HADRONS, FLAVOURS
AND EFFECTIVE THEORIES

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Overview

- Hadronic Physics
- Flavour Physics
- Effective (Field) Theories
- Some recent applications of EFT: not representative, just two of my recent papers

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- Flavour Physics
- Effective (Field) Theories
- Some recent applications of EFT: not representative, just two of my recent papers
  - Hard Pion Chiral Perturbation Theory
    JB+ Alejandro Celis, arXiv:0906.0302 and JB + Ilaria Jemos coming
  - Leading Logarithms to five loop order and large N
    JB + Lisa Carloni, arXiv:0909.5086 and coming

Hadrons

- Hadron: $\alpha\delta\rho o\varsigma$ (hadros: stout, thick)
- Lepton: $\lambda\epsilon\pi\tau o\varsigma$ (leptos: small, thin, delicate) ($\varsigma = \sigma \neq \zeta$)

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- Leptons: those that don't

But they are fundamentally different in other ways too:

- Leptons are known point particles up to about $10^{-19} m \sim \hbar c/(1\ \mathrm{TeV})$
- Hadrons have a typical size of $10^{-15} m$, proton charge radius is 0.875 fm

Hadrons

- Hadrons come in two types:
  - Fermions or half-integer spin: baryons ($\beta\alpha\rho\upsilon\varsigma$ barys, heavy)
  - Bosons or integer spin: mesons ($\mu\epsilon\varsigma o\varsigma$ mesos, intermediate)

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  - Mesons: quark and anti-quark
- Comments:
  - Quarks are as pointlike as leptons
  - Hadrons with different main constituents: glueballs (no quarks), hybrids (with a basic gluon) (probably) exist (mixing is the problem)

Hadron(ic) Physics

The study of the structure and interactions of hadrons

Flavour Physics

- There are six types (Flavours) of quarks in three generations or families
- up, down; strange, charm; bottom and top
- The only (known) interaction that changes quarks into each other (violates the separate quark numbers) is the weak interaction
- Violates also discrete symmetries: Charge conjugation, Parity and Time reversal.

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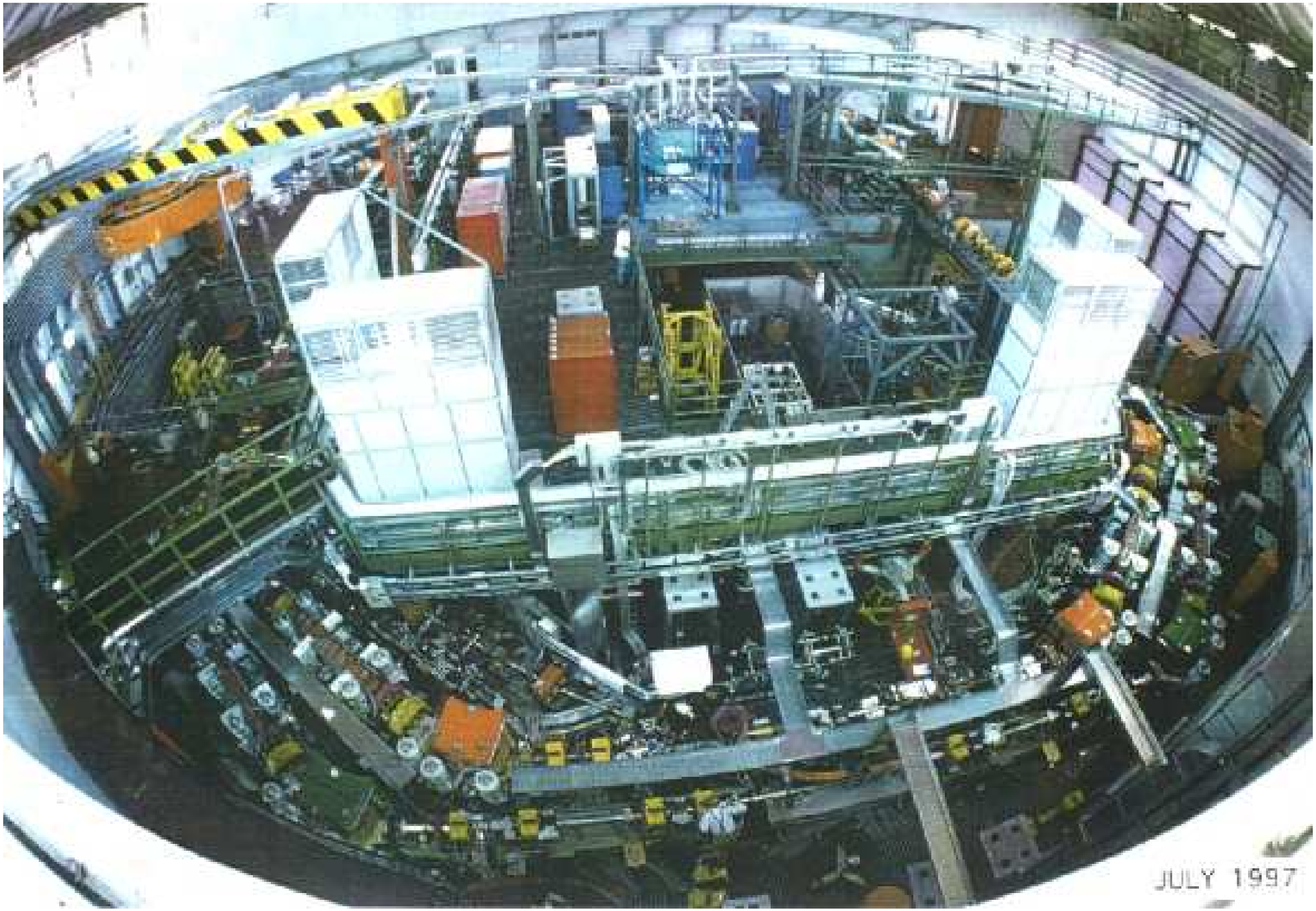
The study of quarks changing flavours (mainly) in decays

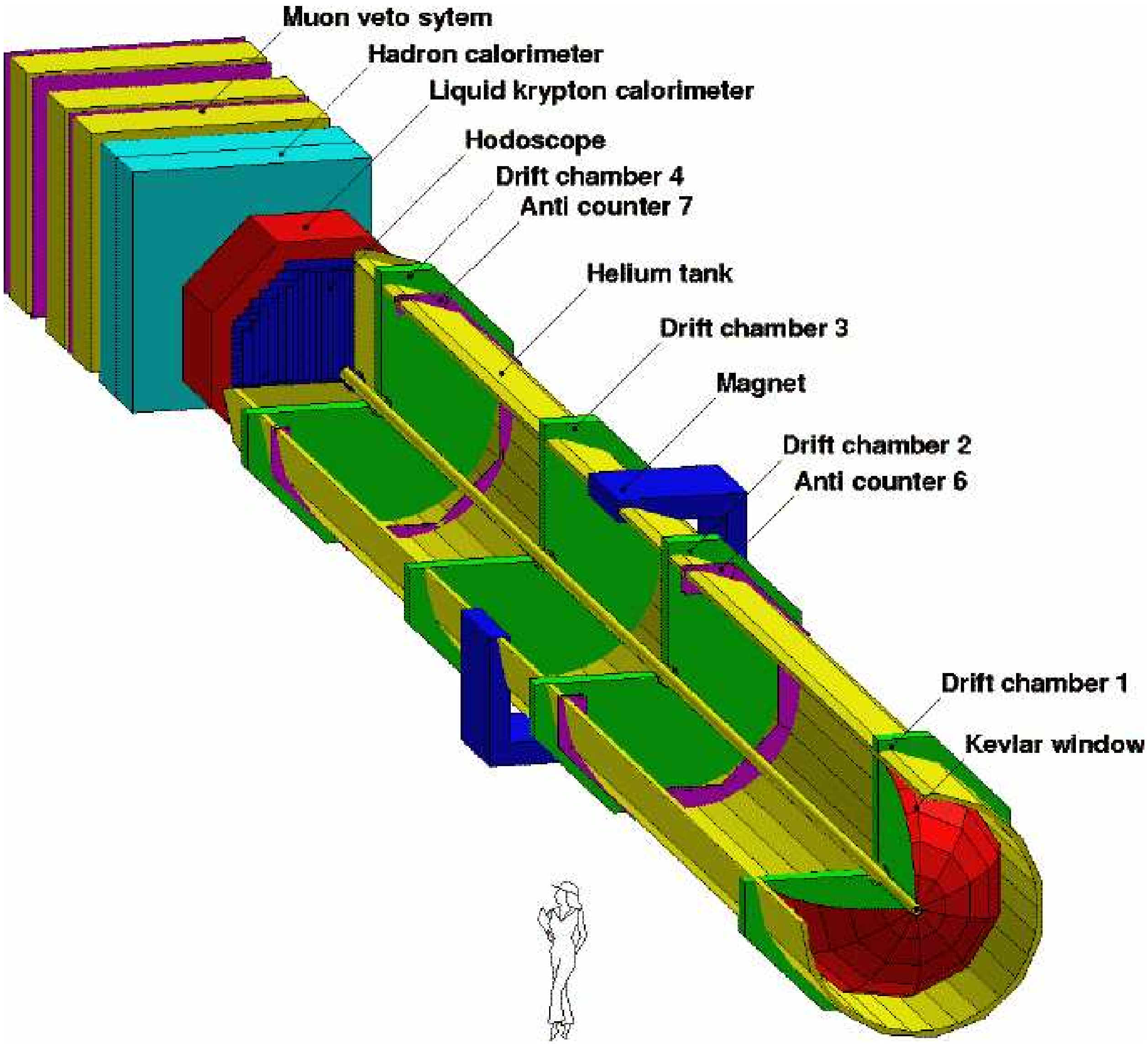
- Experimental research typically done at flavour factories

Flavour Physics: DAΦNE in Frascati



Flavour Physics: KEK B in Tsukuba

Flavour Physics: NA48/62 at CERN



Flavour Physics

- The Standard Model Lagrangian has four parts:

$$\underbrace{\mathcal{L}_H(\phi)}_{\text{Higgs}} + \underbrace{\mathcal{L}_G(W,Z,G)}_{\text{Gauge}}$$

$$\underbrace{\sum_{\psi=\text{fermions}} \bar\psi i \not{D}\, \psi}_{\text{gauge-fermion}} + \underbrace{\sum_{\psi,\psi'=\text{fermions}} g_{\psi\psi'}\bar\psi\phi\psi'}_{\text{Yukawa}}$$

- Last piece: weak interaction and mass eigenstates different
- Many extensions: much more complicated flavour changing sector

Flavour Physics

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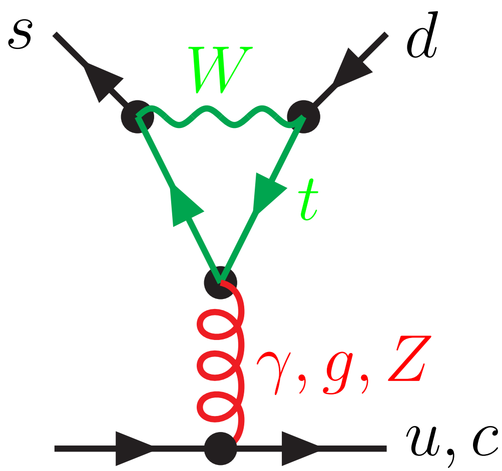

Heavy particles can contribute in loop

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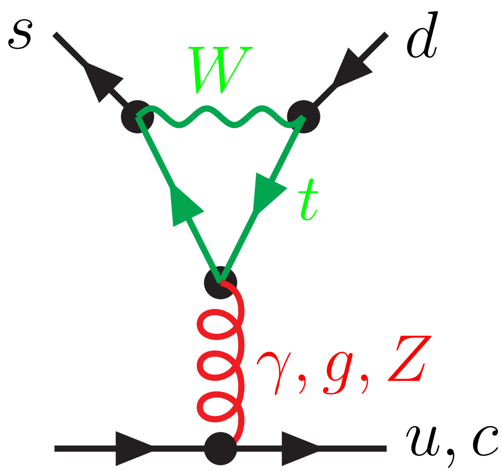


