HADRONTIC LIGHT-BY-LIGHT CONTRIBUTION: THE (RESONANCE) LAGRANGIAN APPROACH

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Hadronic contributions to the muon anomalous magnetic moment: strategies for improvements of the accuracy of the theoretical prediction, Mainz 1-5 April 2014
Overview

1. General

2. $\pi^0$-exchange

3. $\pi$-loop: new stuff is here
HLbL: the main object to calculate

- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks
The overall

\[ a^\text{HLbL}_\mu = -\frac{1}{48 m_\mu} \text{tr}[(\not{p} + m_\mu) M^{\lambda \beta}(0) (\not{p} + m_\mu)[\gamma_\lambda, \gamma_\beta]]. \]

\[ M^{\lambda \beta}(p_3) = |e|^6 \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_2}{(2\pi)^4} \frac{1}{q^2 p_1^2 p_2^2 (p_4^2 - m_\mu^2) (p_5^2 - m_\mu^2)} \times \left[ \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_3^\lambda} \right] \gamma_\alpha(p_4 + m_\mu) \gamma_\nu(p_5 + m_\mu) \gamma_\rho. \]

- We used: \( \Pi^{\rho \nu \alpha \lambda}(p_1, p_2, p_3) = -p_3^\beta \frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_3^\lambda}. \)

- Can calculate at \( p_3 = 0 \) but must take derivative
- derivative: improves convergence

Four point function of \( V_i^\mu(x) \equiv \sum_i Q_i [\bar{q}_i(x) \gamma_\mu q_i(x)] \)

\[ \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3) \equiv i^3 \int d^4 x \int d^4 y \int d^4 z e^{i(p_1 \cdot x + p_2 \cdot y + p_3 \cdot z)} \times \langle 0 | T \left( V_d^\rho(0) V_b^\nu(x) V_c^\alpha(y) V_d^\beta(z) \right) |0 \rangle \]
General properties

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) = \]

Actually we really need

\[ \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \]
General properties

\[ \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3): \]

- In general 138 Lorentz structures (but only 28 contribute to \( g - 2 \))

- Using \( q^\rho \Pi^{\rho\nu\alpha\beta} = p_{1\nu} \Pi^{\rho\nu\alpha\beta} = p_{2\alpha} \Pi^{\rho\nu\alpha\beta} = p_{3\beta} \Pi^{\rho\nu\alpha\beta} = 0 \)

- 43 gauge invariant structures

- Bose symmetry relates some of them

- All depend on \( p_1^2, p_2^2 \) and \( q^2 \), but before derivative and \( p_3 \to 0 \) also \( p_3^2, p_1 \cdot p_2, p_1 \cdot p_3 \)

- Compare HVP: one function, one variable

- General calculation from experiment difficult; but see other contributions at this workshop

- In four photon measurement: lepton contribution
General properties

\[ \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \]  

plus loops inside the hadronic part

- 8 dimensional integral, three trivial,
- 5 remain: \( p_1^2, p_2^2, p_1 \cdot p_2, p_1 \cdot p_\mu, p_2 \cdot p_\mu \)
- Rotate to Euclidean space:
  - Easier separation of long and short-distance
  - Artefacts (confinement) in models smeared out.
- More recent: can do two more using Gegenbauer techniques Knecht-Nyffeler, Jegerlehner-Nyffeler, JB–Zahiri-Abyaneh–Relefors
- \( P_1^2, P_2^2 \) and \( Q^2 \) remain
- study \( a^X_\mu = \int dl_{P_1} dl_{P_2} a^{XLL}_\mu = \int dl_{P_1} dl_{P_2} dl_{Q} a^{XLLQ}_\mu \)
- \( l_P = \ln (P/\text{GeV}) \), to see where the contributions are
- Study the dependence on the cut-off for the photons
Gegenbauer

- $P_1, P_2, Q = P_1 + P_2, P_4 = P_\mu - P_1, P_5 = P_\mu + P_2$
- Average over muon-direction using:

\[
\left\langle \frac{1}{(P_4^2 + m^2)(P_5^2 + m^2)} \right\rangle_\mu = \delta X, \\
\left\langle \frac{P \cdot P_2}{P_4^2 + m^2} \right\rangle_\mu = \frac{1}{8} \delta P_1 \cdot P_2 r_2^2, \\
\left\langle \frac{1}{P_5^2 + m^2} \right\rangle_\mu = \frac{1}{2} \delta r_1, \\
\left\langle \frac{1}{P_4^2 + m^2} \right\rangle_\mu = \frac{1}{2} \delta r_2.
\]

\[
\delta = \frac{1}{m^2}, \quad r_i = 1 - \sqrt{1 + \frac{4m^2}{P_i^2}}, \quad X = \frac{1}{P_1 P_2 \sin \theta} \arctan \left( \frac{z \sin \theta}{1 - z \cos \theta} \right),
\]

