### Overview

<table>
<thead>
<tr>
<th>Decay</th>
<th>Branching Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^+ \rightarrow e^+\nu$</td>
<td>$1.582(7) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \mu^+\nu$</td>
<td>$63.56(11)%$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 e^+\nu$</td>
<td>$5.07(4)%$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 \mu^+\nu$</td>
<td>$3.352(33)%$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 \pi^0 e^+\nu$</td>
<td>$2.55(4) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+\pi^- e^+\nu$</td>
<td>$4.247(24) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+\pi^- \mu^+\nu$</td>
<td>$1.4(9) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \mu^+\nu\gamma$</td>
<td>$6.2(8) \times 10^{-3}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow e^+\nu\gamma$</td>
<td>$9.4(4) \times 10^{-6}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 e^+\nu\gamma$</td>
<td>$2.56(16) \times 10^{-4}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 \mu^+\nu\gamma$</td>
<td>$1.25(25) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow e^+\nu e^+ e^-$</td>
<td>$2.48(20) \times 10^{-8}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow e^+\nu \mu^+ \mu^-$</td>
<td>$1.7(5) \times 10^{-8}$</td>
</tr>
</tbody>
</table>
Overview

<table>
<thead>
<tr>
<th>Decay</th>
<th>Branching Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_S \rightarrow \pi^\pm e^\mp \nu$</td>
<td>$7.04(8) \times 10^{-4}$</td>
</tr>
<tr>
<td>$K_L \rightarrow \pi^\pm e^\mp \nu$</td>
<td>$40.55(11)%$</td>
</tr>
<tr>
<td>$K_L \rightarrow \pi^\pm \mu^\mp \nu$</td>
<td>$27.04(7)%$</td>
</tr>
<tr>
<td>$K_L \rightarrow (\pi\mu)_{\text{atom}} \nu$</td>
<td>$1.05(11) \times 10^{-7}$</td>
</tr>
<tr>
<td>$K_L \rightarrow \pi^0 \pi^\pm e^\mp \nu$</td>
<td>$5.20(11) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K_L \rightarrow \pi^\pm e^\mp \nu e^+ e^-$</td>
<td>$1.26(4) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K_L \rightarrow \pi^\pm e^\mp \nu \gamma$</td>
<td>$3.79(6) \times 10^{-3}$</td>
</tr>
<tr>
<td>$K_L \rightarrow \pi^\pm \mu^\mp \nu \gamma$</td>
<td>$5.65(23) \times 10^{-3}$</td>
</tr>
</tbody>
</table>
You already heard a lot about semileptonic decays:

- Lattice: Sachrajda and Martinelli
- Radiative corrections: Knecht
- $\pi\pi$: Colangelo
- Dispersive work on $K_{\ell 4}$: Stoffer
- Dispersive work on rare decays: Stucki
- Estimate of parameters in rare decays: Greynat
- CKM fits: Descotes-Genon
- Mentioned in a few more talks as well

My talk:

- Chiral Perturbation Theory
- What can we learn/test in the various decays
- Some recent work (to show I do have some)
Earlier reviews of mine

- Da\phi\ne physics handbook, Semileptonic Kaon Decays in ChPT, JB, Ecker, Gasser, hep-ph/920820
- 2nd Da\phi\ne physics handbook, Semileptonic Kaon Decays, JB, Colangelo, Ecker, Gasser, hep-ph/9411311
- . . .
- KAON07, Radiative and semileptonic decays in ChPT, arXiv:0707.0419

But remember also:
Overview

1 Overview

2 Chiral Perturbation Theory
   - Chiral Perturbation Theory
   - A mesonic ChPT program framework
   - Determination of LECs in the continuum

3 Semileptonic decays
   - $K_{\ell 2}$
   - $K_{\ell 2\gamma}$
   - $K \to \ell' \nu \ell^+ \ell^-$
   - $K\pi$ form-factors for $K_{\ell 3}$ and $K \to \pi \nu \bar{\nu}$
   - $K_{\ell 3\gamma}$
   - $K_{\ell 4}$

4 Finite volume

5 Conclusions
Chiral Perturbation Theory

Exploring the consequences of the chiral symmetry of QCD and its spontaneous breaking using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
*On The Foundations Of Chiral Perturbation Theory*,

For references to lectures see:
http://www.thep.lu.se/~bijnens/chpt/
Chiral Perturbation Theory

A general Effective Field Theory:
- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:
- **Degrees of freedom**: Goldstone Bosons from spontaneous breaking of chiral symmetry
- **Powercounting**: Dimensional counting in momenta/masses
- **Breakdown scale**: Resonances, so about $M_\rho$. 
Chiral Perturbation Theory

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Goldstone Bosons

Spontaneous breakdown

- \[ \langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0 \]
- \[ SU(3)_L \times SU(3)_R \] broken spontaneously to \[ SU(3)_V \]
- 8 generators broken \( \implies 8 \) massless degrees of freedom and interaction vanishes at zero momentum
Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg powercounting

\[ \int d^4p \]

\[ p^2 \]

\[ \frac{1}{p^2} \]

\[ p^4 \]

\[ (p^2)^2 \left(\frac{1}{p^2}\right)^2 p^4 = p^4 \]

\[ (p^2) \left(\frac{1}{p^2}\right) p^4 = p^4 \]
Chiral Perturbation Theories

- Which chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$, for $N_f = 2, 3, \ldots$ and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite $T$
- Which interactions to include beyond the strong one
- Which particles included as non Goldstone Bosons
- My general belief: if it involves soft pions (or soft $K, \eta$) some version of ChPT exists
Lagrangians: Lagrangian structure (mesons, strong)