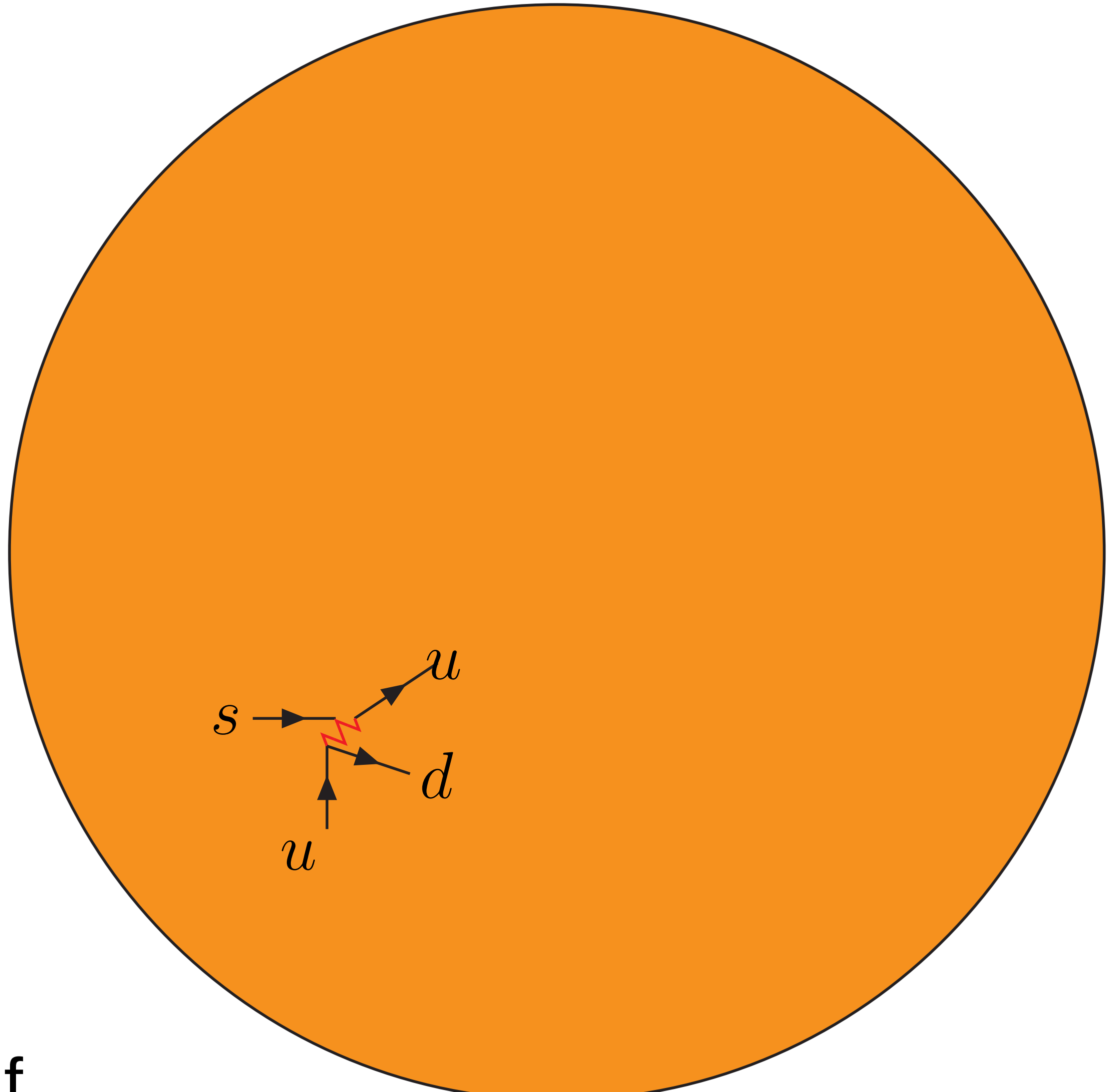
Heavy particles can contribute in loop

- Sometimes need a precise prediction for the standard model effect

Flavour Physics

A weak decay:

Hadron: 1 fm
W-boson: 10^{-3} fm



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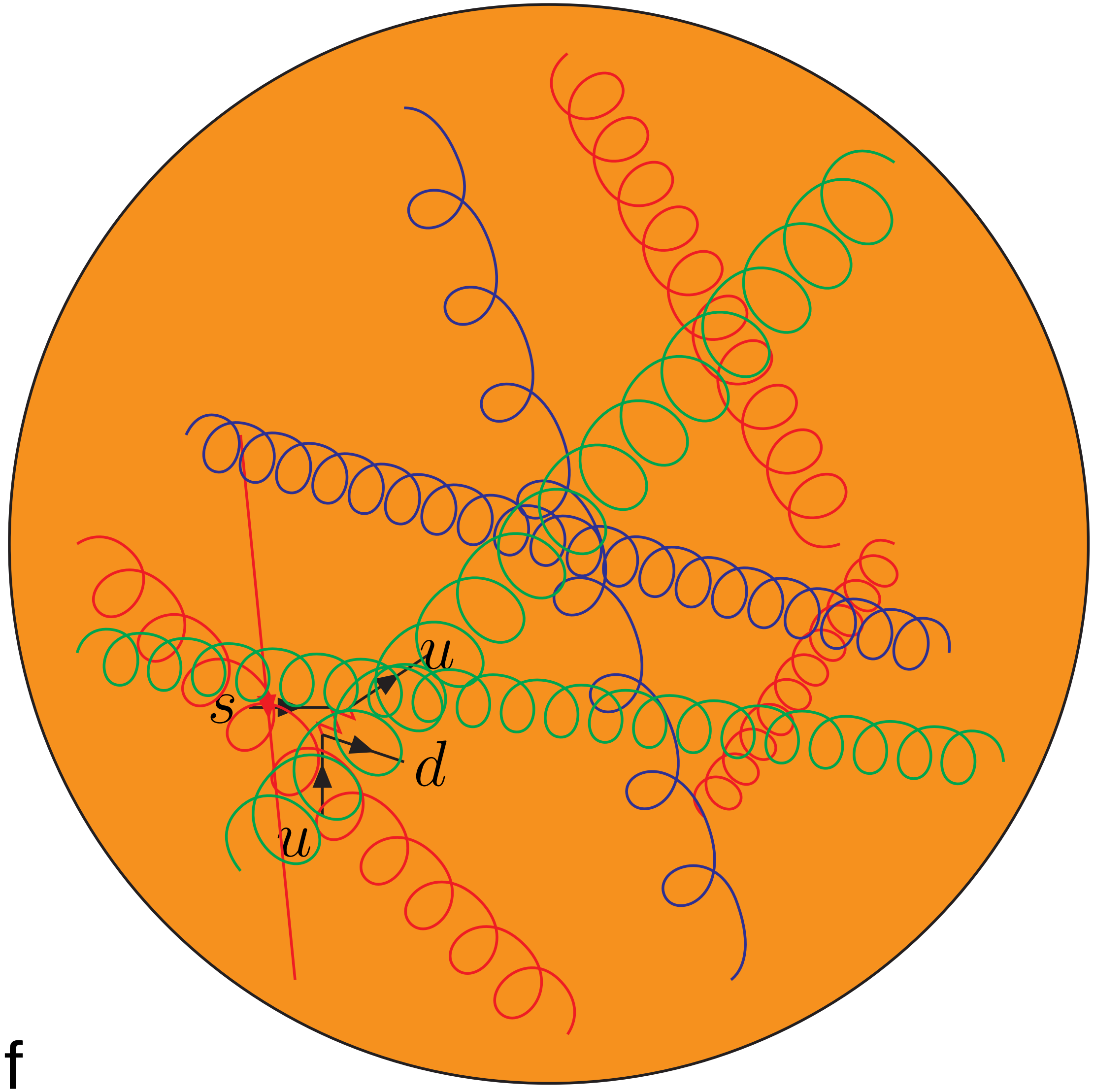

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- QCD: $\overline{q}\gamma_\mu \left(\partial^\mu - i\frac{g}{2}G^\mu\right) q - \frac{1}{8}\mathrm{tr}\left(G_{\mu\nu}G^{\mu\nu}\right)$
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- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig\left(G_\mu G_\nu - G_\nu G_\mu\right)$
- gluons interact with themselves
- $e(\mu)$ smaller for smaller μ, $g(\mu)$ larger for smaller μ
- QCD: low scales no perturbation theory possible

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    - quarks and gluons: $8 \times 2 + 3 \times 4$ d.o.f. per point
    - Do the resulting (very high dimensional) integral numerically
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  - Be less ambitious: try to solve some parts only: EFT

Wikipedia

http://en.wikipedia.org/wiki/Effective_field_theory

In physics, an effective field theory is an approximate theory (usually a quantum field theory) that contains the appropriate degrees of freedom to describe physical phenomena occurring at a chosen length scale, but ignores the substructure and the degrees of freedom at shorter distances (or, equivalently, higher energies).

Effective Field Theory (EFT)

Main Ideas:

- Use right degrees of freedom : essence of (most) physics
- If mass-gap in the excitation spectrum: neglect degrees of freedom above the gap.

Examples: $\left\{ \right.$ Solid state physics: conductors: neglect the empty bands above the partially filled one
Atomic physics: Blue sky: neglect atomic structure

EFT: Power Counting

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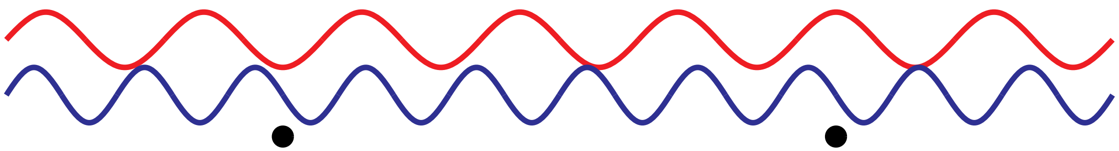
- Taylor series expansion does not work (convergence radius is zero when massless modes are present)
- Continuum of excitation states need to be taken into account

Example: Why is the sky blue ?

System: Photons of visible light and neutral atoms

Length scales: a few 1000 Å versus 1 Å

Atomic excitations suppressed by $\approx 10^{-3}$



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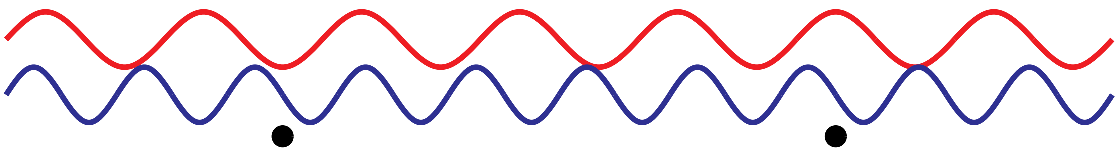

$$\mathcal{L}_A = \Phi_v^\dagger \partial_t \Phi_v + \ldots \qquad \mathcal{L}_{\gamma A} = G F_{\mu\nu}^2 \Phi_v^\dagger \Phi_v + \ldots$$

Units with $\hbar = c = 1$: G energy dimension -3:

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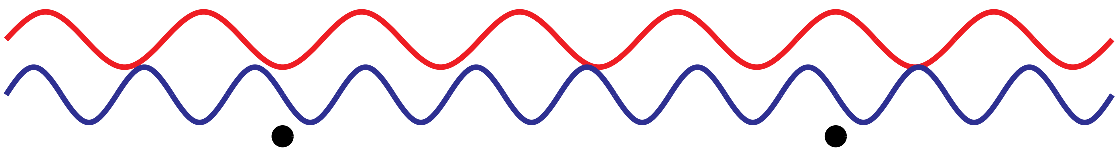

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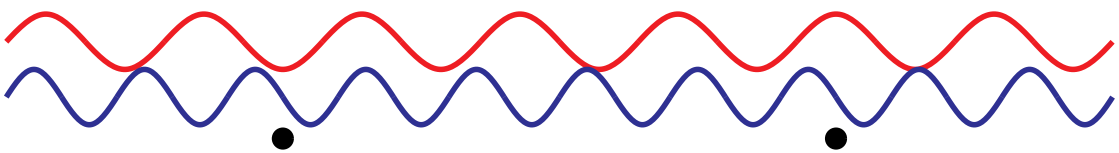

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blue light scatters a lot more than red $\left\{\begin{array}{l} \Longrightarrow \text{red sunsets} \\ \Longrightarrow \text{blue sky} \end{array}\right.$

Higher orders suppressed by 1 Å$/\lambda_\gamma$.