\[
\cos \theta = \frac{P_1 \cdot P_2}{P_1 P_2}, \quad z = \frac{P_1 P_2}{4m^2} r_1 r_2, \quad \rho_1 = P_1^2, \quad \rho_2 = P_2^2, \quad \rho_3 = P_1 \cdot P_2.
\]
\[
\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \equiv \\
\Pi^1(p_1, p_2, p_3)g^{\rho\nu}g^{\alpha\beta} + \Pi^2(p_1, p_2, p_3)g^{\rho\alpha}g^{\nu\beta} + \Pi^3(p_1, p_2, p_3)g^{\rho\beta}g^{\nu\alpha} \\
+ \Pi^{1jk}(p_1, p_2, p_3)g^{\rho\nu}p_j^\alpha p_k^\beta + \Pi^{2jk}(p_1, p_2, p_3)g^{\rho\alpha}p_j^\nu p_k^\beta \\
+ \Pi^{3jk}(p_1, p_2, p_3)g^{\rho\beta}p_j^\nu p_k^\alpha + \Pi^{4jk}(p_1, p_2, p_3)g^{\nu\alpha}p_j^\rho p_k^\beta \\
+ \Pi^{5jk}(p_1, p_2, p_3)g^{\nu\beta}p_j^\rho p_k^\alpha + \Pi^{6jk}(p_1, p_2, p_3)g^{\alpha\beta}p_j^\rho p_k^\nu \\
+ \Pi^{ijkm}(p_1, p_2, p_3)p_i^\rho p_j^\nu p_k^\beta p_m^\alpha
\]

- Use Ward identities to rewrite in the \( \Pi^{ijkm}(p_1, p_2, p_3) \)
- is redundant (81 rather than 43), but easiest to implement and can be done without negative powers of momenta
- \( \delta / \delta p_{3\lambda}, \ p_3 \rightarrow 0 \)
- \( \Pi^{3jk}(p_1, p_2, p_3), \ \Pi^{i3km}(p_1, p_2, p_3), \ \Pi^{ij3m}(p_1, p_2, p_3) \)
- \( (\delta / \delta p_{3\lambda})(\Pi^{ij1}(p_1, p_2, p_3) - \Pi^{ijk2}(p_1, p_2, p_3)): \ 32 \) left
- \( a_\mu = \frac{\alpha^3}{2\pi^2} \int P_1^2 dP_1^2 P_2^2 dP_2^2 \sin \theta d\cos \theta A_\Pi(P_1, P_2, \cos \theta) \)
$A_\Pi(P_1, P_2, \cos \theta)$

$\Pi^{1131}$  
$(-1/6 \rho_3^2 r_2^2 \delta - 2/3 \rho_1 \rho_3 r_2 \delta + 8/3 \rho_1 \rho_3 X - \rho_1^2 r_1 \delta - 4/3 \rho_1^2 \rho_3 X \delta - 2 \rho_1^2 \rho_2 X \delta)$

$\Pi^{1132}$  
$+ (2/3 \rho_3 + 1/3 \rho_2 \rho_3 r_2 \delta - 1/6 \rho_2 \rho_3^2 \delta - 2/3 \rho_1 \rho_3 r_1 \delta - 1/6 \rho_1 \rho_3 r_1^2 \delta - 2/3 \rho_1 \rho_2 r_2 \delta + 1/3 \rho_1 \rho_2 r_1 \delta + 8/3 \rho_1 \rho_2 X - 4/3 \rho_1 \rho_2 \rho_3 X \delta + 2/3 \rho_1^2 \rho_2 X \delta - 4/3 \rho_1^2 \rho_2 X \delta)$

$\Pi^{1231}$  
$(-2/3 \rho_3^2 r_2 \delta - 1/6 \rho_2 \rho_3^2 \delta - 2/3 \rho_1 \rho_3 r_1 \delta - 4/3 \rho_1 \rho_3 X \delta + 1/3 \rho_1 \rho_2 r_2 \delta + 8/3 \rho_1 \rho_2 X - 4/3 \rho_1 \rho_2 \rho_3 X \delta + 2/3 \rho_1^2 \rho_2 X \delta)$

$\Pi^{1232}$  
$(-2/3 \rho_3^2 r_1 \delta - 2/3 \rho_2 - 2/3 \rho_2 \rho_3 r_2 \delta + 8/3 \rho_2 \rho_3 X - 4/3 \rho_2 \rho_3^2 X \delta - 1/3 \rho_2 r_2 \delta - 1/3 \rho_1 \rho_2 r_1 \delta - 4/3 \rho_1 \rho_2 \rho_3 X \delta - 2/3 \rho_1 \rho_2 X \delta)$

$\Pi^{1311}$  
$+ (1/3 \rho_1 \rho_3 r_2 \delta + 1/3 \rho_1^2 r_1 \delta + 2/3 \rho_1^2 \rho_3 X \delta + 2/3 \rho_1^2 \rho_2 X \delta)$

$\Pi^{1312}$  
$(-2/3 \rho_3^2 r_2 \delta + 4/3 \rho_3^2 X - 1/12 \rho_2 \rho_3 r_2 \delta - 4/3 \rho_1 \rho_3 r_1 \delta - 1/12 \rho_1 \rho_3 r_1 \delta - 4/3 \rho_1 \rho_3^2 X \delta + 1/2 \rho_1 \rho_2 r_2 \delta + 1/6 \rho_1 \rho_2 r_1 \delta + 4/3 \rho_1 \rho_2 X - 8/3 \rho_1 \rho_2 \rho_3 X \delta + 1/3 \rho_1 \rho_2 X \delta + \rho_1^2 \rho_2 X \delta)$