<table>
<thead>
<tr>
<th></th>
<th>2 flavour</th>
<th>3 flavour</th>
<th>PQChPT/(N_f) flavour</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p^2)</td>
<td>(F, B)</td>
<td>(F_0, B_0)</td>
<td>2</td>
</tr>
<tr>
<td>(p^4)</td>
<td>(l_i^r, h_i^r)</td>
<td>(L_i^r, H_i^r)</td>
<td>10+2</td>
</tr>
<tr>
<td>(p^6)</td>
<td>(c_i^r)</td>
<td>(C_i^r)</td>
<td>90+4</td>
</tr>
</tbody>
</table>

\(\hat{L}_i^r, \hat{H}_i^r\) 11+2

\(K_i^r\) 112+3

- \(p^2\): Weinberg 1966
- \(p^4\): Gasser, Leutwyler 84,85
- \(p^6\): JB, Colangelo, Ecker 99,00

\(L_i\): LEC = Low Energy Constants = ChPT parameters

\(H_i\): contact terms: value depends on definition of currents/densities

- Finite volume: no new LECs
- Other effects: (many) new LECs
## Mesons: which Lagrangians are known \((n_f = 3)\)

<table>
<thead>
<tr>
<th>Loops</th>
<th>(L_{\text{order}})</th>
<th>LECs</th>
<th>effects included</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L = 0)</td>
<td>(L_p^2)</td>
<td>2</td>
<td>strong (+ external (W, \gamma))</td>
</tr>
<tr>
<td></td>
<td>(L_{e^2p^0})</td>
<td>1</td>
<td>internal (\gamma)</td>
</tr>
<tr>
<td></td>
<td>(L_{\Delta S=1}^{G_F p^2})</td>
<td>2</td>
<td>nonleptonic weak</td>
</tr>
<tr>
<td></td>
<td>(L_{\Delta S=1}^{G_8 e^2p^0})</td>
<td>1</td>
<td>nonleptonic weak + internal (\gamma)</td>
</tr>
<tr>
<td></td>
<td>(L_{\text{odd}}^p)</td>
<td>0</td>
<td>WZW, anomaly</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(L \leq 1)</th>
<th>(L_{\text{order}})</th>
<th>LECs</th>
<th>effects included</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(L_p^4)</td>
<td>10</td>
<td>strong (+ external (W, \gamma))</td>
</tr>
<tr>
<td></td>
<td>(L_{e^2p^2})</td>
<td>13</td>
<td>internal (\gamma)</td>
</tr>
<tr>
<td></td>
<td>(L_{\Delta S=1}^{G_8 F p^4})</td>
<td>22</td>
<td>nonleptonic weak</td>
</tr>
<tr>
<td></td>
<td>(L_{\Delta S=1}^{G_27 p^4})</td>
<td>28</td>
<td>nonleptonic weak</td>
</tr>
<tr>
<td></td>
<td>(L_{\Delta S=1}^{G_8 e^2 p^2})</td>
<td>14</td>
<td>nonleptonic weak + internal (\gamma)</td>
</tr>
<tr>
<td></td>
<td>(L_{\text{odd}}^p)</td>
<td>23</td>
<td>WZW, anomaly</td>
</tr>
<tr>
<td></td>
<td>(L_{\text{leptons}})</td>
<td>5</td>
<td>leptons, internal (\gamma)</td>
</tr>
<tr>
<td></td>
<td>(L_{e^2p^2})</td>
<td>90</td>
<td>strong (+ external (W, \gamma))</td>
</tr>
</tbody>
</table>
Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars/axial currents
- Chiral logarithms
- includes Isospin and the eightfold way \((SU(3)_V)\)
- Unitarity included perturbatively

\[
m^2_\pi = 2B \hat{m} + \left( \frac{2B \hat{m}}{F} \right)^2 \left[ \frac{1}{32\pi^2} \log \frac{2B \hat{m}}{\mu^2} + 2l_3^r(\mu) \right] + \cdots
\]

\[
M^2 = 2B \hat{m}
\]
**LECs and $\mu$**

\[
\bar{l}_i = \frac{32 \pi^2}{\gamma_i} l_i^r(\mu) - \log \frac{M^2}{\mu^2}.
\]

is independent of the scale $\mu$.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

Choice of $\mu$:

- $m_\pi$, $m_K$: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$
  - what about large $N_c$ arguments???
- compromise: $\mu = m_\rho = 0.77$ GeV
Expand in what quantities?