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Advantages

- Calculations are (relatively) simple
- It is general: model-independent
- Theory $\Longrightarrow$ errors can be estimated
- Systematic: ALL effects at a given order can be included
- Even if no convergence: classification of models often useful

Examples of EFT

- Fermi theory of the weak interaction
- Chiral Perturbation Theory: hadronic physics
- NRQCD
- SCET
- General relativity as an EFT
- 2,3,4 nucleon systems from EFT point of view
- Magnons and spin waves

references

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Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

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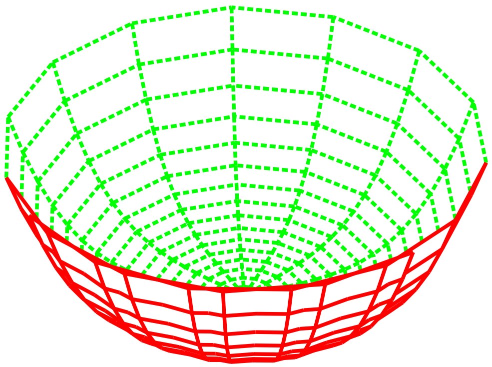
Derivation from QCD:

H. Leutwyler, *On The Foundations Of Chiral Perturbation Theory,*

Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

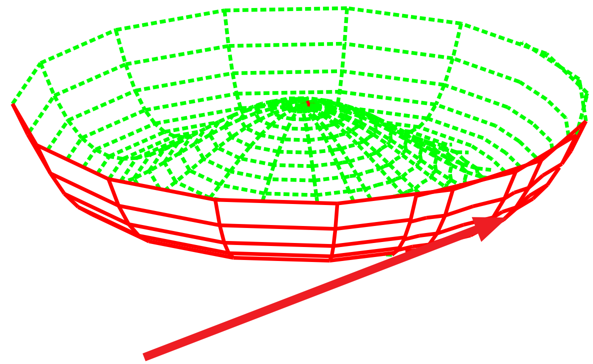
The mass gap: Goldstone Modes

UNBROKEN: $V(\phi)$



Only massive modes around lowest energy state (=vacuum)

BROKEN: $V(\phi)$



Need to pick a vacuum
$\langle\phi\rangle \neq 0$: Breaks symmetry
No parity doublets
Massless mode along bottom

For more complicated symmetries: need to describe the bottom mathematically: $G \to H \Longrightarrow G/H$

The two symmetry modes compared

Wigner-Eckart mode	Nambu-Goldstone mode
Symmetry group G	G spontaneously broken to subgroup H
Vacuum state unique	Vacuum state degenerate
Massive Excitations	Existence of a massless mode
States fall in multiplets of G	States fall in multiplets of H
Wigner Eckart theorem for G	Wigner Eckart theorem for H
	Broken part leads to low-energy theorems
Symmetry linearly realized	Full Symmetry, G, nonlinearly realized
	unbroken part, H, linearly realized

Some clarifications

- $\phi(x)$: orientation of vacuum in every space-time point
- Examples: spin waves, phonons
- Nonlinear: acting by a broken symmetry operator changes the vacuum, $\phi(x) \to \phi(x) + \alpha$
- The precise form of ϕ is *not* important but it must describe the space of vacua (field transformations possible)
- In gauge theories: the *local* symmetry allows the vacua to be different in every point, hence the Goldstone Boson might not be observable as a massless degree of freedom.

The power counting

Very important:

Low energy theorems: Goldstone bosons do not interact at zero momentum

Heuristic proof:

- Which vacuum does not matter, choices related by symmetry
- $\phi(x) \to \phi(x) + \alpha$ should not matter
- Each term in $\mathcal{L}$ must contain at least one $\partial_\mu \phi$

Chiral Perturbation Theory

Degrees of freedom: Goldstone Bosons from Chiral Symmetry Spontaneous Breakdown

Power counting: Dimensional counting

Expected breakdown scale: Resonances, so M_ρ or higher depending on the channel

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Chiral Symmetry

QCD: 3 light quarks: equal mass: interchange: $SU(3)_V$

But $$\mathcal{L}_{QCD} = \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

So if $m_q = 0$ then $SU(3)_L \times SU(3)_R$.

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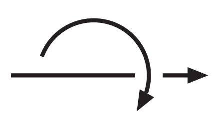
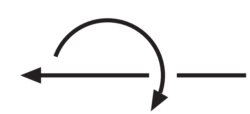
Can also see that via  $v < c,\ m_q \neq 0 \Longrightarrow$ $v = c,\ m_q = 0 \not\Longrightarrow$ 

Chiral Perturbation Theory

$\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
$SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$

8 generators broken $\Longrightarrow$ 8 massless degrees of freedom
and interaction vanishes at zero momentum

We have 8 candidates that are light compared to hte other hadrons: $\pi^0, \pi^+, \pi^-, K^+, K^-, K^0, \overline{K^0}, \eta$

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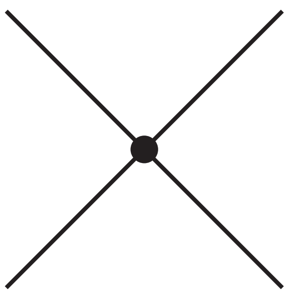
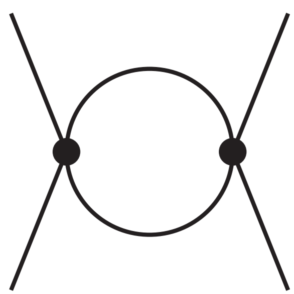
Power counting in momenta (all lines soft):



p^2

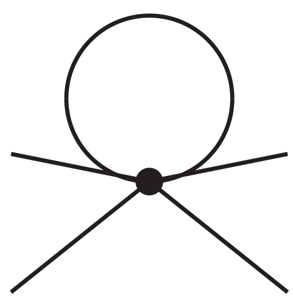


$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$

$1/p^2$

$\int d^4p$

p^4



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Chiral Perturbation Theories

- Baryons
- Heavy Quarks
- Vector Mesons (and other resonances)
- Structure Functions and Related Quantities
- Light Pseudoscalar Mesons
  - Two or Three (or even more) Flavours
  - Strong interaction and couplings to external currents/densities
  - Including electromagnetism
  - Including weak nonleptonic interactions
  - Treating kaon as heavy

Many similarities with strongly interacting Higgs

Hard pion ChPT?

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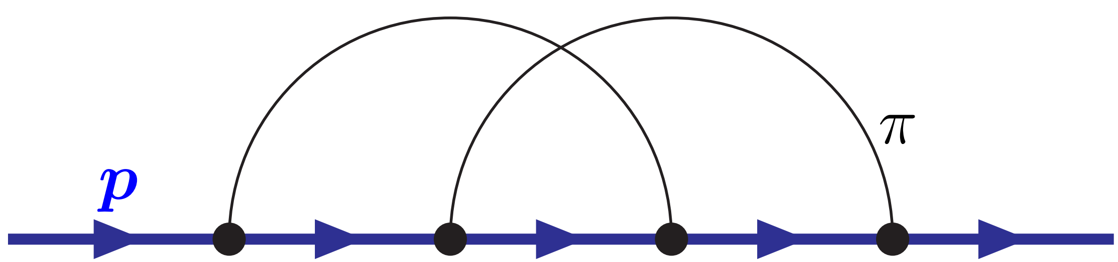
  - Everything else soft
  - Works because baryon or b or c number conserved so the non soft line is continuous



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  - Works because baryon or b or c number conserved so the non soft line is continuous
  - Decay constant works: takes away all heavy momentum
  - General idea: M_p dependence can always be reabsorbed in LECs, is analytic in the other parts k.

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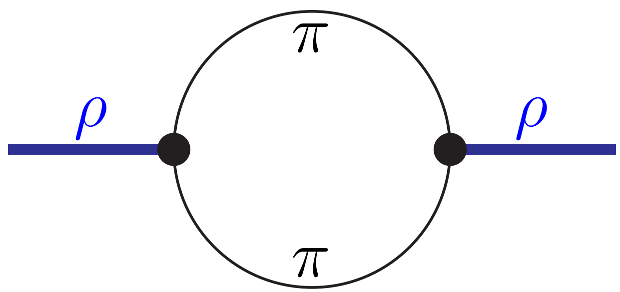

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  - (Vector) Meson: $p = M_V v + k$
  - Everyone else soft or $p = M_V + k$
  - But (Heavy) (Vector) Meson ChPT decays strongly
    - First: keep diagrams where vectors always present
    - Applied to masses and decay constants
    - Decay constant works: takes away all heavy momentum
  - *It was argued that this could be done, the nonanalytic parts of diagrams with pions at large momenta are reproduced correctly*
  - Done both in relativistic and heavy meson formalism
  - General idea: M_V dependence can always be reabsorbed in LECs, is analytic in the other parts k.

Hard pion ChPT?

- Heavy Kaon ChPT:
  - $p = M_K v + k$
  - First: only keep diagrams where Kaon always goes through
  - Applied to masses and πK scattering and decay constant Roessl,Allton et al.,...
  - Applied to $K_{\ell 3}$ at q^2_{max} Flynn-Sachrajda
  - Works like all the previous *heavy* ChPT

Hard pion ChPT?

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  - First: only keep diagrams where Kaon always goes through
  - Applied to masses and πK scattering and decay constant Roessl,Allton et al.,. . .
  - Applied to $K_{\ell 3}$ at q^2_{max} Flynn-Sachrajda
- Flynn-Sachrajda also argued that $K_{\ell 3}$ could be done for q^2 away from q^2_{max}.
- JB-Celis Argument generalizes to other processes with hard/fast pions and applied to $K \to \pi\pi$
- General idea: heavy/fast dependence can always be reabsorbed in LECs, is analytic in the other parts k.

Hard pion ChPT?

- nonanalyticities in the light masses come from soft lines
- soft pion couplings are constrained by current algebra

$$\lim_{q\to 0}\langle\pi^k(q)\alpha|O|\beta\rangle = -\frac{i}{F_\pi}\langle\alpha|\left[Q_5^k,O\right]|\beta\rangle\,,$$

Hard pion ChPT?

