THE MUON ANOMALOUS MAGNETIC MOMENT

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Introduction

- Magnetic moment: \( \vec{\mu} = g \frac{q}{2m} \vec{S} \)
- For angular momentum: \( g = 1 \)
- Dirac equation: \( g = 2 \)
- Structure and/or QFT give different values
- Anomaly: \( a = \frac{g - 2}{2} \)
- QED (Schwinger): \( a = 1 + \frac{\alpha}{2\pi} + \ldots \)
Why do we do this?

The muon \( a_\mu = \frac{g_\mu - 2}{2} \) will be measured more precisely.

J-PARC

Fermilab
Principle of the measurement

Three steps of g-2 measurement

1. Prepare a polarized muon beam.

2. Store in a magnetic field (muon’s spin precesses)

\[
\dot{\omega} = -\frac{e}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \vec{\beta} \times \vec{E} + \frac{e}{2} \left( \vec{\beta} \times \vec{B} + \frac{\vec{E}}{c} \right) \right]
\]

3. Measure decay positron

talk Tsutomu Mibe Seattle g-2 meeting 2019
Principle of the measurement

muon g-2 and EDM measurements

In uniform magnetic field, muon spin rotates ahead of momentum due to $g\cdot 2 \neq 0$

general form of spin precession vector:

$$\vec{\omega} = -\frac{e}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \vec{\beta} \times \vec{E} \right] + \frac{\eta}{2} \left( \vec{\beta} \times \vec{B} + \frac{\vec{E}}{c} \right)$$

BNL E821 approach
$\gamma = 30$ ($P = 3$ GeV/c)

$$\vec{\omega} = -\frac{e}{m} \left[ a_\mu \vec{B} + \frac{\eta}{2} \left( \vec{\beta} \times \vec{B} + \frac{\vec{E}}{c} \right) \right]$$

FNAL E989

talk Tsutomu Mibe Seattle g-2 meeting 2019

J-PARC approach
$E = 0$ at any $\gamma$

$$\vec{\omega} = -\frac{e}{m} \left[ a_\mu \vec{B} + \frac{\eta}{2} \left( \vec{\beta} \times \vec{B} \right) \right]$$

J-PARC E34
Expected results

\[ \tilde{\omega} = -\frac{e}{m} \left[ a_\mu \tilde{B} + \frac{\eta}{2} (\tilde{\beta} \times \tilde{B}) \right] \]

Expected time spectrum of $e^+$ in $\mu \rightarrow e^+\nu\nu$ decay

- $P_\mu = 50\%$  $N_{e^+} = 5.7 \times 10^{11}$
- $200 \text{ MeV} < E_{e^+} < 275 \text{ MeV}$
- $d_\mu = 1 \times 10^{-20} \text{ ecm}$

Talk Tsutomu Mibe Seattle g-2 meeting 2019
The precession frequency is the difference between the spin and cyclotron frequencies, encoded as an oscillation in the positron energy spectrum as seen by the calorimeters.

\[ \omega_a = \omega_s - \omega_c = \left( \frac{g - 2}{2} \right) \frac{eB}{m} \]
Conclusions

• E989 is in the thick of simultaneous Analysis / Hardware Tweaking / Running.
  • We are busy!!!

• People are working hard to get out a much-anticipated result, still in 2019.
  • BUT !!! We cannot afford to rush it so we are making lots and lots of checks.

• As we go “beyond BNL” we are learning a lot.

• Thanks

Principle of the measurement

talk David Hertzog Seattle g-2 meeting 2019
Experiments done for electron and muon: very precise

- \( a_e^{\text{exp}} = 11596521.8073(0.0028) \times 10^{-10} \) Harvard
- \( a_\mu^{\text{exp}} = 11659209.1(5.4)(3.3) \times 10^{-10} \) BNL

Main difference: electron loops enhanced by \( \log(m_e/m_\mu) \)

- \( a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}} \)
- \( a_\mu^{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857425(17) \left( \frac{\alpha}{\pi} \right)^2 + 24.05050996(32) \left( \frac{\alpha}{\pi} \right)^3 \\ + 130.8796(63) \left( \frac{\alpha}{\pi} \right)^4 + 753.3(1.0) \left( \frac{\alpha}{\pi} \right)^5 \)

up to 4 loops essentially analytically, 5-loops numerically

- \( a_\mu^{\text{QED}} = 11658471.895(0.008) \times 10^{-10} \)
- \( a_\mu^{\text{EW}} = 15.36(0.10) \times 10^{-10} \)
- \( a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 7218(63) \times 10^{-11} \)
Why do we do this?