$\Pi^{1322}$  
$(-2/3 \rho_2 - 2/3 \rho_2 \rho_3 r_2 \delta + 8/3 \rho_2 \rho_3 X - 1/3 \rho_2^2 r_2 \delta - 2 \rho_1 \rho_2 r_1 \delta - 4/3 \rho_1 \rho_2 \rho_3 X \delta - 4 \rho_1 \rho_2 X \delta)$

$\Pi^{2131}$  
$(-2/3 \rho_1 - 2/3 \rho_1 \rho_3 r_2 \delta + 8/3 \rho_1 \rho_3 X - 2 \rho_1 \rho_2 r_2 \delta - 4/3 \rho_1 \rho_2 \rho_3 X \delta - 1/3 \rho_1^2 r_1 \delta - 4 \rho_1 \rho_2 X \delta)$

$\Pi^{2231}$  
$+ (1/6 \rho_1 \rho_2 r_2 \delta + 1/2 \rho_1 \rho_2 r_1 \delta + 4/3 \rho_1 \rho_2 X - 8/3 \rho_1 \rho_2 \rho_3 X \delta + \rho_1 \rho_2^2 X \delta + 1/3 \rho_1^2 \rho_2 X \delta)$

$\Pi^{2232}$  
$+ (1/3 \rho_2 \rho_3 r_1 \delta + 1/3 \rho_2^2 r_2 \delta + 2/3 \rho_2^2 \rho_3 X \delta + 2/3 \rho_1 \rho_2 \rho_3 X \delta)$

$\Pi^{2311}$  
$(-2/3 \rho_3^2 r_2 \delta - 2/3 \rho_1 - 2/3 \rho_1 \rho_3 r_1 \delta + 8/3 \rho_1 \rho_3 X - 4/3 \rho_1 \rho_3^2 X \delta - 1/3 \rho_1 \rho_2 r_2 \delta - 4/3 \rho_1 \rho_2 \rho_3 X \delta - 1/3 \rho_1 r_1 \delta - 2/3 \rho_1 \rho_2 X \delta)$