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$$\lim_{q\to 0}\langle \pi^k(q)\alpha|O|\beta\rangle = -\frac{i}{F_\pi}\langle\alpha|\left[Q_5^k, O\right]|\beta\rangle\,,$$

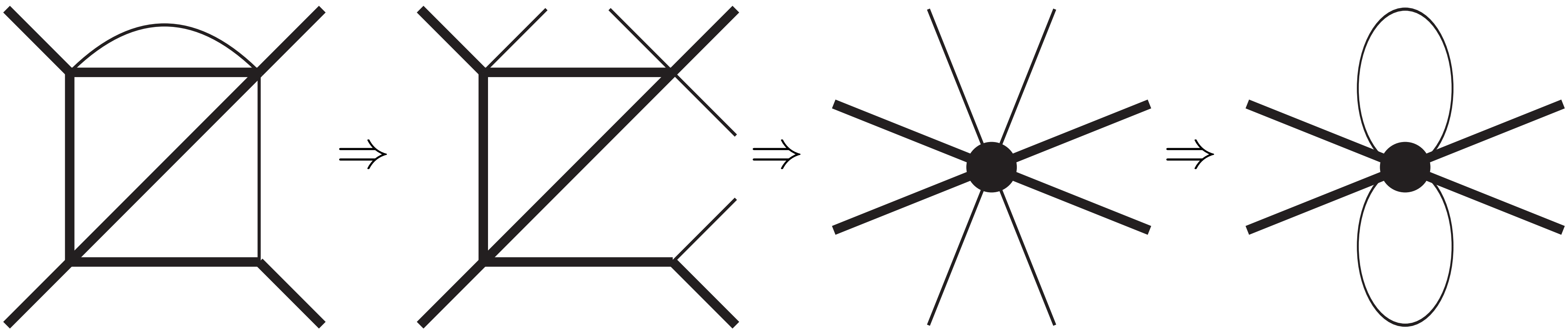
- Nothing prevents hard pions to be in the states α or β
- So by heavily using current algebra I should be able to get the light quark mass nonanalytic dependence

Hard pion ChPT?

Field Theory: a process at given external momenta

- Take a diagram with a particular internal momentum configuration
- Identify the soft lines and cut them
- The result part is analytic in the soft stuff
- So should be describably by an effective Lagrangian with coupling constants dependent on the external given momenta
- If symmetries present, Lagrangian should respect them
- Lagrangian should be complete in *neighbourhood*
- Loop diagrams with this effective Lagrangian *should* reproduce the nonanalyticities in the light masses
  Crucial part of the argument

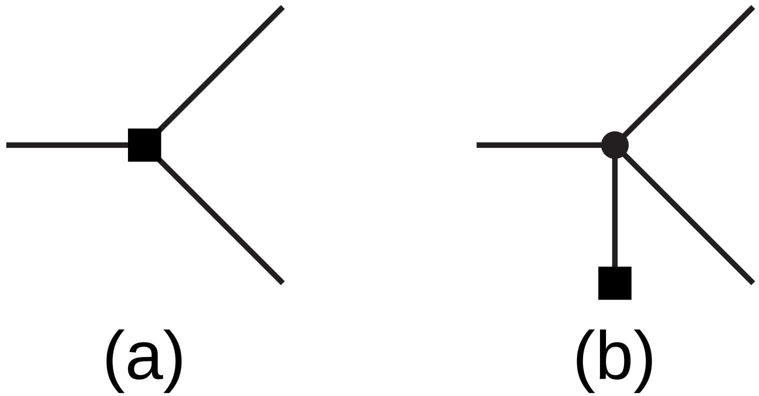
Hard pion ChPT?



This procedure works at one loop level, matching at tree level, nonanalytic dependence at one loop:

- Toy models and vector meson ChPT JB, Gosdzinsky, Talavera
- Recent work on relativistic meson ChPT Gegelia, Scherer et al.
- Extra terms kept in $K \to 2\pi$: a one-loop check
- Some preliminary two-loop checks

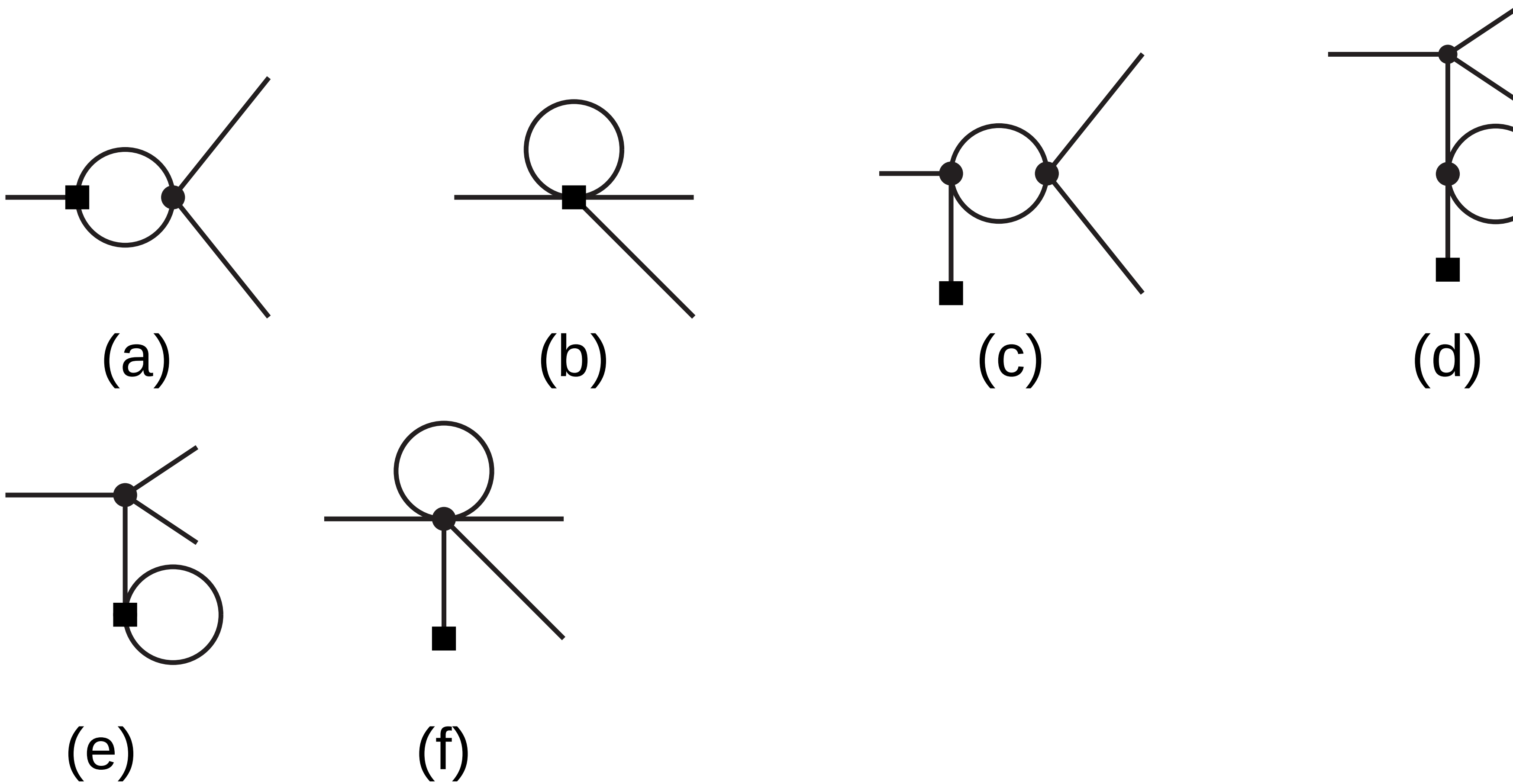
$K \to \pi\pi$: Tree level



$$A_0^{LO} = \frac{\sqrt{3}i}{2F^2}\left[-\frac{1}{2}E_1 + (E_2 - 4E_3)\,\overline{M}_K^2 + 2E_8\overline{M}_K^4 + A_1E_1\right]$$

$$A_2^{LO} = \sqrt{\frac{3}{2}}\,\frac{i}{F^2}\left[(-2D_1 + D_2)\,\overline{M}_K^2\right]$$

$K \to \pi\pi$: One loop



$K \to \pi\pi$: One loop

Diagram	A_0	A_2
Z	$-\frac{2F^2}{3}A_0^{LO}$	$-\frac{2F^2}{3}A_2^{LO}$
(a)	$\sqrt{3}i\left(-\frac{1}{3}E_1+\frac{2}{3}E_2\overline{M}_K^2\right)$	$\sqrt{\frac{3}{2}}i\left(-\frac{2}{3}D_2\overline{M}_K^2\right)$
(b)	$\sqrt{3}i\left(-\frac{5}{96}E_1-\left(\frac{7}{48}E_2+\frac{25}{12}E_3\right)\overline{M}_K^2+\frac{25}{24}E_8\overline{M}_K^4\right)$	$\sqrt{\frac{3}{2}}i\left(-\frac{61}{12}D_1+\frac{77}{24}D_2\right)\overline{M}_K^2$
(e)	$\sqrt{3}i\frac{3}{16}A_1E_1$	
(f)	$\sqrt{3}i\left(\frac{1}{8}E_1+\frac{1}{3}A_1E_1\right)$	

The coefficients of $\overline{A}(M^2)/F^4$ in the contributions to A_0 and A_2. Z denotes the part from wave-function renormalization.

- $\overline{A}(M^2) = -\frac{M^2}{16\pi^2}\log\frac{M^2}{\mu^2}$
- $K\pi$ intermediate state does not contribute, but did for Flynn-Sachrajda

$K \longrightarrow \pi\pi$: One-loop

$$
\begin{aligned}
A_0^{NLO} &= A_0^{LO}\left(1+\frac{3}{8F^2}\overline{A}(M^2)\right)+\lambda_0 M^2+\mathcal{O}(M^4)\,,\\
A_2^{NLO} &= A_2^{LO}\left(1+\frac{15}{8F^2}\overline{A}(M^2)\right)+\lambda_2 M^2+\mathcal{O}(M^4)\,.
\end{aligned}
$$

$K \to \pi\pi$: One-loop

$$A_0^{NLO} = A_0^{LO}\left(1+\frac{3}{8F^2}\overline{A}(M^2)\right)+\lambda_0 M^2+\mathcal{O}(M^4)\,,$$

$$A_2^{NLO} = A_2^{LO}\left(1+\frac{15}{8F^2}\overline{A}(M^2)\right)+\lambda_2 M^2+\mathcal{O}(M^4)\,.$$

Match with three flavour $SU(3)$ calculation Kambor, Missimer, Wyler; JB, Pallante, Prades

$$A_0^{(3)LO} = -\frac{i\sqrt{6}CF_0^4}{\overline{F}_K F^2}\left(G_8+\frac{1}{9}G_{27}\right)\overline{M}_K^2\,, \qquad A_2^{(3)LO} = -\frac{i10\sqrt{3}CF_0^4}{9\overline{F}_K F^2}G_{27}\overline{M}_K^2\,,$$

When using $F_\pi = F\left(1+\frac{1}{F^2}\overline{A}(M^2)+\frac{M^2}{F^2}l_4^r\right)$, $F_K = \overline{F}_K\left(1+\frac{3}{8F^2}\overline{A}(M^2)+\cdots\right)$,

logarithms at one-loop agree with above

Hard Pion ChPT: A two-loop check

- Similar arguments to JB-Celis, Flynn-Sachrajda work for the pion vector and scalar formfactor
- Therefore at any t the chiral log correction must go like the one-loop calculation.
- But note the one-loop log chiral log is with $t >> m_\pi^2$
- Predicts

$$F_V(t,M^2) = F_V(t,0)\,\left(1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\mu^2} + \mathcal{O}(M^2)\right)$$

$$F_S(t,M^2) = F_S(t,0)\,\left(1 - \frac{5}{2}\frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\mu^2} + \mathcal{O}(M^2)\right)$$

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- Note that $F_{V,S}(t,0)$ is now a coupling constant and can be complex
- Take the full two-loop ChPT calculation JB,Colangelo,Talavera and expand in $t >> m_\pi^2$.