- Experiment dominated by BNL, FNAL error down by four
- Theory taken from PDG2018

\[
a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{Had}}
\]

\[
a_{\mu}^{\text{Had}} = a_{\mu}^{\text{LO-HVP}} + a_{\mu}^{\text{HO-HVP}} + a_{\mu}^{\text{HLbL}}
\]
- Impressive agreement with \( g_{\mu} \) to \( 2 \times 10^{-9} \)

<table>
<thead>
<tr>
<th>Part</th>
<th>value</th>
<th>errors</th>
<th>units</th>
</tr>
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<td>( a_{\mu}^{\text{EXP}} )</td>
<td>116 592 091.\text{x}</td>
<td>(54)(33)</td>
<td>( \times 10^{-11} )</td>
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<td>( \Delta a_{\mu} )</td>
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<td>(1.0)</td>
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<td>( a_{\mu}^{\text{LO-HVP}} )</td>
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<td>( \times 10^{-11} )</td>
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<td>( a_{\mu}^{\text{HO-HVP}} )</td>
<td>−86.3</td>
<td>(0.9)</td>
<td>( \times 10^{-11} )</td>
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<td>( a_{\mu}^{\text{HLbL}} )</td>
<td>105.\text{x}</td>
<td>(26)</td>
<td>( \times 10^{-11} )</td>
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</tbody>
</table>
Numerical summary

- Long standing discrepancy with SM
- Theory number has had slowly shrinking error (especially LO HVP)
  \[ a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = (27.0 - 33.6) \pm (7.3 - 7.8) \times 10^{-10} \]
- 3.6 – 4.6σ discrepancy
- Many BSM papers explaining this (experiment has ≥ 1950 citations)
The blobs are hadronic contributions

I will present some results for LO-HVP and HLbL short-distance as well as give an overview

There are higher order contributions of both types: known accurately enough
My recent work related to the muon $g - 2$


[1,2,3]: HLbL, [4,5]: HVP

I will concentrate on [2,5]
Muon $g - 2$: LO hadronic from experiment

\[ a_{\mu}^{\text{LO HVP}} = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_{m_{\pi}^2}^{\infty} ds \frac{K(s)}{s} R^{(0)}(s) \]

- $R^{(0)}(s)$ bare cross-section ratio
  \[ \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \]
- Bare, many different evaluations,\ldots
- $e^+e^-$ versus $\tau$-decays
- Main problem: combining data from many sources and data sometimes not compatible with errors
Numerical summary: HVP

- LO-HVP latest evaluations (talks at workshops KEK 2018/Seattle 2019)
  - Paris (Davier et al.)
    \[ a^{LO}_{\mu} \text{HVP} = 693.9 \pm 1.0 \pm 3.4 \pm 1.6 \pm 0.1\psi \pm 0.7_{QCD} \times 10^{-10} \]
  - Liverpool (Keshavarzi et al.)
    \[ a^{LO}_{\mu} \text{HVP} = 693.8(2.4) \times 10^{-10} \]
  - Jegerlehner
    \[ a^{LO}_{\mu} \text{HVP} = 689.46(3.25) \times 10^{-10} \]
  - HLS (Benayoun et al.)
    \[ a^{LO}_{\mu} \text{HVP} = 686.1(3.8) \times 10^{-10} \]
  - Slight differences in data, how put together plus what theory
  - Discrepancy Babar/KLOE in $\pi\pi$-channel
  - Can be with or without $\tau$-data
  - Numbers vary slightly whenever new experiments included

- NLO HVP
  \[ a^{NLO}_{\mu} \text{HVP} = -9.82(0.04) \times 10^{-10} \]

- NNLO HVP
  \[ a^{NNLO}_{\mu} \text{HVP} = 1.24(0.01) \times 10^{-10} \]

Errors: LO-HVP and HLbL, rest negligible
HVP: Summary of results

Some discrepancies, lattice accuracies improve

Figure from 1807.09370
Status lattice results

- Dispersive has reached below 0.5%
- Lattice accuracies a few %
- Improvement continuous
- Finite volume corrections known to two-loop order in ChPT [4]
- Next: isospin breaking
  - $m_u - m_d$: “easy”
  - Electromagnetism: possibly large finite volume corrections since $1/L^n$ rather than $\exp(-m_\pi L)$