$\Pi^{2312}$  
$(-2/3 \rho_3^2 r_1 \delta - 2/3 \rho_2 \rho_3 r_2 \delta - 4/3 \rho_2 \rho_3^2 X \delta - 1/6 \rho_1 \rho_3 r_1^2 \delta + 1/3 \rho_1 \rho_2 r_1 \delta + 8/3 \rho_1 \rho_2 X - 4/3 \rho_1 \rho_2 \rho_3 X \delta + 2/3 \rho_1 \rho_2 X \delta)$
\[ +\Pi_{321}^{321} ( +2/3 \rho_3 - 2/3 \rho_2 \rho_3 r_2 \delta - 1/6 \rho_2 \rho_3 r_2^2 \delta + 1/3 \rho_1 \rho_3 \rho_1 \delta - 1/6 \rho_1 \rho_3 \rho_1 \delta + 1/3 \rho_1 \rho_2 \rho_2 \delta \\
-2/3 \rho_1 \rho_2 \rho_2 \delta + 8/3 \rho_1 \rho_2 X - 4/3 \rho_1 \rho_2 \rho_3 X \delta - 4/3 \rho_1 \rho_2 X \delta + 2/3 \rho_1 \rho_2 X \delta ) \\
+\Pi_{322}^{322} ( -1/6 \rho_3 \rho_1 \rho_1 \delta - 2/3 \rho_2 \rho_3 \rho_1 \delta + 8/3 \rho_2 \rho_3 X - \rho_2^2 \rho_2 \delta - 4/3 \rho_2 \rho_3 X \delta - 2 \rho_1 \rho_2 \rho_2 \delta ) \\
+\Pi_{311}^{311} ( +1/6 \rho_3 \rho_1 \rho_1 \delta - 2/3 \rho_1 - 4/3 \rho_1 \rho_3 \rho_2 \delta + 1/2 \rho_1 \rho_3 \rho_2 \delta - 1/3 \rho_1 \rho_2 \rho_2 \delta - \rho_2 \rho_2 \delta \\
-1/3 \rho_1 \rho_1 \delta - 8/3 \rho_1 \rho_3 X \delta - 2/3 \rho_2 X \delta - 2 \rho_1 \rho_1 \delta ) \\
+\Pi_{312}^{312} ( +4/3 \rho_3 + 2/3 \rho_2 \rho_3 \rho_2 \delta + 1/6 \rho_2 \rho_3 \rho_2 \delta + 2 \rho_3 \rho_1 + 2 \rho_1 \rho_3 \rho_1 \delta - 1/3 \rho_1 \rho_3 \rho_1 \delta \\
+8 \rho_3 \rho_1 X + 2/3 \rho_1 \rho_2 \rho_2 \delta + 8 \rho_3 \rho_1 X + 4 \rho_1 \rho_2 \rho_3 X \delta + 4 \rho_1 \rho_2 \rho_2 \delta + 1/3 \rho_1 \rho_1 \delta ) \\
+\Pi_{321}^{321} ( +2 \rho_1 + \rho_1 \rho_1 \delta ) + \Pi_{322}^{322} ( +2 \rho_2 + \rho_2 \rho_2 \delta ) \\
+\Pi_{321}^{321} ( +4/3 \rho_3 - 8 \rho_3 \rho_2 \rho_3 \rho_2 \delta + 2 \rho_3 \rho_1 + 2 \rho_3 \rho_3 \rho_1 \delta - 1/6 \rho_3 \rho_3 \rho_1 \delta \\
+8 \rho_3 \rho_1 X + 1/3 \rho_1 \rho_2 \rho_2 \delta + 1/3 \rho_1 \rho_2 \rho_1 \delta + 4 \rho_1 \rho_2 \rho_3 X \delta + 2 \rho_1 \rho_2 \rho_2 \delta + 2 \rho_1 \rho_1 \delta ) \\
+\Pi_{322}^{322} ( +4/3 \rho_3 - 8 \rho_3 \rho_2 \rho_3 \rho_2 \delta + 2 \rho_3 \rho_1 + 2 \rho_3 \rho_3 \rho_1 \delta - 1/6 \rho_3 \rho_3 \rho_1 \delta \\
+8 \rho_3 \rho_1 X + 2 \rho_1 \rho_3 \rho_2 \rho_2 \delta + 8 \rho_3 \rho_1 X + 4 \rho_1 \rho_2 \rho_3 X \delta + 4 \rho_1 \rho_2 \rho_2 \delta + 2 \rho_1 \rho_1 \delta ) \\
+\Pi_{321}^{322} ( +4/3 \rho_3 + 2 \rho_3 + 2 \rho_3 \rho_3 \rho_2 \delta + 1/3 \rho_3 \rho_3 \rho_2 \delta - 8 \rho_3 \rho_3 X + 1 \rho_3 \rho_3 r_2 \delta + 2 \rho_1 \rho_3 \rho_1 \delta \\
+1/6 \rho_3 \rho_3 r_2 \delta + 2 \rho_3 \rho_1 \rho_2 \rho_2 \delta + 1/3 \rho_1 \rho_2 \rho_2 \delta + 1/3 \rho_1 \rho_2 \rho_2 \delta + 4 \rho_1 \rho_2 \rho_3 X \delta + 2 \rho_1 \rho_2 \rho_2 \delta + 2 \rho_1 \rho_1 \delta ) \\
+\Pi_{322}^{311} ( +1/6 \rho_3 \rho_3 r_1 \delta - 2 \rho_3 \rho_3 - 4 \rho_3 \rho_3 \rho_3 \rho_2 \delta + 1/2 \rho_3 \rho_3 r_1 \delta - 1 \rho_3 \rho_3 \rho_2 \delta - \rho_2 \rho_2 \delta - 8 \rho_3 \rho_3 X \delta \\
-2 \rho_3 X \delta - 1/3 \rho_1 \rho_2 \rho_2 \delta + 2 \rho_1 \rho_1 \delta ) \\
+\Pi_{321}^{311} ( -1/3 \rho_1 \rho_3 + 2 \rho_1 \rho_3 \rho_3 X - 1/6 \rho_1 \rho_2 \rho_3 \rho_2 \delta + 1/24 \rho_1 \rho_2 \rho_3 \rho_2 \delta - 1/6 \rho_1 \rho_3 \rho_1 \delta \\
+1/24 \rho_1 \rho_3 \rho_3 r_1 \delta - 1/12 \rho_1 \rho_2 \rho_2 \delta - 1/12 \rho_1 \rho_2 \rho_2 \delta - 2 \rho_1 \rho_2 X - 1/3 \rho_1 \rho_2 \rho_3 X \delta \\
-1/6 \rho_1 \rho_2 X \delta - 1/6 \rho_1 \rho_2 X \delta )
\]
\[ +\Pi_{D121} \left( +1/3 \rho_3^2 - 2/3 \rho_3^3 X + 1/6 \rho_2 \rho_3^2 r_2 \delta - 1/24 \rho_2 \rho_3^2 r_2 \delta + 1/6 \rho_1 \rho_3^2 r_1 \delta - 1/24 \rho_1 \rho_3^2 r_1 \delta + 1/12 \rho_1 \rho_2 \rho_3 r_2 \delta + 1/12 \rho_1 \rho_2 \rho_3 r_1 \delta + 2/3 \rho_1 \rho_2 \rho_3 X + 1/3 \rho_1 \rho_2 \rho_3 X \delta + 1/6 \rho_1 \rho_2 \rho_3 X \delta \right) \]

\[ +\Pi_{D122} \left( +2/3 \rho_2 \rho_3 - 4/3 \rho_2 \rho_3^2 X + 1/3 \rho_2 \rho_3 r_2 \delta - 1/12 \rho_2 \rho_3 r_2 \delta + 1/3 \rho_1 \rho_2 \rho_3 r_1 \delta - 1/12 \rho_1 \rho_2 \rho_3 r_2 \delta + 1/6 \rho_1 \rho_2 \rho_3 r_1 \delta + 4/3 \rho_1 \rho_2 \rho_3 X + 2/3 \rho_1 \rho_2 \rho_3 X \delta + 1/3 \rho_1 \rho_2 \rho_3 X \delta + 1/3 \rho_1 \rho_2 \rho_3 X \delta \right) \]

\[ +\Pi_{D211} \left( -2/3 \rho_1 \rho_3 + 4/3 \rho_1 \rho_3^2 X - 1/3 \rho_1 \rho_2 \rho_3 r_2 \delta \right) \]