- Expansion is in momenta and masses
- But is not unique: relations between masses (Gell-Mann–Okubo) exist
- Express orders in terms of physical masses and quantities \((F_\pi, F_K)\)?
- Express orders in terms of lowest order masses?
- E.g. \(s + t + u = 2m_\pi^2 + 2m_K^2\) in \(\pi K\) scattering
- Note: remaining \(\mu\) dependence can occur at a given order
- Can make quite some difference in the expansion

I prefer physical masses

- Thresholds correct
- Chiral logs are from physical particles propagating
- but sometimes too many masses so very ambiguous
Program availability

Making the programs more accessible for others to use:

- Two-loop results have very long expressions
- Many not published but available from http://www.thep.lu.se/~bijnens/chpt/
- Many programs available on request from the authors
- Idea: make a more general framework
- CHIRON:

  JB,
  “CHIRON: a package for ChPT numerical results at two loops,”
  http://www.thep.lu.se/~bijnens/chiron/
Program availability: CHIRON

- Present version: 0.54
- Classes to deal with $L_i$, $C_i$, $L_i^{(n)}$, $K_i$, standardized in/output, changing the scale, . . .
- Loop integrals: one-loop and sunset integrals
- Included so far (at two-loop order):
  - Masses, decay constants and $\langle \bar{q}q \rangle$ for the three flavour case
  - Masses and decay constants at finite volume in the three flavour case
  - Masses and decay constants in the partially quenched case for three sea quarks
  - Masses and decay constants in the partially quenched case for three sea quarks at finite volume
- A large number of example programs is included
- Manual has already reached 94 pages
- I am continually adding results from my earlier work
LEC determination: (Partial) History/References


Three flavour LECs: uncertainties

- $m_K^2, m_\eta^2 \gg m_\pi^2$
- Contributions from $\rho^6$ Lagrangian are larger
- Reliance on estimates of the $C_i$ much larger
- Typically: $C'_i$: (terms with)
  - kinematical dependence $\equiv$ measurable
  - quark mass dependence $\equiv$ impossible (without lattice)
    - 100\% correlated with $L'_i$
- How suppressed are the $1/N_c$-suppressed terms?
- Are we really testing ChPT or just doing a phenomenological fit?
Three flavour LECs: uncertainties

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Testing if ChPT works: relations


Systematic search for relations between observables that do not depend on the $C_i^r$

Included:

- $m_M^2$ and $F_M$ for $\pi$, $K$, $\eta$.
- 11 $\pi\pi$ threshold parameters
- 14 $\pi K$ threshold parameters
- 6 $\eta \rightarrow 3\pi$ decay parameters,
- 10 observables in $K_{\ell 4}$
- 18 in the scalar formfactors
- 11 in the vector formfactors
- Total: 76

We found 35 relations
Relations at NNLO: summary

- We did numerics for $\pi\pi$ (7), $\pi K$ (5) and $K_{\ell 4}$ (1) 13 relations
- $\pi\pi$: similar quality in two and three flavour ChPT
  - The two involving $a_3^-$ significantly did not work well
- $\pi K$: relation involving $a_3^-$ not OK
  - One more has very large NNLO corrections
- The relation with $K_{\ell 4}$ also did not work: related to that ChPT has trouble with curvature in $K_{\ell 4}$
- Conclusion: Three flavour ChPT “sort of” works
Fits: inputs


- $M_\pi, M_K, M_\eta, F_\pi, F_K/F_\pi$
- $\langle r^2 \rangle_\pi, c_\pi$ slope and curvature of $F_S$
- $\pi\pi$ and $\pi K$ scattering lengths $a_0^0, a_0^2, a_0^{1/2}$ and $a_0^{3/2}$.
- Value and slope of $F$ and $G$ in $K_{4\ell}$
- $\frac{m_s}{\bar{m}} = 27.5$ (lattice)
- $\bar{l}_1, \ldots, \bar{l}_4$

- more variation with $C_i^r$, a penalty for a large $p^6$ contribution to the masses
- 17+3 inputs and 8 $L_i^r + 34 \ C_i^r$ to fit
## Main fit

<table>
<thead>
<tr>
<th></th>
<th>ABT01</th>
<th>BJ12</th>
<th>$L'_4$ free</th>
<th>BE14</th>
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<tbody>
<tr>
<td>$10^3 L'_1$</td>
<td>old data</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^3 L'_2$</td>
<td>0.39(12)</td>
<td>0.88(09)</td>
<td>0.64(06)</td>
<td>0.53(06)</td>
</tr>
<tr>
<td>$10^3 L'_3$</td>
<td>0.73(12)</td>
<td>0.61(20)</td>
<td>0.59(04)</td>
<td>0.81(04)</td>
</tr>
<tr>
<td>$10^3 L'_4$</td>
<td>$-2.34(37)$</td>
<td>$-3.04(43)$</td>
<td>$-2.80(20)$</td>
<td>$-3.07(20)$</td>
</tr>
<tr>
<td>$10^3 L'_5$</td>
<td>$\equiv 0$</td>
<td>0.75(75)</td>
<td>0.76(18)</td>
<td>$\equiv 0.3$</td>
</tr>
<tr>
<td>$10^3 L'_6$</td>
<td>0.97(11)</td>
<td>0.58(13)</td>
<td>0.50(07)</td>
<td>1.01(06)</td>
</tr>
<tr>
<td>$10^3 L'_7$</td>
<td>$\equiv 0$</td>
<td>0.29(8)</td>
<td>0.49(25)</td>
<td>0.14(05)</td>
</tr>
<tr>
<td>$10^3 L'_8$</td>
<td>$-0.30(15)$</td>
<td>$-0.11(15)$</td>
<td>$-0.19(08)$</td>
<td>$-0.34(09)$</td>
</tr>
</tbody>
</table>