A two-loop check

Full two-loop ChPT JB,Colangelo,Talavera, expand in $t >> m_\pi^2$:

$$F_V(t,M^2) = F_V(t,0)\left(1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\mu^2} + \mathcal{O}(M^2)\right)$$
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with

$$F_V(t,0) = 1 + \frac{t}{16\pi^2F^2}\left(\frac{5}{18} - 16\pi^2 l_6^r + \frac{i\pi}{6} - \frac{1}{6}\ln\frac{t}{\mu^2}\right)$$
$$F_S(t,0) = 1 + \frac{t}{16\pi^2F^2}\left(1 - 16\pi^2 l_4^r + i\pi - \ln\frac{t}{\mu^2}\right)$$

- The needed coupling constants are complex
- Both calculations have two-loop diagrams with overlapping divergences
- The chiral logs should be valid for any t where a pointlike interaction is a valid approximation

Hard Pion ChPT

Why is this useful:

- Lattice works actually around the strange quark mass
- need only extrapolate in m_u and m_d.
- Applicable in momentum regimes where usual ChPT might not work
- In progress: $B \to \pi$ semileptonic decays

Leading Logarithms

- Take a quantity with a single scale: $F(M)$
- The dependence on the scale in field theory is typically logarithmic
- $L = \log(\mu/M)$
- $F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + F_3^3 L^3 + \cdots$
- Leading Logarithms: The terms $F_m^m L^m$

The F_m^m can be more easily calculated than the full result

- $\mu\,(dF/d\mu) \equiv 0$
- Ultraviolet divergences in Quantum Field Theory are always local

Renormalizable theories

- Loop expansion $\equiv \alpha$ expansion
- $F = \alpha + f_1^1 \alpha^2 L + f_0^1 \alpha^2 + f_2^2 \alpha^3 L^2 + f_1^2 \alpha^3 L + f_0^2 \alpha^3 + f_3^3 \alpha^4 L^3 + \cdots$
- f_i^j are pure numbers

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- $\mu \dfrac{dF}{d\mu} \equiv F'$, $\mu \dfrac{d\alpha}{d\mu} \equiv \alpha'$, $\mu \dfrac{dL}{d\mu} = 1$
- $$F' = \alpha' + f_1^1 \alpha^2 + f_1^1 2\alpha'\alpha L + f_2^2 \alpha^3 2L + f_2^2 3\alpha'\alpha^2 L^2 + f_1^2 \alpha^3 + f_1^2 3\alpha'\alpha^2 L + f_0^2 3\alpha'\alpha^2 + f_3^3 \alpha^3 3L^2 + f_3^3 4\alpha'\alpha^3 L^3 + \cdots$$

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- $\alpha' = \beta_0\alpha^2 + \beta_1\alpha^3 + \cdots$

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- $\alpha' = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$
- $$0 = F' = \left(\beta_0 + f_1^1\right)\alpha^2 + \left(2\beta_0 f_1^1 + 2 f_2^2\right)\alpha^3 L + \left(\beta_1 + 2\beta_0 f_0^1 + f_1^2\right)\alpha^3 + \left(3\beta_0 f_2^2 + 3 f_3^3\right)\alpha^4 L^2 + \cdots$$

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- $\boxed{\beta_0 = -f_1^1 = f_2^2 = -f_3^3 = \cdots}$

Renormalization Group

- Can be extended to other operators as well
- Underlying argument always $\mu\dfrac{dF}{d\mu} = 0$.
- Gell-Mann–Low, Callan–Symanzik, Weinberg–'t Hooft
- In great detail: J.C. Collins, *Renormalization*
- Relies on the α the same in all orders
- LL one-loop β_0
- NLL two-loop β_1, f_0^1

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- In effective field theories: different Lagrangian at each order
- The recursive argument does not work

Weinberg's argument

- Weinberg, Physica A96 (1979) 327
- Two-loop leading logarithms can be calculated using only one-loop
- Weinberg consistency conditions
- $\pi\pi$ at 2-loop: Colangelo, hep-ph/9502285
- General at 2 loop: JB, Colangelo, Ecker, hep-ph/9808421
- Proof at all orders using β-functions
  Büchler, Colangelo, hep-ph/0309049
- Proof with diagrams: present work

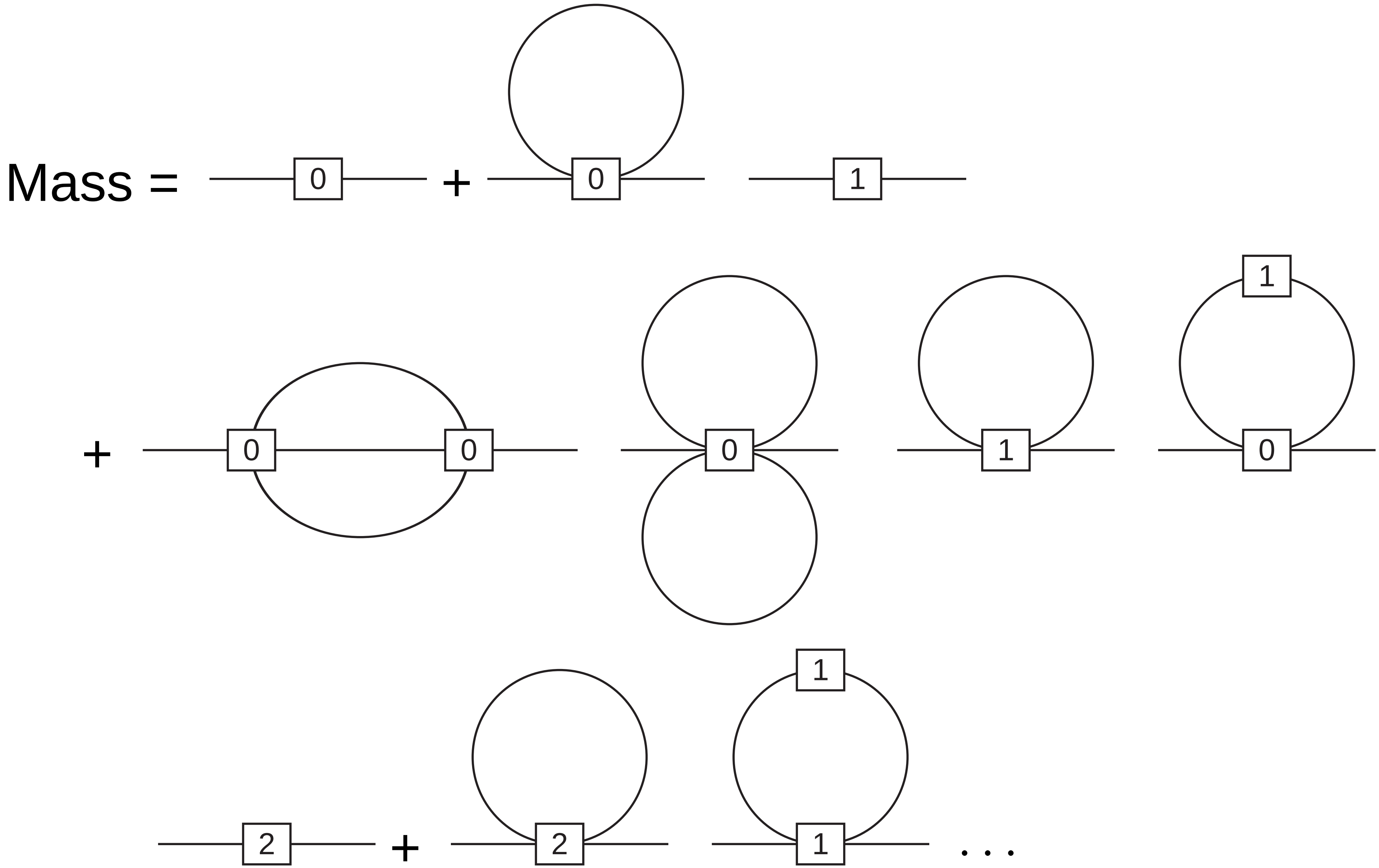
Weinberg's argument

- μ: dimensional regularization scale
- $d = 4 - w$
- loop-expansion $\equiv \hbar$-expansion
- $$\mathcal{L}^{\text{bare}} = \sum_{n\geq 0} \hbar^n \mu^{-nw} \mathcal{L}^{(n)}$$
- $$\mathcal{L}^{(n)} = \sum_i \left(\sum_{k=0,n} \frac{c_{ki}^{(n)}}{w^k} \right) \mathcal{O}_i^{(n)}$$
- $c_{0i}^{(n)}$ have a direct μ-dependence
- $c_{ki}^{(n)}$ $k \geq 1$ only depend on μ through $c_{0i}^{(m<n)}$

Weinberg's argument

- L_l^n l-loop contribution at order $\hbar^n$
- Expand in divergences from the loops (not from the counterterms) $L_l^n = \sum_{k=0,l} \frac{1}{w^k} L_{kl}^n$
- Neglected positive powers: not relevant here, but should be kept in general
- $\{c\}_l^n$ all combinations $c_{k_1 j_1}^{(m_1)} c_{k_2 j_2}^{(m_2)} \ldots c_{k_r j_r}^{(m_r)}$ with $m_i \geq 1$, such that $\sum_{i=1,r} m_i = n$ and $\sum_{i=1,r} k_i = l$.
- $\{c_n^n\} \equiv \{c_{ni}^{(n)}\}$, $\{c\}_2^2 = \{c_{2i}^{(2)}, c_{1i}^{(1)} c_{1k}^{(1)}\}$
- $\mathcal{L}^{(n)} = \boxed{\text{n}}$

Weinberg's argument



Weinberg's argument

- $\hbar^0$: L_0^0
- $\hbar^1$: $\frac{1}{w}\left(\mu^{-w}L^1_{00}(\{c\}^1_1) + L^1_{11}\right) + \mu^{-w}L^1_{00}(\{c\}^1_0) + L^1_{10}$

Weinberg's argument

- $\hbar^0$: L_0^0
- $\hbar^1$: $\dfrac{1}{w}\left(\mu^{-w}L^1_{00}(\{c\}^1_1)+L^1_{11}\right)+\mu^{-w}L^1_{00}(\{c\}^1_0)+L^1_{10}$
  - Expand $\mu^{-w}=1-w\log\mu+\dfrac{1}{2}w^2\log^2\mu+\cdots$
  - $1/w$ must cancel: $L^1_{00}(\{c\}^1_1)+L^1_{11}=0$
    this determines the c^1_{1i}
  - Explicit $\log\mu$: $-\log\mu\, L^1_{00}(\{c\}^1_0)\equiv\log\mu\, L^1_{11}$

Weinberg's argument

- $\hbar^2$:
$$\frac{1}{w^2}\left(\mu^{-2w}L^2_{00}(\{c\}^2_2)+\mu^{-w}L^2_{11}(\{c\}^1_1)+L^2_{22}\right)$$
$$+\frac{1}{w}\left(\mu^{-2w}L^2_{00}(\{c\}^2_1)+\mu^{-w}L^2_{11}(\{c\}^1_0)+\mu^{-w}L^2_{10}(\{c\}^1_1)+L^2_{21}\right)$$
$$+\left(\mu^{-2w}L^2_{00}(\{c\}^2_0)+\mu^{-w}L^2_{10}(\{c\}^1_0)+L^2_{20}\right)$$
- $1/w^2$ and $\log\mu/w$ must cancel
$$L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)+L^2_{22}=0$$
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$$L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)+L^2_{22}=0$$
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- Solution: $L^2_{00}(\{c\}^2_2)=-\dfrac{1}{2}L^2_{11}(\{c\}^1_1)\qquad L^2_{11}(\{c\}^1_1)=-2L^2_{22}$

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- Explicit $\log\mu$ dependence (one-loop is enough)
$$\frac{1}{2}\log^2\mu\left(4L^2_{00}(\{c\}^2_2)+L^2_{11}(\{c\}^1_1)\right)=-\frac{1}{2}L^2_{11}(\{c\}^1_1)\log^2\mu\,.$$

All orders

- $\hbar^n$:
$$\frac{1}{w^n}\left(\mu^{-nw}L^n_{00}(\{c\}^n_n)+\mu^{-(n-1)w}L^n_{11}(\{c\}^{n-1}_{n-1})+\cdots\right.$$
$$\left.+\mu^{-w}L^n_{n-1\ n-1}(\{c\}^1_1)+L^n_{nn}\right)+\frac{1}{w^{n-1}}\cdots$$