Presented at $g - 2$ meetings in KEK February 2018, Mainz June 2018

They are expected to be small
Conventions

- Main object: \[ \Pi_{ab}^{\mu\nu}(q) = i \int d^4xe^{iq\cdot x}\langle 0| T(j_a^{\mu}(x)j_b^{\nu\dagger}(0)|0\rangle \]

- Continuum/infinite volume:
  \[ \Pi_{EM}^{\mu\nu}(q) = (q^{\mu}q^{\nu} - q^2g^{\mu\nu}) \Pi_{EM}(q^2) \]

- Known positive weight functions \( v, w \) and \( Q^2 = -q^2 \):
  - \[ a_{\mu} = \int_{\text{threshold}}^{\infty} dq^2 w(q^2) \frac{1}{\pi} \text{Im} \Pi_{EM}(q^2) \]
  - \[ a_{\mu} = \int_{0}^{\infty} dQ^2 v(Q^2) (-\Pi(Q^2) + \Pi(0)) \]

- Dispersion relation:
  \[ \Pi(q^2) = \Pi(0) + \frac{q^2}{\pi} \int_{\text{threshold}}^{\infty} ds \frac{1}{s(s - q^2)} \frac{1}{\pi} \text{Im} \Pi(s) \]
First: Scalar QED

- Pion loops: finite volume effects suppressed by $e^{-m_\pi L}$ (if off-shell)
- Photon loops have suppression only by powers of $1/L$
- Dynamical photons: large finite volume effects possible
- Scalar QED in usual $\overline{MS}$
  \[ (\mu^2_{\text{ChPT}} = \mu^2_{\overline{MS}}e, \ e = 2.71 \ldots) \]
- $\mathcal{L} = (\partial_\mu \Phi^* + ieA_\mu \Phi^*)(\partial_\mu \Phi - ieA_\mu \Phi) - m_0^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
  $(-\lambda (\phi^* \phi)^2$ not needed)
Integrals at finite volume

- Finite volume for photons started in earnest with
- Get the explicit $1/L$ behaviour from one-loop integrals:
- $S = \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2 - m^2)((k+l-p)^2 - m^2))}$
  do $l^0$, $k^0$ integrals via contour integration
  $\vec{k} = \frac{2\pi}{L} \vec{n}$ and expand in $1/L$
- Write the $\vec{k}$ part as
  $\frac{1}{L^{d-1}} \sum_{\vec{n} \neq \vec{0}} = \int \frac{d^{d-1} k}{(2\pi)^{d-1}} + \left[ \frac{1}{L^{d-1}} \sum_{\vec{n} \neq \vec{0}} - \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \right]$
  First term: infinite volume contribution
  Call the quantity in brackets $(1/L^{d-1}) \Delta'_{\vec{n}}$
  Define $c_m = \Delta_{\vec{n}} \frac{1}{|\vec{n}|^m}$
Example: mass

\[ \frac{\Delta V m^2}{m^2} = e^2 \left( \frac{4c_2}{16\pi^2 mL} + \frac{2c_1}{16\pi^2 m^2 L^2} + \mathcal{O} \left( \frac{1}{L^4}, e^{-mL} \right) \right) \]

- Agrees with earlier results
- \(c_2\) shows up since BOTH propagators can be 'on-shell'
- For \(\Pi_{\mu\nu}\) below threshold only photon line goes 'on-shell'
  \[\implies\] corrections start only at \(1/L^2\)
- Only true if infinite volume mass is used in the expressions at LO
- The finite volume correction is 30% of the total electromagnetic for “standard” pion masses
Diagrams

- Lowest order:

- NLO:

contributions from counterterms and \[ \text{counterterms and } \]
Twopoint function