$\chi^2$  

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3 L'_4$</td>
<td>0.26</td>
<td>1.28</td>
<td>0.48</td>
<td>1.04</td>
</tr>
<tr>
<td>dof</td>
<td>1</td>
<td>4</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$F_0$ [MeV]</td>
<td>87</td>
<td>65</td>
<td>64</td>
<td>71</td>
</tr>
</tbody>
</table>

$\equiv (17 + 3) - (8 + 34)$
Main fit: Comments

- All values of the $C_i^r$ we settled on are “reasonable”
- Leaving $L_4^r$ free ends up with $L_4^r \approx 0.76$
- Keeping $L_4^r$ small: also $L_6^r$ and $2L_1^r - L_2^r$ small (large $N_c$ relations)
- Compatible with lattice determinations
- Not too bad with resonance saturation both for $L_i^r$ and $C_i^r$, including from the scalars
- Decent convergence (but enforced for masses)
- Many prejudices went in: large $N_c$, resonance model, quark model estimates, . . .
Some results of this fit

Mass:
\[
\begin{align*}
m^2_{\pi}/m^2_{\pi^{\text{phys}}} &= 1.055(p^2) - 0.005(p^4) - 0.050(p^6), \\
m^2_K/m^2_{K^{\text{phys}}} &= 1.112(p^2) - 0.069(p^4) - 0.043(p^6), \\
m^2_\eta/m^2_{\eta^{\text{phys}}} &= 1.197(p^2) - 0.214(p^4) + 0.017(p^6),
\end{align*}
\]

Decay constants:
\[
\begin{align*}
F_\pi/F_0 &= 1.000(p^2) + 0.208(p^4) + 0.088(p^6), \\
F_K/F_\pi &= 1.000(p^2) + 0.176(p^4) + 0.023(p^6).
\end{align*}
\]

Scattering:
\[
\begin{align*}
a^0_0 &= 0.160(p^2) + 0.044(p^4) + 0.012(p^6), \\
a^{1/2}_0 &= 0.142(p^2) + 0.031(p^4) + 0.051(p^6).
\end{align*}
\]
- $K^+ \rightarrow \mu^+ \nu$: determining $F_K |V_{us}|$
- $K^+ \rightarrow e^+ \nu$: Lepton universality, NA62/1 or NA48/3
- ChPT known to two loops: JB, Amoros, Talavera, hep-ph/9907264, with $m_u - m_d$ hep-ph/0101127
- Radiative corrections: talk by Knecht
Semileptonic Decays

Overview

ChPT

Semileptonic decays

\( K \ell_2 \gamma \)

- \( K^+ \rightarrow \mu^+ \nu \gamma \): determining \( F_K|V_{us}| \) (radiative corrections to \( K^+ \mu^+ \nu \))
- \( K^+ \rightarrow e^+ \nu \gamma \) and \( K^+ \rightarrow e^+ \nu \gamma \): Bremsstrahlung and structure dependent parts
- Structure dependent parts: Vector and axial vector form factor and the former is related to \( \pi^0 \gamma^* \gamma \)
- ChPT one-loop: JB, Gasser, Ecker, 1993
- ChPT two-loop: Geng, Ho, Wu 2004

Conclusions
\[ K^+(p) \rightarrow l^+(p_l)\nu_l(p_\nu)\gamma(q) \quad [K_{\ell 2\gamma}] \]

\[
T = -iG_F e V_{us}^* \epsilon_\mu \{ F_K L^\mu - H^{\mu\nu} l_\nu \}
\]

\[
L^\mu = m_l \bar{u}(p_\nu)(1 + \gamma_5) \left( \frac{2p^\mu}{2pq} - \frac{2p_l^\mu + q \gamma^\mu}{2p_l q} \right) \nu(p_l)
\]

\[
l^\mu = \bar{u}(p_\nu)\gamma^\mu (1 - \gamma_5) \nu(p_l)
\]

\[
H^{\mu\nu} = i V(W^2) \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta - A(W^2)(qWg^{\mu\nu} - W^\mu q^\nu)
\]

\[
W^\mu = (p - q)^\mu = (p_l + p_\nu)^\mu.
\]

\( L_\mu \): IB or inner Bremsstrahlung part

\( V \) and \( A \): SD or structure dependent part, starts at \( p^4 \)

\( V \): anomaly at \( p^4 \), known to \( p^6 \): Ametller, JB, Bramon, Cornet 1993

\( A \): \( p^4 \) JB, Ecker, Gasser 1993, \( p^6 \) Geng, Ho, Wu 2004
Semileptonic Decays

Johan Bijnens

Overview

ChPT

Semileptonic decays

$K^+ \ell^2 \gamma$

$K^+ \ell^2$ form-factors for $K^+ \ell^3$ and $K^+ \to \pi \nu \bar{\nu}$

Finite volume

Conclusions

From Geng, Ho, Wu 2004
From Geng, Ho, Wu 2004

dotted: $p^4$
solid $p^6$ $C_i^V$ from VMD, dashed $p^6$ $C_i^V$ from CQM
Same formfactors as previous decay but now depend on both $m_{l'\nu}^2$ and $m_{l^+l^-}^2$.