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- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\;n-i}(\{c\}^i_i)=0 \qquad\qquad j=0,..,n-1.$$

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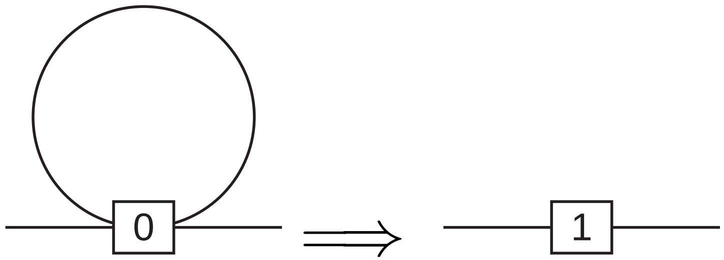
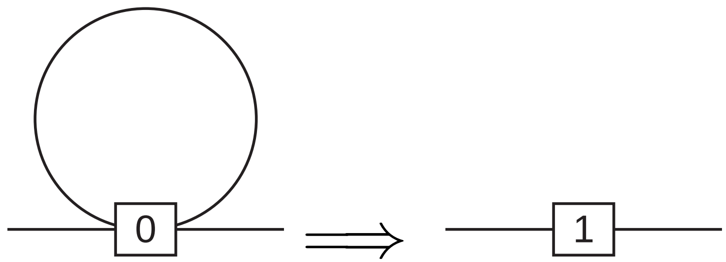
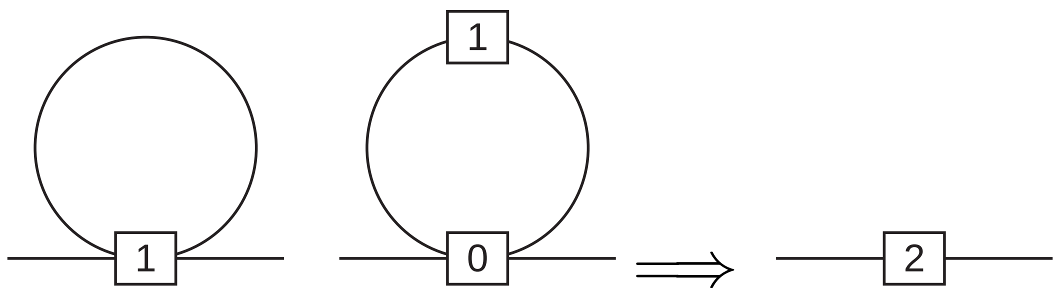
- $1/w^n, \log\mu/w^{n-1},\ldots,\log^{n-1}\mu/w$ cancel:
$$\sum_{i=0}^{n} i^j L^n_{n-i\;n-i}(\{c\}^i_i)=0 \qquad j=0,..,n-1.$$
- Solution: $L^n_{n-i\;n-i}(\{c\}^i_i)=(-1)^i\begin{pmatrix} n\\ i\end{pmatrix}L^n_{nn}$
- explicit leading $\log\mu$ dependence and divergence
$$\log^n\mu\frac{(-1)^{n-1}}{n}L^n_{11}(\{c\}^{n-1}_{n-1}) \qquad L^n_{00}(\{c\}^n_n)=-\frac{1}{n}L^n_{11}(\{c\}^{n-1}_{n-1})$$

Mass to $\hbar^2$

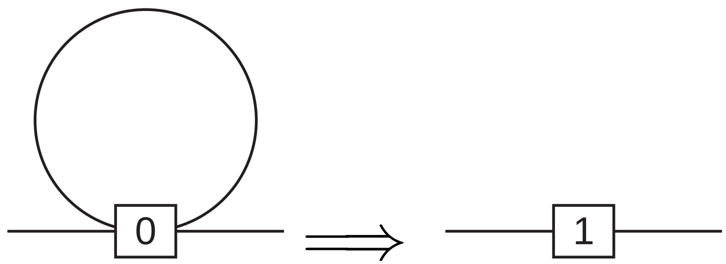
- $\hbar^1$: 

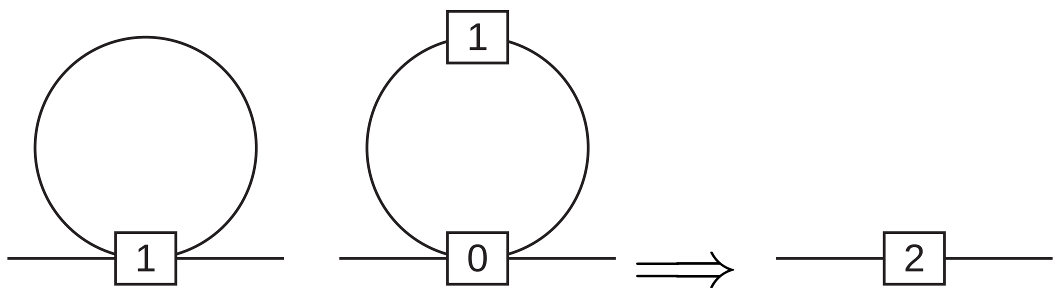
Mass to $\hbar^2$

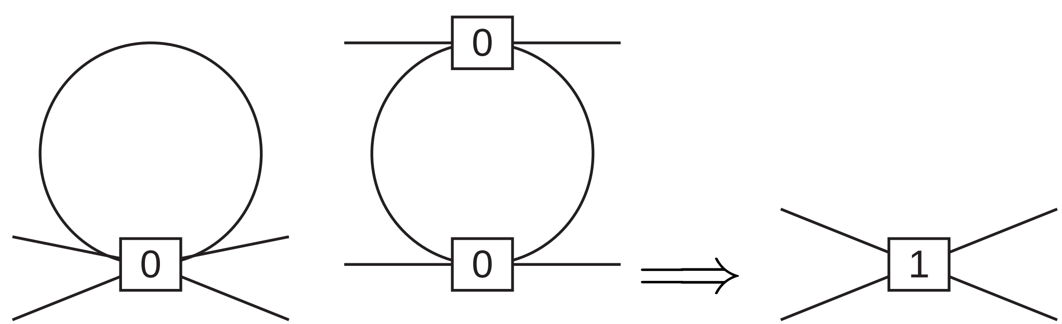
- $\hbar^1$: 

- $\hbar^2$: 

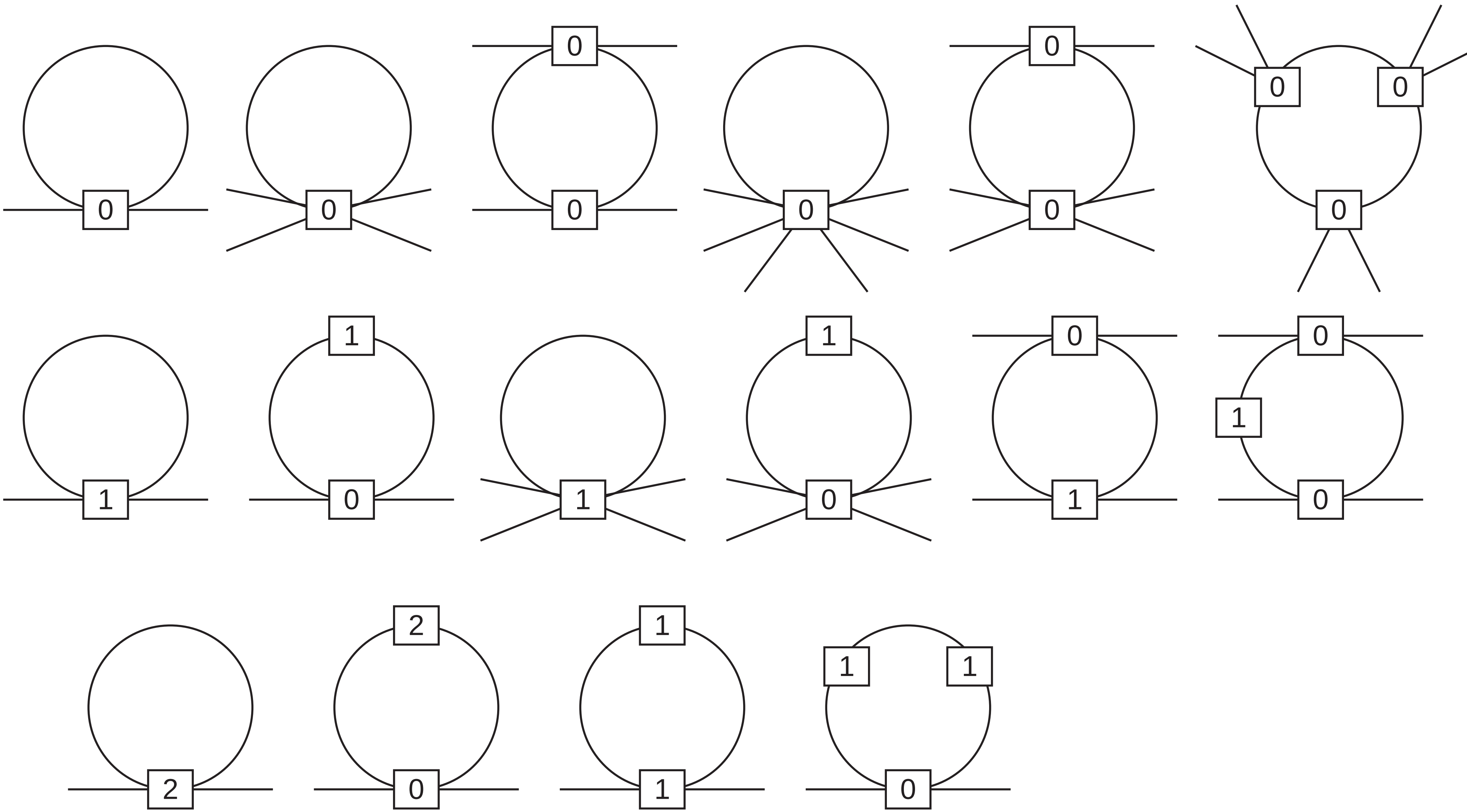
Mass to $\hbar^2$

- $\hbar^1$: 

- $\hbar^2$: 

- but also needs $\hbar^1$: 

Mass to order $\hbar^3$



Mass+decay to $\hbar^5$

- $\hbar^1$: 18 + 27
- $\hbar^2$: 26 + 45
- $\hbar^3$: 33 + 51
- $\hbar^4$: 26 + 33
- $\hbar^5$: 13 + 13

- Calculate the divergence
- rewrite it in terms of a local Lagrangian
- Luckily: symmetry kept: we know result will be symmetrical, hence do not need to explicitly rewrite the Lagrangians in a nice form
- We keep all terms to have all 1PI (one particle irreducible) diagrams finite

Massive $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model
- $\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu\Phi^T\partial^\mu\Phi + F^2\chi^T\Phi\,.$
- Φ is a real $N+1$ vector; $\Phi \to O\Phi$; $\Phi^T\Phi = 1$.
- Vacuum expectation value $\langle\Phi^T\rangle = (1\;0\ldots0)$
- Explicit symmetry breaking: $\chi^T = \left(M^2\;0\ldots0\right)$
- Both spontaneous and explicit symmetry breaking
- N-vector ϕ
- N (pseudo-)Nambu-Goldstone Bosons
- $N=3$ is two-flavour Chiral Perturbation Theory

Massive $O(N)$ sigma model: Φ vs ϕ

- $$\Phi_1 = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi^1}{F} \\ \vdots \\ \frac{\phi^N}{F} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}$$ Gasser, Leutwyler

- $$\Phi_2 = \frac{1}{\sqrt{1+\frac{\phi^T\phi}{F^2}}}\begin{pmatrix} 1 \\ \frac{\phi}{F} \end{pmatrix}$$ Weinberg

  $$\Phi_3 = \begin{pmatrix} 1-\frac{1}{2}\frac{\phi^T\phi}{F^2} \\ \sqrt{1-\frac{1}{4}\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{F} \end{pmatrix}$$ only mass term