- Do the $k^0, l^0$ integral
- expand for large $L$ as explained earlier
- Rest can be expressed in terms of
  \[ Z_{ij}(m^2, p^2) = \int \frac{d^{d-1}l}{(2\pi)^{d-1}} \frac{1}{(l^2 + m^2)^{i/2}(4l^2 + 4m^2 - p^2)^j} \]
- these correspond to one-loop integrals with masses
- $\Omega_{ij} \equiv Z_{ij}(m^2, p^2)m^{i+2j-d+1}$
- Calculate in center of mass frame: $p = (p^0, \vec{0})$
- QED$_L$: no disconnected contribution
- $t_{\mu\nu}$ spatial part of $g_{\mu\nu}$
- $\tilde{\Pi}((p^0)^2) \equiv \frac{-1}{3p^2} t_{\mu\nu} (\Pi^{\mu\nu}(p) - \Pi^{\mu\nu}(p = 0))$
- Infinite volume: $\tilde{\Pi}(p^2) = \Pi(p^2)$
Results

\[ \tilde{\Pi}(p^2) = \frac{c_1}{\pi m^2 L^2} \left( \frac{16}{3} \Omega_{-1,3} - \frac{1}{3} \Omega_{1,2} - \frac{32}{3} \Omega_{1,3} - \frac{2}{3} \Omega_{3,2} + \frac{16}{3} \Omega_{3,3} - \frac{1}{8} \Omega_{5,1} + \Omega_{5,2} \right) \]

\[ + \frac{c_0}{m^3 L^3} \left( - \frac{128}{3} \Omega_{-2,4} + \frac{256}{3} \Omega_{0,4} - \frac{5}{3} \Omega_{2,2} + \frac{8}{3} \Omega_{2,3} - \frac{128}{3} \Omega_{2,4} \right. \]

\[ \left. - \frac{3}{8} \Omega_{4,1} + \frac{7}{6} \Omega_{4,2} - \frac{8}{3} \Omega_{4,3} \right) + \mathcal{O} \left( \frac{1}{L^4}, e^{-mL} \right) \]

Simplify using relations of the $\Omega_{ij}$ to

\[ \tilde{\Pi}(p^2) = + \frac{c_0}{m^3 L^3} \left( - \frac{16}{3} \Omega_{0,3} - \frac{5}{3} \Omega_{2,2} + \frac{40}{9} \Omega_{2,3} - \frac{3}{8} \Omega_{4,1} + \frac{7}{6} \Omega_{4,2} + \frac{8}{9} \Omega_{4,3} \right) \]

\[ + \mathcal{O} \left( \frac{1}{L^4}, e^{-mL} \right) \]

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect
Scalar QED on a lattice

Correction for $mL \sim 5$ less than 1%
Generality

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect

- The blob (hadrons) is a four-point function of electromagnetic currents and has no singularities in the regime needed
- The conclusion about $1/L^3$ is general
HLbL: the main object to calculate

- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks
- Two main evaluations around 1995
Numerical summary: HLbL

- The largest contribution is $\pi^0$ (and $\eta, \eta'$) exchange/pole
  - Beware: pole/exchange not quite the same
  - Most evaluations are in reasonable agreement
- The pion loop can be sizable but a large difference between the two evaluations
  - For the pure pion loop part, even larger numbers have been proposed by Engel, Ramsey-Musolf
  - Scalar exchange versus $\pi\pi$-rescattering
- There are other contributions but the sum is smaller than the leading pseudo-scalar exchange
- BPP: $(8.3 \pm 3.2) \times 10^{-10}$  HKS: $(8.96 \pm 1.54) \times 10^{-10}$
Numerical summary: HLbL

- $\pi^0$-exchange (in units of $10^{-10}$)
  - BPP: $5.9(0.9)$
  - pointlike VMD: $5.6$
  - Dispersive ($1808.04823$): $6.26^{+0.30}_{-0.25}$

- $\eta, \eta'$ add about 1.5 each

- Pion loop (in units of $10^{-10}$)
  - BPP: $-1.9(1.3)$ large error to include other model
  - dispersive ($1702.07347$): $-2.4(1)$

- $a_1$: $0.7$

- Scalars: $-0.7$

- Quark loop: here is the main double counting problem 0–2

- Short distance constraints might enhance a bit

- Everything else

- $a_{\mu}^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$ (error up to $4 \times 10^{-10}$ defendable)
General properties

\[ \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \delta q_{4\rho} \frac{\delta \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{q_4=0} \]

Actually we really need
General properties

\( \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) \):