\[ +\Pi_{D221} \left( -1/3 \rho_3^2 + 2/3 \rho_3^3 X - 1/6 \rho_2 \rho_3^2 r_2 \delta + 1/24 \rho_2 \rho_3^2 r_2 \delta - 1/6 \rho_1 \rho_3^2 r_1 \delta + 1/24 \rho_1 \rho_3^2 r_1 \delta - 1/12 \rho_1 \rho_2 \rho_3 r_2 \delta - 1/12 \rho_1 \rho_2 \rho_3 r_1 \delta - 2/3 \rho_1 \rho_2 \rho_3 X - 1/3 \rho_1 \rho_2 \rho_3 X \delta - 1/6 \rho_1 \rho_2 \rho_3 X \delta \right) \]

\[ +\Pi_{D222} \left( +1/3 \rho_2 \rho_3 - 2/3 \rho_2 \rho_3^2 X + 1/6 \rho_2 \rho_3 r_2 \delta - 1/24 \rho_2 \rho_3 r_2 \delta + 1/6 \rho_1 \rho_2 \rho_3 r_1 \delta - 1/24 \rho_1 \rho_2 \rho_3 r_2 \delta + 1/12 \rho_1 \rho_2 \rho_3 r_2 \delta + 1/12 \rho_1 \rho_2 \rho_3 r_1 \delta + 2/3 \rho_1 \rho_2 \rho_3 X + 1/3 \rho_1 \rho_2 \rho_3 X \delta + 1/6 \rho_1 \rho_2 \rho_3 X \delta \right) \].

- “Only” 28 contribute
- Full formula fairly “short”
\( \pi^0 \) exchange

- \( \text{“}\pi^0\text{”} = 1/(p^2 - m^2_\pi) \)
- The blobs need to be modelled, and in e.g. ENJL contain corrections also to the \( 1/(p^2 - m^2_\pi) \)
- Pointlike has a logarithmic divergence
- Numbers \( \pi^0 \), but also \( \eta, \eta' \)
\( \pi^0 \) exchange

- **BPP:**
  \[ a_\mu^{\pi^0} = 5.9(0.9) \times 10^{-10} \]

- **Nonlocal quark model:**
  \[ a_\mu^{\pi^0} = 6.27 \times 10^{-10} \]

- **DSE model:**
  \[ a_\mu^{\pi^0} = 5.75 \times 10^{-10} \]
  Goecke, Fischer and Williams, Phys.Rev.D83(2011)094006[1012.3886]

- **LMD+V:**
  \[ a_\mu^{\pi^0} = (5.8 - 6.3) \times 10^{-10} \]

- **Formfactor inspired by AdS/QCD:**
  \[ a_\mu^{\pi^0} = 6.54 \times 10^{-10} \]
  Cappiello, Cata and D'Ambrosio, Phys.Rev.D83(2011)093006 [1009.1161]

- **Chiral Quark Model:**
  \[ a_\mu^{\pi^0} = 6.8 \times 10^{-10} \]

- **Constraint via magnetic susceptibility:**
  \[ a_\mu^{\pi^0} = 7.2 \times 10^{-10} \]

- All in reasonable agreement
\( \pi^0 \) exchange: most recent addition

- Kampf Novotny 1104.3137, Roig, Guevara, Lópe
z Castro, 1401.4099

- \( R\chi T \): study \( VVP \) Green function, \( e^+ e^- \rightarrow \omega \pi^0 \) and \( \pi \gamma^* \gamma \) transition form-factor

- \( VVP, V\gamma P \) vertices.

- Lagrangians Kampf Novotny 1104.3137 Roig Sanz-Cillero 1312.6206

- Small violation of Brodsky-Lepage in \( \pi \gamma^* \gamma \)

- Include vector and pseudo-scalar nonet

- Short distance constraints require \( F_V = \sqrt{3} F \) (KSRF \( \sqrt{2} \))

\[
F_V = \sqrt{3} F, \quad c_{125} = 0, \quad c_{1256} = -\frac{N_C M_V}{32\sqrt{2}\pi^2 F_V} \sim -3.26 \cdot 10^{-2}, \quad c_{1235} = 0
\]

\[
d_{123} = \frac{F^2}{8 F_V^2} = \frac{1}{24}, \quad d_3 = -\frac{N_C M_V^2}{64\pi^2 F_V^2} \sim -0.112, \quad d_s = \frac{\sqrt{2} M_V c_{1256} - 2 d_3 F_V}{F_V^1} = 0
\]

