops

Relative correction from loops (absolute value)
Conclusions

Showed you results for:

- HVP: ChPT at two-loops including partially quenched
  - Connected versus disconnected at two-loops
  - Connected: twisting and finite volume at two-loops
- Two flavour ChPT correction at three loops for the pion mass and decay constant
- Two flavour ChPT correction at one-loop for $n\bar{n}$-oscillations
- Be careful: ChPT must exactly correspond to your lattice calculation
- Programs available (for published ones) via CHIRON
  Those for this talk: sometime later this year (I hope) (or ask me)