$V(m_{l'\nu}^2, m_{l^+l^-}^2)$: anomalous part: related by ChPT to the $\pi^0 \rightarrow \gamma^* \gamma^*$ physics and thus the same questions.

$A(m_{l'\nu}^2, m_{l^+l^-}^2)$: allows to study mixed axial and vector terms. But need precision away from constant form-factors.
Definition of $K\pi$ form-factors

Needed for $K_{\ell 3}^+, K_{\ell 3}^0, K^{+,0} \rightarrow \pi^{+,0}\nu\bar{\nu}$

We have four transitions:

\[
\langle \pi^0(p')|\bar{s}\gamma_\mu u|K^+(p)\rangle = \frac{1}{\sqrt{2}} \left[ (p + p')f_{+}^{K^0\pi^0} + (p - p')f_{-}^{K^0\pi^0} \right]
\]

\[
\langle \pi^-(p')|\bar{s}\gamma_\mu u|K^0(p)\rangle = \left[ (p + p')f_{+}^{K^0\pi^0} + (p - p')f_{-}^{K^0\pi^0} \right]
\]

\[
\langle \pi^+(p')|\bar{s}\gamma_\mu d|K^+(p)\rangle = \left[ (p + p')f_{+}^{K^0\pi^0} + (p - p')f_{-}^{K^0\pi^0} \right]
\]

\[
\langle \pi^0(p')|\bar{s}\gamma_\mu d|K^0(p)\rangle = -\frac{1}{\sqrt{2}} \left[ (p + p')f_{+}^{K^0\pi^0} + (p - p')f_{-}^{K^0\pi^0} \right]
\]

- Scalar formfactor: $f_0^{K^i\pi^i} = f_+^{K^i\pi^i} + \frac{(p-p')^2}{m^2_{K^i} - m^2_{\pi^i}} f_-^{K^i\pi^i}$

- In the isospin limit: all cases have the same form-factors

- Behrends-Sirlin-Ademollo-Gatto:

  $f_{+,0} = 1 + a(m_s - \hat{m})^2 + \cdots$
Measurements

- Both neutral and charged decay
- $f_+$ and $f_0$
  
  $f_+(t) = f_+(0) \left( 1 + \lambda_+ t + \lambda'_+ t^2 + \cdots \right)$
  
  $f_0(t) = f_+(0) \left( 1 + \lambda_0 t + \cdots \right)$

- Alternatively use dispersive parametrizations

- KLOE, NA48, ISTRA, KTeV: large number of recent measurements

- Correlations in the form-factor measurements **very important**

- $f_+$: VMD and $SU(3)$ breaking

- $f_0$: Scalar meson dominance? (or dispersive better?)
Isospin breaking: general results

To first order: insert $\frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d)$ once (JB, Ghorbani, 0711.0148)

$$f^K_{K^+\pi^0} = f^A_k(t) + \delta f^B_k(t) + \cdots$$
$$f^K_{K^0\pi^-} = f^A_k(t) - \delta f^D_k(t) + \cdots$$
$$f^K_{K^+\pi^+} = f^A_k(t) + \delta f^D_k(t) + \cdots$$
$$f^K_{K^0\pi^0} = f^A_k(t) - \delta f^B_k(t) + \cdots$$

$\delta = m_u - m_d$, $t = (p - p')^2$

Valid for $k = +, -, 0$ and for scalar current matrix elements

- $f^K_{K^+\pi^0}(t) - f^K_{K^0\pi^-}(t) - f^K_{K^+\pi^+}(t) + f^K_{K^0\pi^0}(t) = \mathcal{O}(\delta^2)$
- $r(t) = \frac{f^K_{K^+\pi^0}(t)f^K_{K^0\pi^0}(t)}{f^K_{K^0\pi^-}(t)f^K_{K^+\pi^+}(t)} = 1 + \mathcal{O}(\delta^2)$
Chiral Perturbation Theory

- $p^2$: $f_+ = 1, f_- = 0$ (current algebra)
- $p^4$: $f_+(0)$ Leutwyler-Roos 1984
- $p^4$: Gasser-Leutwyler 1985 and isospin corrections for the weak decays
- Radiative corrections: talk by Knecht
- $p^4$ and radiative corrections for rare decays: Mescia-Smith, arXiv:0705.2025
- Numbers for $K_{\ell 3}$: see Kastner, Neufeld, arXiv:0805.2222
\[ f_0(t) = 1 - \frac{8}{F_\pi^4} (C_{12}^r + C_{34}^r) (m_K^2 - m_\pi^2)^2 \]
\[ + \frac{8}{F_\pi^4} t (2C_{12}^r + C_{34}^r) (m_K^2 + m_\pi^2) + \frac{t}{m_K^2 - m_\pi^2} (F_K/F_\pi - 1) \]
\[ - \frac{8}{F_\pi^4} t^2 C_{12}^r + \bar{\Delta}(t) + \Delta(0). \]

\[ \bar{\Delta}(t) \text{ and } \Delta(0) \text{ contain NO } C_i^r \text{ and only depend on the } L_i^r \text{ at order } p^6 \]
\[ \implies \text{All needed parameters can be determined experimentally} \]