- Note short-distance matching must be done in many channels, JB,Gamiz,Lipartia,Prades, hep-ph/0304222: with finite number of resonances this requires compromises
\[ \mathcal{F}_{\pi^0\gamma\gamma}(p^2, q^2, 0) = \frac{2}{3F} \left[ -\frac{N_C}{8\pi^2} + \frac{4F_V^2 d_3(p^2 + q^2)}{(M_V^2 - p^2)(M_V^2 - q^2)} + 2\sqrt{2} \frac{F_V}{M_V} \frac{p^2 c_{1256} - q^2 c_{125}}{M_V^2 - p^2} \right. \\
+ 2\sqrt{2} \frac{F_V}{M_V} \frac{q^2 c_{1256} - p^2 c_{125}}{M_V^2 - q^2} \right]. \]

\[ \mathcal{F}_{\pi^0\gamma\gamma}(p^2, q^2, r^2) = \frac{2r^2}{3F} \left[ -\frac{N_C}{8\pi^2 r^2} + 4F_V^2 \frac{d_3(p^2 + q^2)}{(M_V^2 - p^2)(M_V^2 - q^2)r^2} + \frac{4F_V^2 d_{123}}{(M_V^2 - p^2)(M_V^2 - q^2)} \right. \\
-2\sqrt{2} \frac{F_V}{M_V} \frac{r^2 c_{1235} - p^2 c_{1256} + q^2 c_{125}}{(M_V^2 - p^2)(M_V^2 - q^2)r^2} - 2\sqrt{2} \frac{F_V}{M_V} \frac{r^2 c_{1235} - q^2 c_{1256} + p^2 c_{125}}{(M_V^2 - q^2)r^2} + \frac{64P_1}{M_P^2 - r^2} \\
- \frac{16\sqrt{2}P_2 F_V}{(M_V^2 - p^2)(M_P^2 - r^2)} - \frac{16\sqrt{2}P_2 F_V}{(M_V^2 - q^2)(M_P^2 - r^2)} + \frac{16F_V^2 P_3}{(M_V^2 - p^2)(M_V^2 - q^2)(M_P^2 - r^2)} \right]. \]

- Kampf Novotny 1104.3137
- Roig, Guevara, López Castro, 1401.4099

No \( r^2 \) (i.e. pole)

\[ a_{\mu}^{\pi^0} = (6.58 \pm 0.12) \times 10^{-10} \]

\[ a_{\mu}^{\pi^0} = (6.65 \pm 0.19) \times 10^{-10} \]

\[ a_{\mu}^{\pi^0} = (5.75 \pm 0.05) \times 10^{-10} \]
MV short-distance: $\pi^0$ exchange

- take $P_1^2 \approx P_2^2 \gg Q^2$: Leading term in OPE of two vector currents is proportional to axial current
  \[ \Pi^{\rho\nu\alpha\beta} \propto \frac{P_\rho}{P_1^2} \langle 0| T (J_A \nu J_V \alpha J_V \beta) |0 \rangle \]
- $J_A$ comes from
- AVV triangle anomaly: extra info
  - Implemented via setting one blob = 1
- $a_{\mu}^{\pi^0} = 7.7 \times 10^{-10}$
The pointlike vertex implements shortdistance part, not only $\pi^0$-exchange

Are these part of the quark-loop? See also in


BPP quarkloop + $\pi^0$-exchange $\approx$ MV $\pi^0$-exchange
\( \pi^0 \) exchange


- \( a_\mu = \int dl_1 dl_2 a^{LL}_\mu \) with \( l_i = \log(P_i/\text{GeV}) \)

Which momentum regions do what:
volume under the plot \( \propto a_\mu \)
Pseudoscalar exchange

- Point-like VMD: \( \pi^0 \eta \) and \( \eta' \) give 5.58, 1.38, 1.04.
- Roig et al. 6.65, 2.03, 1.75
- Models that include \( U(1)_A \) breaking give similar ratios
- Pure large \( N_c \) models use this ratio
- The MV argument should give some enhancement over the full VMD like models
- Total pseudo-scalar exchange is about
  \[ a_{\mu}^{PS} = 8 - 10 \times 10^{-10} \]
- AdS/QCD estimate (includes excited pseudo-scalars)
  \[ a_{\mu}^{PS} = 10.7 \times 10^{-10} \]

A bare $\pi$-loop (sQED) give about $-4 \cdot 10^{-10}$

The $\pi\pi\gamma^*$ vertex is always done using VMD

$\pi\pi\gamma^*\gamma^*$ vertex two choices:
- Hidden local symmetry model: only one $\gamma$ has VMD
- Full VMD
- Both are chirally symmetric
- The HLS model used has problems with $\pi^+ - \pi^0$ mass difference (due to not having an $a_1$)

Final numbers quite different: $-0.45$ and $-1.9 \times 10^{-10}$

For BPP stopped at 1 GeV but within 10% of higher $\Lambda$
\[\pi\] loop: Bare vs VMD

- Plotted \( a_{\mu}^{LLQ} \) for \( P_1 = P_2 \)
- \[
  a_{\mu} = \int dl_{P_1} dl_{P_2} dl_Q \ a_{\mu}^{LLQ}
\]
- \( l_Q = \log(Q/1 \text{ GeV}) \)
\[ \pi \text{ loop: VMD vs HLS} \]

Usual HLS, \( a = 2 \)
HLbL: Lagrangian Approach
Johan Bijnens
General
$\pi^0$-exchange
$\pi$-loop

$\pi$ loop: VMD vs HLS

HLS with $a = 1$, satisfies more short-distance constraints
\( \pi \) loop

- \( \pi\pi\gamma^*\gamma^* \) for \( q_1^2 = q_2^2 \) has a short-distance constraint from the OPE as well.
- HLS does not satisfy it
- full VMD does: so probably better estimate