- In general 138 Lorentz structures (136 in 4 dimensions)
- Using \( q_{1\mu} \Pi^{\mu\nu\lambda\sigma} = q_{2\nu} \Pi^{\mu\nu\lambda\sigma} = q_{3\lambda} \Pi^{\mu\nu\lambda\sigma} = q_{4\sigma} \Pi^{\mu\nu\lambda\sigma} = 0 \)
- 43 (41) gauge invariant structures
- 41 helicity amplitudes
- Bose symmetry relates some of them
- Compare HVP: one function, one variable
- General calculation from experiment via dispersion relations: recent progress
  - Colangelo, Hoferichter, Kubis, Procura, Stoffer, . . .
- Well defined separation between different contributions
- Theory initiative: paper under preparation
- One remaining problem: intermediate- and short-distances
General properties

- **Formalism of** Colangelo et al., JHEP 1704 (2017) 161 [1702.07347]
  \[ \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1,54} \hat{T}_{\mu\nu\lambda\sigma} \hat{\Pi}_i \]

- \[ Q_i^2 = -q_i^2 \]
- \[ Q_3^2 = Q_1^2 + Q_2^2 + 2\tau Q_1 Q_3 \]
- \[ a_\mu = \frac{2\alpha^3}{3\pi^2} \int_{-1}^{1} d\tau \sqrt{1-\tau^2} \sum_{i=1,12} \hat{T}_i \hat{\Pi}_i \]

- The 12 \( \bar{\Pi}_i \) are related to 6 \( \hat{\Pi}_i \) with \( q_4 \to 0 \).
  \[
  \bar{\Pi}_1 = \hat{\Pi}_1, \quad \bar{\Pi}_2 = C_{23} \left[ \hat{\Pi}_1 \right], \quad \bar{\Pi}_3 = \hat{\Pi}_4, \quad \bar{\Pi}_4 = C_{23} \left[ \hat{\Pi}_4 \right], \\
  \bar{\Pi}_5 = \hat{\Pi}_7, \quad \bar{\Pi}_6 = C_{12} \left[ C_{13} \left[ \hat{\Pi}_7 \right] \right], \quad \bar{\Pi}_7 = C_{23} \left[ \hat{\Pi}_7 \right], \\
  \bar{\Pi}_8 = C_{13} \left[ \hat{\Pi}_{17} \right], \quad \bar{\Pi}_9 = \hat{\Pi}_{17}, \quad \bar{\Pi}_{10} = \hat{\Pi}_{39}, \\
  \bar{\Pi}_{11} = -C_{23} \left[ \hat{\Pi}_{54} \right], \quad \bar{\Pi}_{12} = \hat{\Pi}_{54},
\]
General properties

- Colangelo et al.
- Dispersion relations can be written for the $\hat{\Pi}_i$
- Can cleanly separate different contributions: $\pi^0, \eta, \eta', \pi\pi, \ldots$
- Well defined separations (other ways of doing it can be fine as well)
- Caveat: separate contributions must satisfy certain sum rules so the dispersion relations work
- White-paper in progress (actually both HVP and HLbL)
\( \pi^0 \)-pole

- Depends on the double off-shell form-factor 
  \[ F_{\pi^0 \gamma^* \gamma^*}(Q_1^2, Q_2^2) \]
- Old days: VMD
- Now: normalization from experiment, form-factors mixture, experiment/theory/lattice
- Short-distance constraints important
- Fully dispersive Hoferichter et al., 1808.04823
  \[ a_{\mu \pi^0\text{-pole}} = 63.0^{+2.7}_{-2.1} \times 10^{-11} \]
- Padé/Canterbury approximants
  Masjuan, Sanchez-Puertas 1701.05829
  \[ a_{\mu \pi^0\text{-pole}} = 63.6(2.7) \times 10^{-11} \]
- Compatible with all other estimates (except MV)
$\eta, \eta'$-pole

- Depends on the double off-shell form-factor
  \[ F_{\eta(\gamma^*\gamma^*)}(Q_1^2, Q_2^2) \]
- Old days: VMD
- Now: normalization from experiment, form-factors mixture
  experiment/theory/lattice
- Short-distance constraints important
- Padé/Canterbury approximants
  \[
  a_{\mu}\eta\text{-pole} = 16.3(1.4) \times 10^{-11} \\
  a_{\mu}\eta'\text{-pole} = 14.5(1.9) \times 10^{-11}
  \]
- Compatible with all other estimates (except MV)
\( \pi \)-loop