Now update input with the new results:
\[ \Delta(0) = -0.02276 (p^4) + 0.01140 (p^6 \text{ pure loop}) + 0.0504 (p^6 L_i^r) \]
Update on $K_{\ell 3}$

- Take Bijnens-Talavera 2003 result but update for BE14 parameters
- $f^K_{+,K^0\pi^-}(0) = 1 - 0.02276 - 0.00754 = 0.970 \pm 0.008$
- in good agreement with the latest lattice numbers (Juettner, lattice 2015, preliminary FLAG)
  - $K_{\ell 2}$: $0.9677(37)$
  - $K_{\ell 3}$: $0.9704(32)$
- Note original JB-Talavera: $0.976(10)$
- FLAG1: $0.956(08)$
- Jamin, Oller, Pich, hep-ph/0401080: $0.978(09)$
Callan-Treiman: $f_0(m_K^2 - m_\pi^2) = \frac{F_K}{F_\pi} + O(m_\pi^2)$

SU(2) current algebra must hold to all orders in SU(3) ChPT

Define $\Delta_{CT} = f_0(m_K^2 - m_\pi^2) - \frac{F_K}{F_\pi}$

$p^4$ Gasser, Leutwyler 1985: $\Delta_{CT} = -3.5 \times 10^{-3}$

$p^6$ Using JB, Talavera, 2003

$p^6$ isospinbreaking

JB, Ghorbani, 2007

Add $p^6$ contribution from $C_i$:

$\Delta_{CT}^{C_i} = \frac{16}{F_\pi^4} (2C_{12} + C_{34}) m_\pi^2 (m_K^2 - m_\pi^2)$

$\Delta_{CT}^{C_i} = 1.3 \times 10^{-3}$
$$r_{0-} = \frac{f_+^{K^+\pi^0}}{f_+^{K^0\pi^-}}$$
Semileptonic Decays

Overview

ChPT

Semileptonic decays

$K\to\ell\nu\ell^+\ell^-$

$K\pi$ form-factors for $K\ell_3$ and $K\to\pi\nu\bar{\nu}$

$K\ell_4$

Finite volume

Conclusions

$r_K$

$r_0^- = f^{K^+\pi^+}_{+}/f^{K^0\pi^-}_{+}$
$K_{\ell 3\gamma}$ or $K \rightarrow \pi \ell \nu \gamma$

$p^2$: Fearing, Fischbach, Smith 1970 IB only

$p^4$: JB, Ecker, Gasser, 1993

$p^6$: Axial form-factors fully known

$p^6$: Vector form-factors: approximately known

Gasser, Kubis, Paver, Verbeni hep-ph/0412130: $K_{Le\nu\gamma}$

Müller, Kubis, Meißner hep-ph/0607151: T-odd correlations

Kubis, Müller, Gasser, Schmid hep-ph/0611366: $K_{e\nu\gamma}^+$

Approximately known: structure functions smooth cuts: $p$-wave or far away: approximate by polynomials
$K_{\ell 3 \gamma}$

![Diagram of $K_{\ell 3 \gamma}$ decay processes](image)

**Remainder is from Kubis et al. 2006**

$$T(K_{e3 \gamma}^+) =$$

$$G_F \frac{e}{\sqrt{2}} V_{us}^* \epsilon^{\mu}(q)^* \left[ (V_{\mu\nu} - A_{\mu\nu}) \bar{u}(p_\nu) \gamma^\nu (1 - \gamma_5) v(p_e) \right. + \frac{F_\nu}{2p_e q} \bar{u}(p_\nu) \gamma^\nu (1 - \gamma_5) (m_e - p_e - q) \gamma_\mu v(p_e) \]$$
$$K_{\ell 3\gamma}$$

$$A_{\mu \nu} = \frac{i}{\sqrt{2}} \left[ \epsilon_{\mu \nu \rho \sigma} (A_1 p^\rho q^\sigma + A_2 q^\rho W^\sigma) + \epsilon_{\mu \lambda \rho \sigma} p^\lambda q^\rho W^\sigma \left( \frac{A_3}{M_K^2 - W^2} W_\nu + A_4 p'_\nu \right) \right] ,$$

$$V_{\mu \nu} = V_{\mu \nu}^{IB} + V_{\mu \nu}^{SD}$$

$$V_{\mu \nu}^{SD}$$ has again 4 structure function $$V_i$$

$$V_{\mu \nu}^{IB}$$: IB part, mainly determined by Low's theorem and from the $$K_{\ell 3}$$ form-factors

$$R \left( E_\gamma^{\text{cut}}, \theta_\gamma^{\text{cut}} \right) = \frac{\Gamma \left( K_{e3\gamma}^\pm, E_\gamma^{\ast} > E_\gamma^{\text{cut}}, \theta_\gamma^{\ast} > \theta_\gamma^{\text{cut}} \right)}{\Gamma \left( K_{e3}^\pm \right)} ,$$

Many uncertainties drop out
\[ R \left( \bar{\lambda}_+, \bar{\lambda}_+'' \right) = R(1, 0) \left\{ 1 + c_1 \left( \bar{\lambda}_+ - 1 \right) + c_2 \left( \bar{\lambda}_+ - 1 \right)^2 + c_3 \bar{\lambda}_+'' + \ldots \right\} \]

\[ R^{IB} \text{ accordingly (with expansion coefficients } c_i^{IB}) \]