Ramsey-Musolf suggested to do pure ChPT for the \( \pi \) loop


- Later added \( a_1 \) Engel and Ramsey-Musolf, arXiv:1309.2225

- Polarizability \((L_9 + L_{10})\) up to 10%, charge radius 30% at low energies
- Both HLS and VMD have charge radius effect but not polarizability
\( \pi \) loop

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π loop: \( L_9, L_{10} \)

- ChPT for muon \( g - 2 \) at order \( p^6 \) is not powercounting finite so no prediction for \( a_\mu \) exists.
- But can be used to study the low momentum end of the integral over \( P_1, P_2, Q \)
- The four-photon amplitude is finite still at two-loop order (counterterms start at order \( p^8 \))
- Add \( L_9 \) and \( L_{10} \) vertices to the bare pion loop

JB-Zahiri-Abyaneh

- Program the Euler-Heisenberg plus NLO result of Ramsey-Musolf et al. into our programs for \( a_\mu \)
- Bare pion-loop and \( L_9, L_{10} \) part in limit \( p_1, p_2, q \ll m_\pi \) agree with Euler-Heisenberg plus next order analytically
\( \pi \) loop: \( L_9, L_{10} \)

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π loop: VMD vs charge radius

low scale, charge radius effect well reproduced
\( \pi \) loop: VMD vs \( L_9 \) and \( L_{10} \)

- \( L_9 + L_{10} \neq 0 \) gives an enhancement of 10-15\%
- To do it fully need to get a model: include \( a_1 \)
Include $a_1$

- $L_9 + L_{10}$ effect is from

- But to get gauge invariance correctly need
Include $a_1$

- Consistency problem: full $a_1$-loop?
- Treat $a_1$ and $\rho$ classical and $\pi$ quantum: there must be a $\pi$ that closes the loop
  - Argument: integrate out $\rho$ and $a_1$ classically, then do pion loops with the resulting Lagrangian
- To avoid problems: representation without $a_1-\pi$ mixing
- Check for curiosity what happens if we add $a_1$-loop
Include $a_1$

- Use antisymmetric vector representation for $a_1$ and $\rho$
- Fields $A_{\mu\nu}$, $V_{\mu\nu}$ (nonets)
- Kinetic terms: $-\frac{1}{2} \left\langle \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{1}{2} V_{\mu\nu} V^{\mu\nu} \right\rangle$
  $-\frac{1}{2} \left\langle \nabla^\lambda A_{\lambda\mu} \nabla_\nu A^{\nu\mu} - \frac{1}{2} A_{\mu\nu} A^{\mu\nu} \right\rangle$
- Terms that give contributions to the $L_i^r$:
  \[
  \frac{F_V}{2\sqrt{2}} \left\langle f_{+\mu\nu} V^{\mu\nu} \right\rangle + \frac{iG_V}{\sqrt{2}} \left\langle V^{\mu\nu} u_\mu u_\nu \right\rangle + \frac{F_A}{2\sqrt{2}} \left\langle f_{-\mu\nu} A^{\mu\nu} \right\rangle
  \]
- $L_9 = \frac{F_V G_V}{2M_V^2}$, $L_{10} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}$
- Weinberg sum rules: (Chiral limit)
  \[
  F_V^2 = F_A^2 + F_\pi^2
  \]
  \[
  F_V^2 M_V^2 = F_A^2 M_A^2
  \]
- VMD for $\pi\pi\gamma$:
  \[
  F_V G_V = F_\pi^2
  \]
\( V_{\mu\nu} \) only

- \( \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \) is not finite
  (but was also not finite for HLS)
- But \( \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0} \) also not finite
  (but was finite for HLS)
- Derivative one finite for \( G_V = F_V / 2 \)
- Surprise: \( g - 2 \) identical to HLS with \( a = \frac{F_V^2}{F_\pi^2} \)
- Yes I know, different representations are identical BUT they do differ in higher order terms and even in what is higher order
- Same comments as for HLS numerics
$$V_{\mu\nu} \text{ only}$$

- $\Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)$ is not finite
  (but was also not finite for HLS)
- But $\left. \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \right|_{p_3=0}$ also not finite
  (but was finite for HLS)
- Derivative one finite for $G_V = F_V/2$
- Surprise: $g - 2$ identical to HLS with $a = \frac{F_V^2}{F_\pi^2}$
- Yes I know, different representations are identical BUT they do differ in higher order terms and even in what is higher order
- Same comments as for HLS numerics
\( V_{\mu\nu} \) and \( A_{\mu\nu} \)

- Add \( a_1 \)
- Calculate a lot

\[
\left. \frac{\delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \right|_{p_3=0}
\]

finite for:

- \( G_V = F_V = 0 \) and \( F_A^2 = -2F_\pi^2 \)
- If adding full \( a_1 \)-loop \( G_V = F_V = 0 \) and \( F_A^2 = -F_\pi^2 \)

Clearly unphysical (but will show some numerics anyway)
Add $a_1$

Calculate a lot

$$\frac{\delta \Pi^{\rho \nu \alpha \beta}(p_1, p_2, p_3)}{\delta p_{3\lambda}} \bigg|_{p_3=0}$$

finite for:

- $G_V = F_V = 0$ and $F_A^2 = -2F_{\pi}^2$
- If adding full $a_1$-loop $G_V = F_V = 0$ and $F_A^2 = -F_{\pi}^2$