- BPP: \(-1.9 \times 10^{-10}\) (ENJL/Full-VMD)
- HKS: \(-0.5 \times 10^{-10}\) (HLS model)
- Since then:
  - HLS does not satisfy shortdistance JB, Prades 2007
  - Polarizability expected to give small enhancement
    \(-20(5) \times 10^{-11}\) JB, Relefors, 2016
  - rescattering as parametrized by scalars expected to give enhancement beyond this \(-6.8(2.0) \times 10^{-11}\) JB, Relefors, 2016
- Dispersive: Pion-box and rescattering has all contributions
- Well defined separation
- Pion box: needs \(F_V(Q^2)\): the pion form-factor
  Colangelo, Hoferichter, Procura, Stoffer, 1702.07347
  \[ a_{\mu}^{\pi\text{-box}} = -15.8(2) \times 10^{-11} \]
- Pion S-wave rescattering
  \[ a_{\mu}^{\pi\text{-box}} = -8(1) \times 10^{-11} \]
- Other rescatterings: work in progress
Other hadrons

- There are very many states above 1 GeV
- Not that much is known about their photon couplings
- Even less (essentially nothing) about form-factors in the couplings to photons
- Many estimates
  - Pauk-Vanderhaeghen, Jegerlehner, BPP, Melnikov-Vainshtein, ...
- Estimates:
  - Scalars/tensors: $a^{\text{scalars+tensors}}_\mu = -2(3) \times 10^{-11}$
  - Axials: $a^{\text{axials}}_\mu = +8(?) \times 10^{-11}$
- Melnikov-Vainshtein, 2003 have a much large estimate ($\sim 22$)
MV short-distance


- take $Q_1^2 \approx Q_2^2 \gg Q_3^2$: Leading term in OPE of two vector currents is proportional to axial current

$$\Pi^{\rho\nu\alpha\beta} \propto \frac{P_\rho}{Q_1^2} \langle 0 | T (J_A \nu J_{V\alpha} J_{V\beta}) | 0 \rangle$$

- $J_A$ comes from

- AVV triangle anomaly: extra info

- Implemented via setting one blob = 1

- $a_{\mu}^{\pi^0} = 7.7 \times 10^{-10}$
Short-distance: MV

- The pointlike vertex implements short-distance part, not only $\pi^0$-exchange

$\pi^0$-loop

Other hadrons

Short-distance: MV

BPP quarkloop + $\pi^0$-exchange $\approx$ MV $\pi^0$-exchange

Important short-distance constraint but numerical effect probably overestimated

Are these part of the quark-loop? JB, Prades, 2007

See also in Dorokhov,Broniowski, Phys.Rev. D78(2008)07301
Quarkloop

- Use (constituent) quark loop
- Used for full estimates since the beginning (1970s)
- Used for short-distance estimates with mass as a cut-off
  
  JB, Pallante, Prades, 1996

- We recalculated:

  - In agreement with quarkloop formulae from
    Hoferichter, Stoffer, private communication
  - In agreement with known numerics

\[ a_\mu = \left( \frac{\alpha}{\pi} \right)^3 N_c e_q^4 \left[ \frac{m_\mu^2}{M^2} \left( \frac{3}{3} \zeta_3 - \frac{19}{16} \right) + \cdots \right] \]

Up to \( m_\mu^{10}/M^{10} \) in paper

\( m_c = 1.27 \text{ GeV} \)

\( a^{\text{HLbLc}}_\mu = (3.165 - 0.0786 - 0.00033 + \cdots) \times 10^{-11} \)

\( = 3.1(1) \times 10^{-11} \)

\( m_b = 4.18 \text{ GeV} \)

\( a^{\text{HLbLb}}_\mu = 1.8 \times 10^{-13} \)
Quarkloop: $u, d, s$

- $M_Q$ provides an infrared cut-off, $M_Q \rightarrow 0$ divergent
- About $12 \times 10^{-11}$ from above 1 GeV for $M_Q = 0.3$ GeV
- About $17 \times 10^{-11}$ from above 1 GeV for $M_Q = 0$

$Q_1, Q_2, Q_3 > Q_{\text{min}}$

$M_Q = 0$: full
$M_Q = 0.3$ GeV
dashed

$\alpha_{\mu}^{\text{HLbLQ}} = 54 \times 10^{-11}$

$Q_{\text{min}} [\text{GeV}]$

$\bar{\Pi}_{1-12}$
$\bar{\Pi}_{1-2}$
$\bar{\Pi}_{3-12}$

figure: Hoferichter
Quarkloop

- Is it a first term in a systematic OPE?
- OPE has been used as constraints on specific contributions
  - $\pi^0 \gamma^* \gamma^*$ asymptotic behaviour
  - Constraints on many other hadronic formfactors
  - $q_1^2 \approx q_2^2 \gg q_3^2$ Melnikhov, Vainshtein 2003
  - These are discussed in the next two talks

Is it a first term in a systematic OPE?