<table>
<thead>
<tr>
<th>(E_{\gamma}^{\text{cut}})</th>
<th>(\theta_{e^- \gamma}^{\text{cut}})</th>
<th>(R^{IB} \cdot 10^2)</th>
<th>(R \cdot 10^2)</th>
<th>(c_1 \cdot 10^3)</th>
<th>(c_2 \cdot 10^4)</th>
<th>(c_3 \cdot 10^4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 MeV</td>
<td>20°</td>
<td>0.640</td>
<td>0.633 ± 0.002</td>
<td>12.5 ± 0.4</td>
<td>−5.4 ± 0.3</td>
<td>16.9 ± 0.4</td>
</tr>
<tr>
<td>30 MeV</td>
<td>10°</td>
<td>0.925</td>
<td>0.918 ± 0.002</td>
<td>11.1 ± 0.3</td>
<td>−4.7 ± 0.2</td>
<td>15.0 ± 0.3</td>
</tr>
<tr>
<td>10 MeV</td>
<td>20°</td>
<td>1.211</td>
<td>1.204 ± 0.002</td>
<td>7.5 ± 0.2</td>
<td>−3.2 ± 0.2</td>
<td>10.1 ± 0.2</td>
</tr>
<tr>
<td>10 MeV</td>
<td>10°</td>
<td>1.792</td>
<td>1.785 ± 0.002</td>
<td>6.7 ± 0.2</td>
<td>−2.8 ± 0.1</td>
<td>9.0 ± 0.1</td>
</tr>
<tr>
<td>10 MeV</td>
<td>26° − 53°</td>
<td>0.554</td>
<td>0.553 ± 0.001</td>
<td>5.7 ± 0.1</td>
<td>−2.4 ± 0.1</td>
<td>7.5 ± 0.1</td>
</tr>
</tbody>
</table>
$K_{\ell 3\gamma}$

\[
\frac{d\Gamma}{dE_\gamma^*} = \frac{d\Gamma_{IB}}{dE_\gamma^*} + \sum_{i=1}^{4} \left( \langle V_i \rangle \frac{d\Gamma_{V_i}}{dE_\gamma^*} + \langle A_i \rangle \frac{d\Gamma_{A_i}}{dE_\gamma^*} \right) + O\left( |T^{SD}|^2, \Delta V_i, \Delta A_i \right)
\]
Semileptonic Decays

Johan Bijnens

Overview

ChPT

Semileptonic decays

\[ K^\ell_4 \]

\[
K^+(p) \rightarrow \pi^+(p_+)\pi^-(p_-)\ell^+(p_\ell)\nu_\ell(p_\nu),
\]

\[
K^+(p) \rightarrow \pi^0(p_+)\pi^0(p_-)\ell^+(p_\ell)\nu_\ell(p_\nu),
\]

\[
K^0(p) \rightarrow \pi^-(p_+)\pi^0(p_-)\ell^+(p_\ell)\nu_\ell(p_\nu).
\]

Kinematical variables for hadronic system: \( t, u, s_\pi, s_\ell \)

\[
T^{+-} = \frac{G_F}{\sqrt{2}} V^*_u s(p_\nu) \gamma_\mu (1 - \gamma_5) \nu(p_\ell) (V^\mu - A^\mu),
\]

\[
V_\mu = - \frac{H}{m^3_K} \epsilon_{\mu\nu\rho\sigma} (p_\ell + p_\nu)^\nu (p_+ + p_-)^\rho (p_+ - p_-)^\sigma,
\]

\[
A_\mu = - \frac{i}{m_K} [(p_+ + p_-)_\mu F + (p_+ - p_-)_\mu G + (p_\ell + p_\nu)_\mu R].
\]
Semileptonic Decays

Johan Bijnens

Overview

ChPT

Semileptonic decays

\( K \to \pi \nu \bar{\nu} \)

Finite volume

Conclusions

**\( K_{\ell 4} \)**

\[
T^{+-} = \frac{T^{-0}}{\sqrt{2}} + T^{00}
\]

\( T^{-0} \) is anti-symmetric under \( t \leftrightarrow u \)

\( T^{00} \) is symmetric.

Lowest order: Weinberg: 
\[
F = G = \frac{m_K}{\sqrt{2} F_\pi},
\]

Order \( p^4 \): JB 1990, Riggenbach et al. 1991

Could fit data with reasonable corrections:

Determine \( L^r_i \), \( i = 1, 2, 3 \)

Dispersive estimate of \( p^6 \) corrections:

JB, Colangelo, Gasser 1994
Parametrization for experiment: Amoros, JB, 1999
Full $p^6$ calculation: Amoros, JB, Talavera 2000
Ametller, JB, Bramon, Cornet 1993 (H only)
Isospin breaking at $p^4$: Nehme et al.
Isospin breaking and radiative corrections: Colangelo, Gasser, Rusetsky
Radiative corrections: Stoffer
Dispersive (most recent): Stoffer
Semileptonic Decays