Clearly unphysical (but will show some numerics anyway)
$V_{\mu\nu}$ and $A_{\mu\nu}$

- Start by adding $\rho a_1 \pi$ vertices
  
  $\lambda_1 \langle [V_{\mu\nu}, A_{\mu\nu}] \chi_- \rangle + \lambda_2 \langle [V_{\mu\nu}, A_{\nu\alpha}] h_{\mu}^{\nu} \rangle$
  $+ \lambda_3 \langle i \left[ \nabla^\mu V_{\mu\nu}, A_{\nu\alpha} \right] u_\alpha \rangle + \lambda_4 \langle i \left[ \nabla_\alpha V_{\mu\nu}, A_{\alpha\nu} \right] u^\mu \rangle$
  $+ \lambda_5 \langle i \left[ \nabla^\alpha V_{\mu\nu}, A_{\mu\nu} \right] u_\alpha \rangle + \lambda_6 \langle i \left[ V_{\mu\nu}, A_{\mu\nu} \right] f_{-}^{\alpha} \rangle$
  $+ \lambda_7 \langle i V_{\mu\nu} A^{\mu\rho} A^{\nu}_{\rho} \rangle$

- All lowest dimensional vertices of their respective type
- Not all independent, there are three relations
- Follow from the constraints on $V_{\mu\nu}$ and $A_{\mu\nu}$ (thanks to Stefan Leupold)
$V_{\mu\nu}$ and $A_{\mu\nu}$: big disappointment

- Work a whole lot
  \[ \delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \Bigr|_{\delta p_3\lambda} \]
  \[ p_3 = 0 \]
  not obviously finite

- Work a lot more
  \[ \delta \Pi^{\rho\nu\alpha\beta}(p_1, p_2, p_3) \Bigr|_{\delta p_3\lambda} \]
  \[ p_3 = 0 \]
  finite, only same solutions as before

- Try the combination that show up in $g - 2$ only

- Work a lot

- Again, only same solutions as before

- Small loophole left: after the integration for $g - 2$ could be finite but many funny functions of $m_\pi$, $m_\mu$, $M_V$ and $M_A$ show up.
\[ \pi \text{ loop: add } a_1 \text{ and } F_A^2 = -2F_\pi^2 \]

- Lower at low energies, \( L_9 + L_{10} < 0 \) here
- Funny peak at \( a_1 \) mass
$a_1$-loop: cases with good $L_9$ and $L_{10}$

- Add $F_V$, $G_V$, and $F_A$
- Fix values by Weinberg sum rules and VMD in $\gamma^* \pi \pi$
- no $a_1$-loop
$a_1$-loop: cases with good $L_9$ and $L_{10}$

Add $a_1$ with $F_A^2 = +F_\pi^2$ and $a_1$-loop

Add the full VMD as done earlier for the bare pion loop
Integration results

\[P_1, P_2, Q \leq \Lambda\]
Integration results

\[ a_1 F_A^2 = -2F^2 \]
\[ a_1 F_A^2 = -1 \ a_1\text{-loop} \]

HLS

HLS \(a=1\)

VMD

\(a_1\) VMD

\(a_1\) Weinberg

\(P_1, P_2, Q \leq \Lambda\)
Integration results with $a_1$

- Problem: get high energy behaviour good enough
- But all models with reasonable $L_9$ and $L_{10}$ fall way inside the error quoted earlier ($-1.9 \pm 1.3 \times 10^{-10}$)
- Tentative conclusion: Use hadrons only below about 1 GeV: $a_{\mu}^{\pi - \text{loop}} = (-2.0 \pm 0.5) \times 10^{-10}$
- Note that Engel and Ramsey-Musolf, arXiv:1309.2225 is a bit more pessimistic quoting numbers from ($-1.1 \text{ to } -7.1) \times 10^{-10}$
## Summary: ENJL vs PdRV

<table>
<thead>
<tr>
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<th>BPP</th>
<th>PdRV arXiv:0901.0306</th>
</tr>
</thead>
<tbody>
<tr>
<td>quark-loop</td>
<td>$(2.1 \pm 0.3) \cdot 10^{-10}$</td>
<td>$____$</td>
</tr>
<tr>
<td>pseudo-scalar</td>
<td>$(8.5 \pm 1.3) \cdot 10^{-10}$</td>
<td>$(11.4 \pm 1.3) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>axial-vector</td>
<td>$(0.25 \pm 0.1) \cdot 10^{-10}$</td>
<td>$(1.5 \pm 1.0) \cdot 10^{-10}$</td>
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<tr>
<td>scalar</td>
<td>$(-0.68 \pm 0.2) \cdot 10^{-10}$</td>
<td>$(-0.7 \pm 0.7) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>$\pi K$-loop</td>
<td>$(-1.9 \pm 1.3) \cdot 10^{-10}$</td>
<td>$(-1.9 \pm 1.9) \cdot 10^{-10}$</td>
</tr>
<tr>
<td>errors</td>
<td>linearly</td>
<td>quadratically</td>
</tr>
<tr>
<td>sum</td>
<td>$(8.3 \pm 3.2) \cdot 10^{-10}$</td>
<td>$(10.5 \pm 2.6) \cdot 10^{-10}$</td>
</tr>
</tbody>
</table>

But now with a smaller error on the $\pi$-loop.