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- These are discussed in the next two talks
Short-distance: first attempt

\[ \Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle T(j^\mu(x)j^\nu(y)j^\lambda(z)j^\sigma(0)) \rangle \]

- Usual OPE: \(x, y, z\) all small
- First term in the expansion is the quark-loop
  - no problem with \(\partial/\partial q_4^\rho\) and \(q_4 \to 0\)
  - \(p\) in loop \(\Rightarrow\) no singular propagators:

\[ q_4 \to 0 \text{ propagator diverges:} \]
Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, 1984
- For the $q_4$-leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge: $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$
  whole calculation is immediately with $q_4 = 0$.
- First term is exactly the usual quark loop
  (even including quark masses)
Short-distance: next term(s)

- Do the usual QCD sum rule expansion in terms of vacuum condensates

- There are new condensates, induced by the constant magnetic field: \( \langle \bar{q} \sigma_{\alpha\beta} q \rangle \equiv e_q F_{\alpha\beta} X_q \)

- Lattice QCD Bali et al., arXiv:1206.4205

  \[ X_u = 40.7 \pm 1.3 \text{ MeV}, \]
  \[ X_d = 39.4 \pm 1.4 \text{ MeV}, \]
  \[ X_s = 53.0 \pm 7.2 \text{ MeV} \]

- Could have started at order \( 1/Q \), only starts at \( 1/Q^2 \) via \( m_q X_q \) corrections to the leading quark-loop result

- \( X_q \) and \( m_q \) are very small, only a very small correction

- Next order: very many condensates contribute, work in progress
Short-distance

- Result derived from:

\[
\hat{\Pi}_1 = m_q X_q e_q \frac{-4(Q_1^2 + Q_2^2 - Q_3^2)}{Q_1^2 Q_2^2 Q_3^4}
\]

\[
\hat{\Pi}_7 = 0
\]

\[
\hat{\Pi}_4 = m_q X_q e_q \frac{8}{Q_1^2 Q_2^2 Q_3^2}
\]

\[
\hat{\Pi}_{17} = m_q X_q e_q \frac{8}{Q_1^2 Q_2^2 Q_3^4}
\]

\[
\hat{\Pi}_{54} = m_q X_q e_q \frac{-4(Q_1^2 - Q_2^2)}{Q_1^4 Q_2^4 Q_3^2}
\]

\[
\hat{\Pi}_{39} = 0
\]

- \( N_c = 3 \) and one quark
Short-distance: numerical results

- preliminary
- \(Q_1, Q_2, Q_3 \geq Q_{\text{min}}\)
- \(m_u = m_d = m_s = 0\) for quark-loop
- \(m_u = m_d = 5\) MeV and \(m_s = 100\) MeV for \(m_q X_q\)

<table>
<thead>
<tr>
<th>(Q_{\text{min}})</th>
<th>quarkloop</th>
<th>(m_u X_u + m_d X_d)</th>
<th>(m_s X_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 GeV</td>
<td>(17.3 \times 10^{-11})</td>
<td>(5.40 \times 10^{-13})</td>
<td>(8.29 \times 10^{-13})</td>
</tr>
<tr>
<td>2 GeV</td>
<td>(4.35 \times 10^{-11})</td>
<td>(3.40 \times 10^{-14})</td>
<td>(5.22 \times 10^{-14})</td>
</tr>
</tbody>
</table>

Above 1 GeV still 15% of total value of HLbL

- Quarkloop goes as \(1/Q_{\text{min}}^2\)
- \(m_q X_q\) goes as \(1/Q_{\text{min}}^4\)
  (dimensional analysis)
- Naive suppression is \(m_q X_q/Q_{\text{min}}^2 \sim 2 \times 10^{-3}\)
- Observed is roughly that
Numerical results

![Graph showing numerical results for the muon g-2](image-url)
Short-distance: $1/Q_{min}^2$

- Can we understand scaling with $Q_{min}$?

$$a_{\mu} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^{1} d\tau \sqrt{1-\tau^2} \sum_{i=1,12} \hat{\pi}_i \Pi_i$$