Johan Bijnens

Overview

ChPT

Semileptonic decays

$K \rightarrow \ell^+ \ell^- \nu \bar{\nu}$

Finite volume

Conclusions

$K_{\ell 4}$

|F| (p^6)

|F| (p^4)

|F| (p^2)

|f|_R Omnès

$s_{\pi} [\text{GeV}^2]$
Semileptonic Decays

Johan Bijnens

Overview

ChPT

Semileptonic decays

$K\ell_2$

$K\ell_2\gamma$

$K \to \ell^+\ell^-\nu\bar{\nu}$

$K\pi$ form-factors for $K\ell_3$ and $K \to \pi\nu\bar{\nu}$

$K\ell_3\gamma$

$K\ell_4$

Finite volume

Conclusions
Finite volume

- Lattice QCD calculates at different quark masses, volumes boundary conditions...
- A general result by Lüscher: relate finite volume effects to scattering (1986)
- Chiral Perturbation Theory is also useful for this
  - $M_\pi$, $F_\pi$, $\langle \bar{q}q \rangle$ one-loop equal mass case
- I will stay with ChPT and the $p$ regime ($M_\pi L >> 1$)
- $1/m_\pi = 1.4$ fm
  - may need to go beyond leading $e^{-m_\pi L}$ terms
- Convergence of ChPT is given by $1/m_\rho \approx 0.25$ fm
Finite volume: selection of ChPT results

- masses and decay constants for $\pi, K, \eta$ one-loop

- $M_\pi$ at 2-loops (2-flavour)

- $\langle \bar{q}q \rangle$ at 2 loops (3-flavour)

- Twisted mass at one-loop

- Twisted boundary conditions

- This talk:
  - Twisted boundary conditions and some funny effects
  - Some results on masses 3-flavours at two loop order
Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i / L$: very few momenta directly accessible.
- Put a constraint on certain quark fields in some directions:
  \[ q(x^i + L) = e^{i\theta^i} q(x^i) \]
- Then momenta are $p^i = \theta^i / L + 2\pi n^i / L$. Allows to map out momentum space on the lattice much better Bedaque, …
- But:
  - Box: Rotation invariance $\rightarrow$ cubic invariance
  - Twisting: reduces symmetry further

Consequences:
- $m^2(\vec{p}^2) = E^2 - \vec{p}^2$ is not constant
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum
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Twisted boundary conditions: Two-point function


\[ \int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0 \]

\[ \langle \bar{u} \gamma^\mu u \rangle \neq 0 \]

\[ j^\pi_\mu = \bar{d} \gamma_\mu u \]

satisfies \( \partial^\mu \langle T(j^\pi_\mu(x)j^\pi_\nu(0)) \rangle = \delta^{(4)}(x) \langle \bar{d} \gamma_\nu d - \bar{u} \gamma_\nu u \rangle \)

\[ \Pi^{\pi +}_{\mu\nu}(q) \equiv i \int d^4 x e^{i q \cdot x} \langle T(j^a_\mu(x)j^a_\nu(0)) \rangle \]

Satisfies WT identity. \( q^\mu \Pi^{\pi +}_{\mu\nu} = \langle \bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d \rangle \)

ChPT at one-loop satisfies this

Twisted boundary conditions: volume correction masses

\[ m_\pi L = 2, \quad \vec{\theta}_u = (\theta, 0, 0), \quad \vec{\theta}_d = \vec{\theta}_s = 0 \]

\[ \Delta V X = X^V - X^\infty \quad (\text{dip is going through zero}) \]
Volume correction decay constants: $F_{\pi^+}$

- $\langle 0| A^M_\mu | M(p) \rangle = i\sqrt{2} F_M p_\mu + i\sqrt{2} F^V_M \mu$
- Extra terms are needed for Ward identities

![Graphs showing the relative and extra terms for $F_{\pi^+}$ compared to $F_\pi$](graph.png)
Volume correction electromagnetic formfactor

- earlier two-flavour work:

\[ \langle M'(p')|j_\mu|M(p)\rangle = f_\mu = f_+(p_\mu + p'_\mu) + f_- q_\mu + h_\mu \]

- Extra terms are again needed for Ward identities

- Note that masses have finite volume corrections
  - \( q^2 \) for fixed \( \vec{p} \) and \( \vec{p}' \) has corrections
    - small effect
  - This also affects the ward identities, e.g.
    \[ q^\mu f_\mu = (p^2 - p'^2)f_+ + q^2 f_- + q^\mu h_\mu = 0 \]
    - is satisfied but all effects should be considered
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Volume correction electromagnetic formfactor

\[ f_\mu = -\frac{1}{\sqrt{2}} \langle \pi^0(p')|\bar{d}\gamma_\mu u|\pi^+(p)\rangle \]

\[ = (1 + f_+^\infty + \Delta^V f_+)(p + p')_\mu + \Delta^V f_- q_\mu + \Delta^V h_\mu \]

Pure loop plotted: \( f_+^\infty(p + p'), \Delta^V f_+(p + p') \) and \( \Delta^V f_\mu \)

Finite volume corrections large, different for different \( \mu \)
Conclusions

- Short introduction to ChPT
- A very fast overview of all semileptonic modes and what they are useful for
- Looking forward on possible improvements from NA62
- Lattice: finite volume now limiting factor for \( f_+(0) \) (MILC), relevant ChPT calculation in progress