- $$\Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\,\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$ CCWZ

Massive $O(N)$ sigma model: Checks

Need (many) checks:

- use the four different parametrizations
- compare with known results:

$$M^2_{phys} = M^2\left(1 - \frac{1}{2}L_M + \frac{17}{8}L_M^2 + \cdots\right)\,,$$

$$L_M = \frac{M^2}{16\pi^2F^2}\log\frac{\mu^2}{\mathcal{M}^2}$$

  Usual choice $\mathcal{M} = M$.
- large N (but known results only for massless case)
  Coleman, Jackiw, Politzer 1974

Results

$$M^2_{\text{phys}} = M^2(1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \ldots)$$

$$L_M = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2}$$

i	a_i, $N = 3$	a_i for general N
1	$-\frac{1}{2}$	$1 - \frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4} - \frac{7N}{4} + \frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12} - \frac{113N}{24} + \frac{15\ N^2}{4} - N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144} - \frac{1601\ N}{144} + \frac{695\ N^2}{48} - \frac{135\ N^3}{16} + \frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400} - \frac{1151407\ N}{43200} + \frac{197587\ N^2}{4320} - \frac{12709\ N^3}{300} + \frac{6271\ N^4}{320} - \frac{7\ N^5}{2}$

Results

$$F_{\text{phys}} = F(1 + b_1 L_M + b_2 L_M^2 + b_3 L_M^3 + \ldots)$$

i	b_i for $N = 3$	b_i for general N
1	1	$-\frac{1}{2} + \frac{N}{2}$
2	$-\frac{5}{4}$	$-\frac{1}{2} + \frac{7N}{8} - \frac{3N^2}{8}$
3	$\frac{83}{24}$	$-\frac{7}{24} + \frac{21N}{16} - \frac{73N^2}{48} + \frac{1N^3}{2}$
4	$-\frac{3013}{288}$	$\frac{47}{576} + \frac{1345N}{864} - \frac{14077N^2}{3456} + \frac{625N^3}{192} - \frac{105N^4}{128}$
5	$\frac{2060147}{51840}$	$-\frac{23087}{64800} + \frac{459413N}{172800} - \frac{189875N^2}{20736} + \frac{546941\ N^3}{43200} - \frac{1169\ N^4}{160} + \frac{3\ N^5}{2}$

Results

$$\langle \bar{q}_i q_i \rangle = -BF^2(1 + c_1 L_M + c_2 L_M^2 + c_3 L_M^3 + \ldots)$$

$M^2 = 2B\hat{m}$ $\qquad$ $\chi^T = 2B(s\ 0\ \ldots 0)$

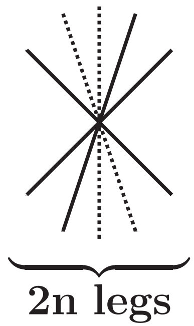
s corresponds to $\bar{u}u + \bar{d}d$ current

i	c_i for $N = 3$	c_i for general N
1	$\frac{3}{2}$	$\frac{N}{2}$
2	$-\frac{9}{8}$	$\frac{3N}{4} - \frac{3N^2}{8}$
3	$\frac{9}{2}$	$\frac{3N}{2} - \frac{3N^2}{2} + \frac{N^3}{2}$
4	$-\frac{1285}{128}$	$\frac{145N}{48} - \frac{55N^2}{12} + \frac{105N^3}{32} - \frac{105N^4}{128}$
5	46	$\frac{3007N}{480} - \frac{1471N^2}{120} + \frac{557\ N^3}{40} - \frac{1191\ N^4}{160} + \frac{3\ N^5}{2}$

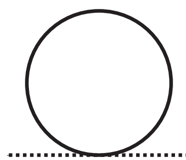
Anyone recognize any funny functions?

Large N

Power counting: pick $\mathcal{L}$ extensive in $N \Rightarrow F^2 \sim N$, $M^2 \sim 1$

- 

  $\Leftrightarrow F^{2-2n} \sim \frac{1}{N^{n-1}}$

  $\Leftrightarrow N$

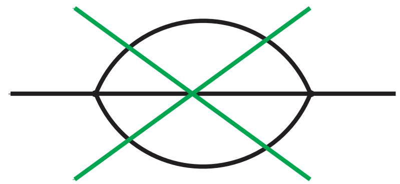
- 1PI diagrams:

$$\left.\begin{array}{l} N_L = N_I - \sum_n N_{2n} + 1 \\ 2N_I + N_E = \sum_n 2nN_{2n} \end{array}\right\} \Rightarrow N_L = \sum_n (n-1)N_{2n} - \frac{1}{2}N_E + 1$$

- diagram suppression factor: $\dfrac{N^{N_L}}{N^{N_E/2-1}}$

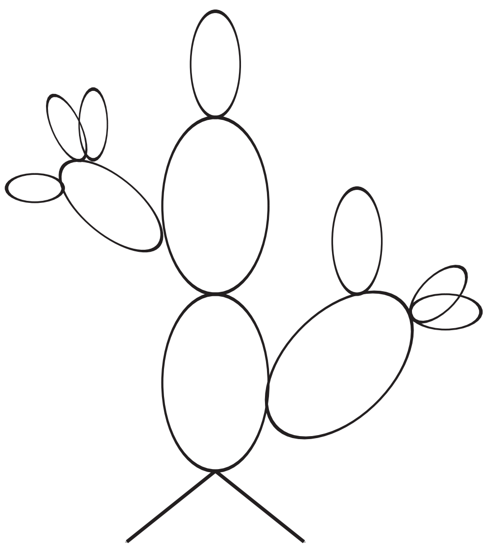
Large N

- diagrams with shared lines are suppressed



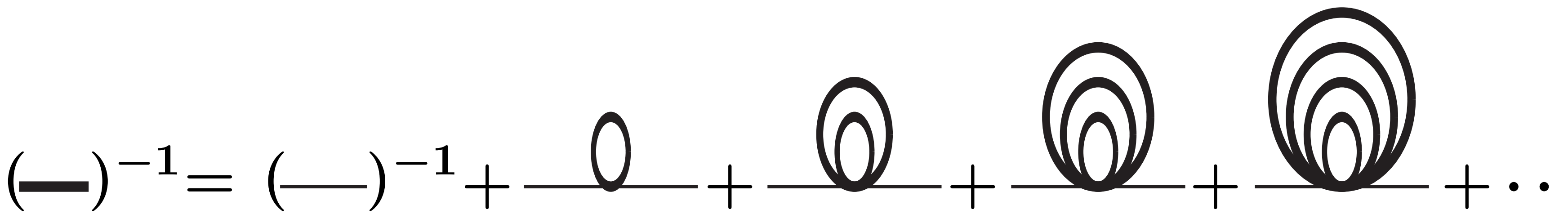
  each new loop needs also a new flavour loop

- in the large N limit only "*cactus*" diagrams survive:



large N: propagator

Generate recursively via a Gap equation



$\Rightarrow$ resum the series and look for the pole

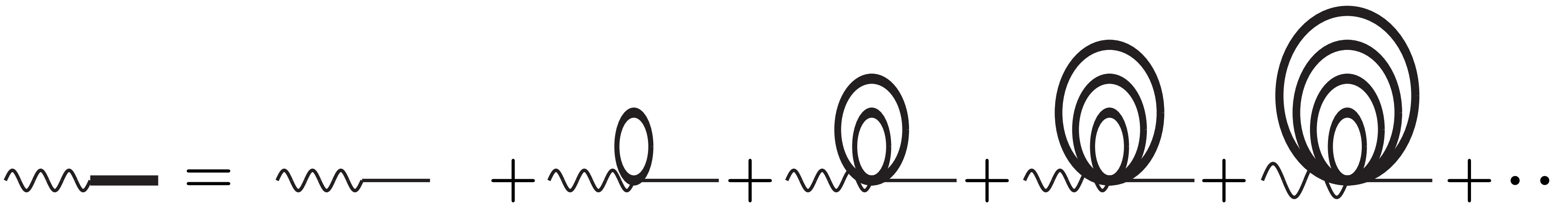
$$\boxed{M^2 = M^2_{\mathrm{phys}}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\mathrm{phys}})}}$$

$$\overline{A}(m^2) = \frac{m^2}{16\pi^2}\log\frac{\mu^2}{m^2}\,.$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N: Decay constant



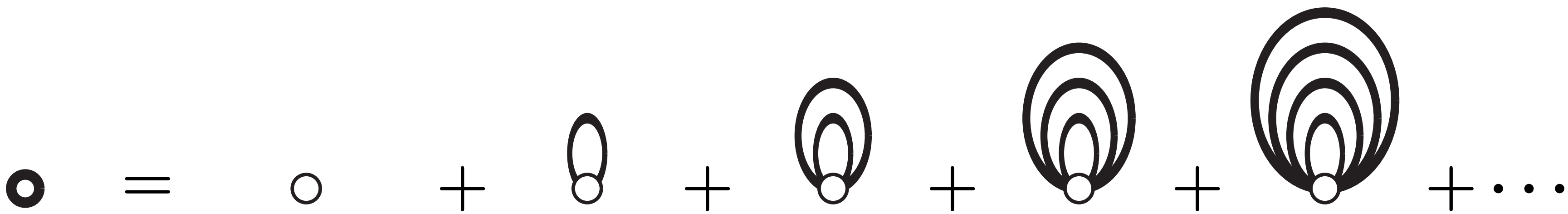
$\Rightarrow$ and include wave-function renormalization

$$\boxed{F_{\text{phys}} = F\sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}}$$

Solve recursively, agrees with other result

Note: can be done for all parametrizations

large N: Vacuum Expectation Value



$$\boxed{\langle \bar{q}q\rangle_{\mathrm{phys}} = \langle \bar{q}q\rangle_0 \sqrt{1 + \frac{N}{F^2}\overline{A}(M^2_{\mathrm{phys}})}}$$

Comments:

- These are the full* leading N results, not just leading log
- But depends on the choice of N-dependence of higher order coefficients
- Assumes higher LECs zero ($< N^{n+1}$ for $\hbar^n$)
- Large N as in $O(N)$ not large N_c

Large N: Checking expansions

$$M^2 = M^2_{\text{phys}}\sqrt{1+\frac{N}{F^2}\overline{A}(M^2_{\text{phys}})}$$

much smaller expansion coefficients than the table, try

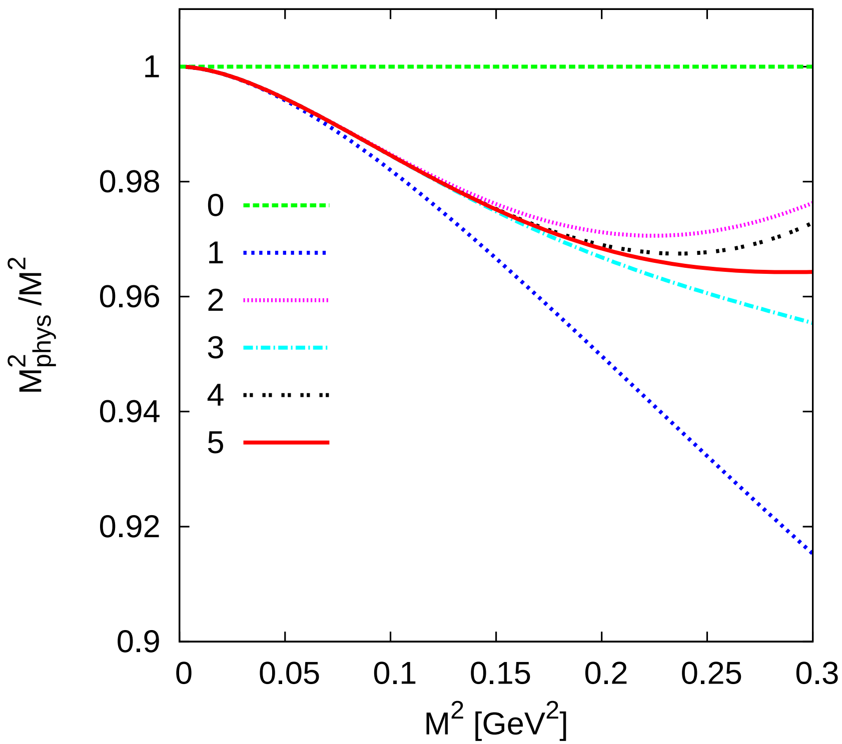
$$M^2 = M^2_{\text{phys}}(1 + d_1 L_{M_{\text{phys}}} + d_2 L^2_{M_{\text{phys}}} + d_3 L^3_{M_{\text{phys}}} + \ldots)$$