- Do $Q_i \rightarrow \lambda Q_i$

- Overall factor goes as $\lambda^8$

- Quark loop has no scale thus $\Pi_i$ scale with their dimension
$$\hat{\Pi}_1, \hat{\Pi}_4 \sim \lambda^{-4}, \quad \hat{\Pi}_7, \hat{\Pi}_{17}, \hat{\Pi}_{39}, \hat{\Pi}_{54} \sim \lambda^{-6}$$

$$\Rightarrow \Pi_1,\ldots,4 \sim \lambda^{-4} \quad \Pi_5,\ldots,12 \sim \lambda^{-6}$$

- Expand the $T_i$ for $Q_i \gg m_\mu$: $T_1 \sim m_\mu^4$, $T_i \neq 1 \sim m_\mu^2$

$$T_1 \sim \lambda^{-8}, \quad T_2,3,4 \sim \lambda^{-6}, \quad T_5,\ldots,12 \sim \lambda^{-4}$$

- Put all together: quark-loop scales as $a_{\mu}^{SD \ ql} \sim \lambda^{-2}$

$m_q X_q$ adds an overall factor $\Rightarrow a_{\mu}^{SD \ X_q} \sim \lambda^{-4}$
Short distance HLbL Conclusions

- We have shown that the quarkloop really is the first term of a proper OPE expansion for the HLbL
- We have calculated the next term which is suppressed by quark masses and a small $X_q$: negligible
- The next term contains both the usual vacuum and a large number of induced condensates but will not be suppressed by small quark masses
- Why do this: matching of the sum over hadronic contributions to the expected short distance domain
- Finding the onset of the asymptotic domain
- Still to be worked out: overlap with what is in the other parts
Putting together

- LO-HVP:
  - Paris (Davier et al.)
    \[ a_{\mu}^{LO \ HVP} = 6939 \pm 10 \pm 34 \pm 16 \pm 1_{\psi} \pm 7_{QCD} \times 10^{-11} \]
  - Liverpool (Keshavarzi et al)
    \[ a_{\mu}^{LO \ HVP} = 6938(24) \times 10^{-11} \]
- HO-HVP: \[ a_{\mu}^{HO \ HVP} = -85.8(4) \times 10^{-11} \]
- HLbL: put all numbers together
  - \( \pi^0, \eta, \eta' \): \[ 93.8(4.0) \times 10^{-11} \]
  - \( \pi \)-loop: \[ -24(1) \times 10^{-11} \]
  - Kaon-loop: \[ -0.5 \times 10^{-11} \]
  - HO-HLbL \[ 3(2) \times 10^{-11} \]
  - Charm loop:\[ a_{\mu}^{HLbLc} = 3.1(1) \times 10^{-11} \]
  - Other hadronic: \[ 6(?) \times 10^{-11} \]
  - Short-distance: \[ 10(10) \times 10^{-11} \]
  - Sum: \[ a_{\mu}^{HLbL} = 91(17?) \times 10^{-11} \]
Putting together

- So what is the status?
  \[ a_{\mu}^{\text{exp}} - a_{\mu}^{\text{QED}} - a_{\mu}^{\text{EW}} = 7218(63) \times 10^{-11} \]

- LO-HVP: \[ a_{\mu}^{\text{LO HVP}} = 6939(40?) \times 10^{-11} \]
  Note: Babar/KLOE discrepancy

- HO-HVP: \[ a_{\mu}^{\text{HO HVP}} = -85.8(4) \times 10^{-11} \]

- HLbL: \[ a_{\mu}^{\text{HLbL}} = 91(17?) \times 10^{-11} \]
  Note: error from overlap and guessing putting together

- Sum: \[ 6952(44?) \times 10^{-11} \]

- Difference: \[ \Delta a_{\mu} = 274(77?) \times 10^{-11} \]
Putting together

- Difference: $\Delta a_\mu = 274(77?) \times 10^{-11}$
- HVP: Babar/KLOE: Belle II, Novosibirsk
- Lattice HVP: not competitive now but will get there
- HLbL: improvements have happened
- Lattice HLbL: numbers compatible
- **Theory**: errors are slowly going down
  no large changes in numbers expected, in fact numbers
  quite stable since 2000
- **Experiment**: FNAL is running;
  first result expected late this year or early next year
- Thank you for listening