Large N: Checking expansions

i	d_i, $N=3$	d_i for general N
1	$\frac{1}{2}$	$-1+\frac{1}{2}\ N$
2	$-\frac{13}{8}$	$\frac{1}{4}-\frac{1}{4}\ N-\frac{1}{8}\ N^2$
3	$-\frac{19}{48}$	$\frac{2}{3}-\frac{11}{12}\ N+\frac{1}{16}\ N^3$
4	$-\frac{5773}{1152}$	$-\frac{8}{9}+\frac{107}{144}\ N-\frac{1}{6}\ N^2-\frac{1}{16}\ N^3-\frac{5}{128}\ N^4$
5	$-\frac{3343}{768}$	$-\frac{18383}{7200}+\frac{130807}{43200}\ N-\frac{2771}{2160}\ N^2-\frac{527}{1600}\ N^3+\frac{23}{640}\ N^4+\frac{7}{256}\ N^5$

i	a_i, $N=3$	a_i for general N
1	$-\frac{1}{2}$	$1-\frac{N}{2}$
2	$\frac{17}{8}$	$\frac{7}{4}-\frac{7N}{4}+\frac{5\ N^2}{8}$
3	$-\frac{103}{24}$	$\frac{37}{12}-\frac{113N}{24}+\frac{15\ N^2}{4}-N^3$
4	$\frac{24367}{1152}$	$\frac{839}{144}-\frac{1601\ N}{144}+\frac{695\ N^2}{48}-\frac{135\ N^3}{16}+\frac{231\ N^4}{128}$
5	$-\frac{8821}{144}$	$\frac{33661}{2400}-\frac{1151407\ N}{43200}+\frac{197587\ N^2}{4320}-\frac{12709\ N^3}{300}+\frac{6271\ N^4}{320}-\frac{7\ N^5}{2}$

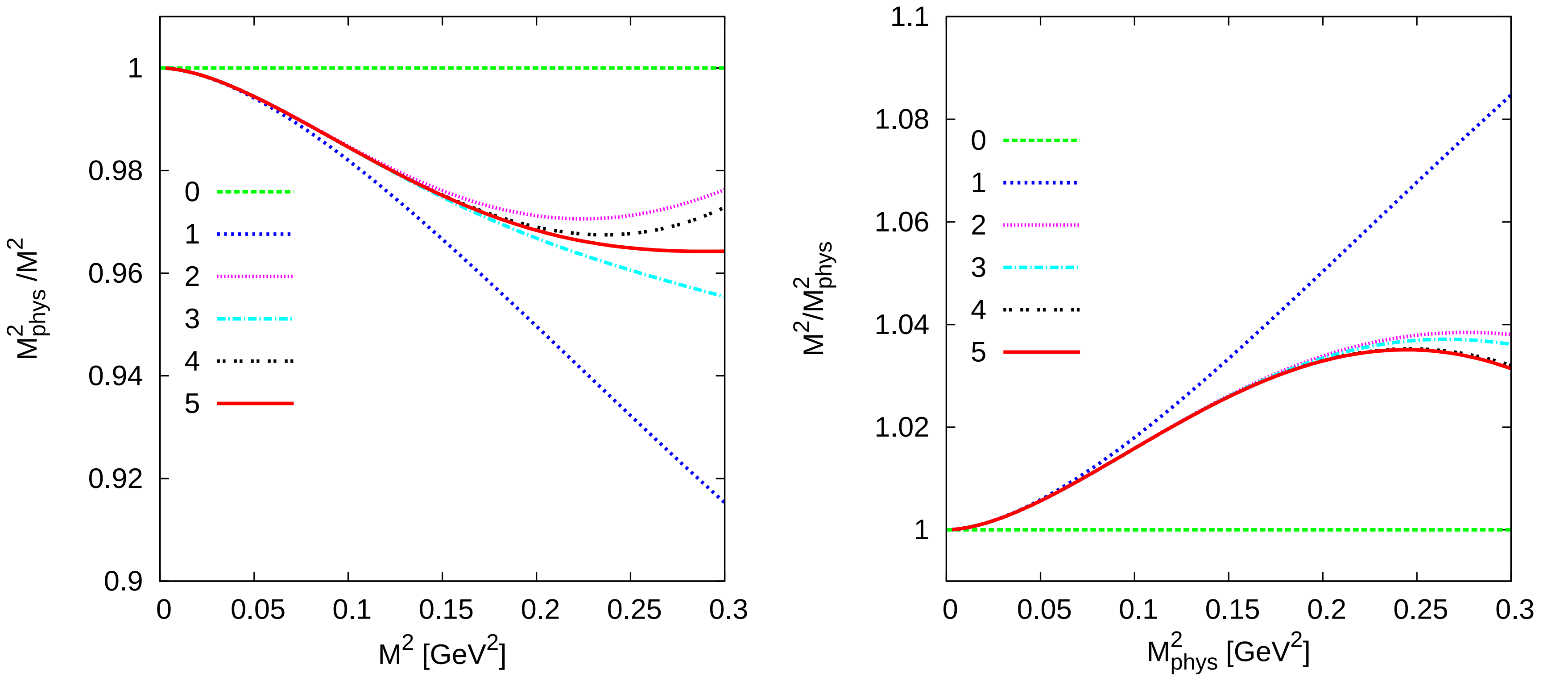
Numerical results



Left: $\frac{M^2_{\text{phys}}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

F = 90 MeV, $\mu = 1$ GeV

Numerical results



Left: $\frac{M^2_{\rm phys}}{M^2} = 1 + a_1 L_M + a_2 L_M^2 + a_3 L_M^3 + \cdots$

Right: $\frac{M^2}{M^2_{\rm phys}} = 1 + d_1 L_{M_{\rm phys}} + d_2 L^2_{M_{\rm phys}} + d_3 L^3_{M_{\rm phys}} + \cdots$

F = 90 MeV, $\mu = 1$ GeV

Other results

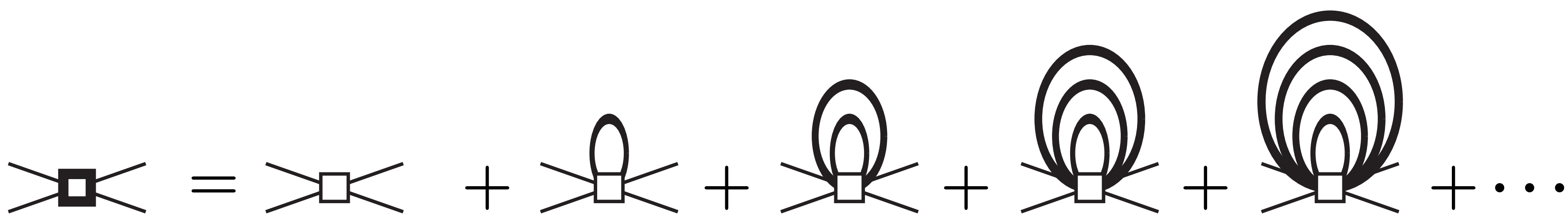
- JB,Carloni to be published
  - massive case: $\pi\pi$, F_V and F_S to 4-loop order
  - large N for these cases also for massive $O(N)$.
  - done using bubble resummations or recursion eqation which can be solved analytically (extension similar to gap equation)

Large N: $\pi\pi$-scattering

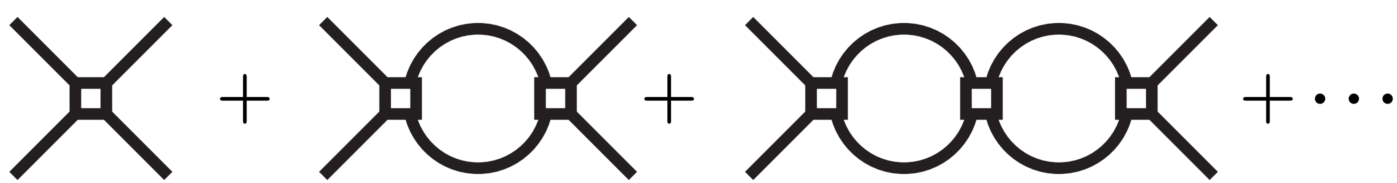
- Semiclassical methods Coleman, Jackiw, Politzer 1974
- Diagram resummation Dobado, Pelaez 1992
- $A(\phi^i\phi^j \longrightarrow \phi^k\phi^l) =$
  $A(s,t,u)\delta^{ij}\delta^{kl} + A(t,u,s)\delta^{ik}\delta^{jl} + A(u,s,t)\delta^{il}\delta^{jk}$
- $A(s,t,u) = A(s,u,t)$
- Proof same as Weinberg's for $O(4)/O(3)$, group theory and crossing

Large N: $\pi\pi$-scattering

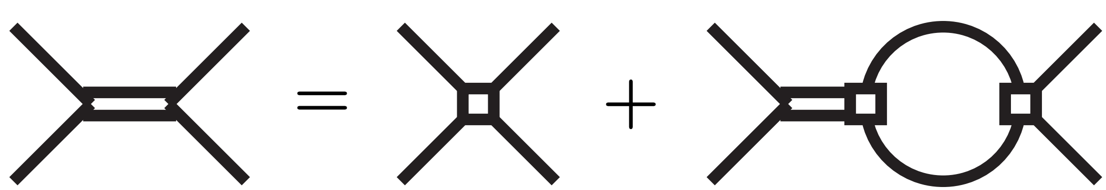
- Cactus diagrams for $A(s,t,u)$
- Branch with no momentum: resummed by
- Branch starting at vertex: resum by



- The full result is then



- Can be summarized by a recursive equation



Large N: $\pi\pi$ scattering

$$y = \frac{N}{F^2}\overline{A}(M^2_{\text{phys}})$$

$$A(s,t,u) = \frac{\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}}{1 - \frac{1}{2}\left(\frac{s}{F^2(1+y)} - \frac{M^2}{F^2(1+y)^{3/2}}\right)\overline{B}(M^2_{\text{phys}}, M^2_{\text{phys}}, s)}$$

or

$$A(s,t,u) = \frac{\frac{s-M^2_{\text{phys}}}{F_{\text{phys}}}}{1 - \frac{1}{2}\frac{s-M^2_{\text{phys}}}{F^2_{\text{phys}}}\overline{B}(M^2_{\text{phys}}, M^2_{\text{phys}}, s)}$$

- $M^2 \to 0$ agrees with the known results
- Agrees with our 4-loop results

Conclusions Leading Logs

- Several quantities in massive $O(N)$ LL known to high loop order
- Large N in massive $O(N)$ model solved
- Had hoped: recognize the series also for general N
- Limited essentially by CPU time and size of intermediate files
- Some first studies on convergence etc.
- In progress: $\pi\pi$, F_V and F_S to four-loop order
- The technique can be generalized to other models/theories
  - $SU(N) \times SU(N)/SU(N)$
  - One nucleon sector

Conclusions

- A general introduction to Effective Field Theory
- Two applications:
  - Hard Pion ChPT: a new application domain for EFT and a first result
  - Leading Logarithms and large N: some progress in getting results at high loop orders