CHIRAL EFT AND LATTICE

Johan Bijnens
Lund University

bijnens@thep.lu.se
http://thep.lu.se/~bijnens
http://thep.lu.se/~bijnens/chpt/
http://thep.lu.se/~bijnens/chiron/

MIAPP – Interface of Effective Field Theories and Lattice Gauge Theory – Garching 31 October 2018
Overview

1. Chiral Perturbation Theory
2. $p^8$ results
   - Lagrangian
   - Mass/decay constant
3. Masses and decay constants: 2 loop
   - Older results
   - Idea
   - Fitting lattice
   - Example fit
4. $g - 2$: HVP
   - To ChPT or not to ChPT
   - sQED
     - Infinite volume
     - Finite volume
     - Lattice
   - HVP: chiral corrections
   - Finite Volume
5. Conclusions
A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- **Degrees of freedom**: Goldstone Bosons from spontaneous breaking of chiral symmetry
- **Powercounting**: Dimensional counting in momenta/masses
- **Breakdown scale**: Resonances, so about $M_\rho$. 
Chiral Perturbation Theory

A general Effective Field Theory:
- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:
- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about $M_\rho$. 
Goldstone Bosons

Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\implies$ 8 massless degrees of freedom and interaction vanishes at zero momentum
Chiral Perturbation Theories

- Which chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$, for $N_f = 2, 3, \ldots$ and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite $T$
- Which interactions to include beyond the strong one
- Which particles included as non Goldstone Bosons
- Here: restrict to standard ChPT but with extensions for the lattice
Lagrangians: Lowest order

\[ U(\phi) = \exp(i\sqrt{2}\Phi/F_0) \] parametrizes Goldstone Bosons

\[ \Phi(\chi) = \begin{pmatrix}
\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+
\pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0
K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}}
\end{pmatrix} . \]

LO Lagrangian: \( \mathcal{L}_2 = \frac{F_0^2}{4} \left\{ \langle D_\mu U^\dagger D^\mu U \rangle + \langle \chi^\dagger U + \chi U^\dagger \rangle \right\} , \)

\[ D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu , \]

left and right external currents: \( r(l)_\mu = \nu_\mu + (-)a_\mu \)

Scalar and pseudoscalar external densities: \( \chi = 2B_0(s + ip) \) quark masses via scalar density: \( s = M + \cdots \)

\[ \langle A \rangle = Tr_F (A) \]
Lagrangians: structure (mesons, strong)

<table>
<thead>
<tr>
<th></th>
<th>2 flavour</th>
<th>3 flavour</th>
<th>PQChPT/(N_f) flavour</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p^2)</td>
<td>(F, B)</td>
<td>(F_0, B_0)</td>
<td>(F_0, B_0)</td>
</tr>
<tr>
<td>(p^4)</td>
<td>(l_i^r, h_i^r)</td>
<td>(L_i^r, H_i^r)</td>
<td>(\hat{L}_i^r, \hat{H}_i^r)</td>
</tr>
<tr>
<td>(p^6)</td>
<td>(c_i^r)</td>
<td>(C_i^r)</td>
<td>(K_i^r)</td>
</tr>
<tr>
<td>(p^8)</td>
<td>(x_i^r)</td>
<td>(X_i^r)</td>
<td>(Y_i^r)</td>
</tr>
</tbody>
</table>

- \(p^2\): Weinberg 1966
- \(p^4\): Gasser, Leutwyler 84,85
- \(p^6\): JB, Colangelo, Ecker 99,00
- \(p^8\): JB, Hermansson-Truedsson, Wang 1810.06834

\(L_i\) LEC = Low Energy Constants = ChPT parameters
\(H_i\): contact terms: value depends on definition of currents/densities
Finite volume: no new LECs
Other effects: (many) new LECs
How to find a Lagrangian

- Find building blocks with an easy transformation law
- $u_\mu, \chi_\pm, f^{\mu\nu}_{\pm}$
- all transform simply under the compensating transformation $h \in SU(N_f)_V$
- and covariant derivatives of them (uses $\Gamma_\mu$)
- make all invariant traces up to the order you want
- Combine into $C, P$ and hermitian conjugation invariant combinations
- relatively easy to handle
How to find a **minimal** Lagrangian

- integration-by-parts: construct all total derivatives
- field redefinitions: construct all terms containing the LO EOM
- Bianchi of $\Gamma_{\mu\nu} = \frac{1}{4} [u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}$,
- the identities $f_{-\mu\nu} - \nabla_\nu u_\mu + \nabla_\mu u_\nu = 0$ and $[\nabla_\mu, \nabla_\nu] X = [\Gamma_{\mu\nu}, X]$,
- the Cayley-Hamilton theorem (for $N_f = 2, 3$).
- 9740 monomials and 12444 different relations plus Cayley-Hamilton
- positive intrinsic parity: Schouten identity not needed ($p^8$)
- Contact terms: many simplifications not applicable
- Remember $\tilde{\chi}^{\dagger} \equiv \det(\chi)\chi^{-1}$
Methods/Checks/results

- generate all terms and relations using FORM
- generate from the output back identities etc using Python
- Final diagonalization done using C++ with GMP
- reproduce all known $\rho^6$ results (Dipl. thesis J. Weber)
- have two independent sets of programs
- Determined for no external fields, $s, p$-only, $\nu, a$-only, all
- Basis with all terms as supplementary material (57 pages)
### Lagrangian: all external fields set to zero

<table>
<thead>
<tr>
<th></th>
<th>$N_f$</th>
<th>$N_f = 3$</th>
<th>$N_f = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$p^4$</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$p^6$</td>
<td>19</td>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td>$p^8$</td>
<td>135</td>
<td>56</td>
<td>16</td>
</tr>
</tbody>
</table>

Fits with what can be determined from meson-meson scattering
Two flavour ChPT: mass and decay constant

- First step towards finding out why hard-pion ChPT does not work at three-loops
- Lowest order: Gell-Mann, Oakes, Renner (1968)
- Chiral logarithm Langacker, Pagels (1973)
- Full NLO (and properly starting ChPT) Gasser-Leutwyler (1984)
- NNNLO JB, Hermansson-Truedsson (1710.01901)
Methods used

- LO and Chiral logs: current algebra
- NLO: Feynman diagrams (by hand) and direct expansion of functional integral (with REDUCE)
- NNLO: Feynman diagrams (with a little help from FORM)
- NNLO: Feynman diagrams purely with FORM
- Main stumbling block: integrals
  - Reduction to master integrals with REDUZE Studerus (2009)
  - Master Integrals known
    Laporta-Remiddi (1996); Melnikov, van Ritbergen (2001)
- Lots of book-keeping: FORM
- Checks:
  - All nonlocal divergences must cancel
  - Use different parametrizations of the Lagrangian
  - Agree with known leading log result JB, Carloni (2009)
Methods used

- LO and Chiral logs: current algebra
- NLO: Feynman diagrams (by hand) and direct expansion of functional integral (with REDUCE)
- NNLO: Feynman diagrams (with a little help from FORM)
- NNLO: Feynman diagrams purely with FORM
- Main stumbling block: integrals
  - Reduction to master integrals with REDUCE Studerus (2009)
  - Master Integrals known
    - Laporta-Remiddi (1996); Melnikov, van Ritbergen (2001)
- Lots of book-keeping: FORM
- Checks:
  - All nonlocal divergences must cancel
  - Use different parametrizations of the Lagrangian
  - Agree with known leading log result JB, Carloni (2009)
Diagrams

\[ p^2; \quad p^4; \quad p^6; \quad p^8 \]
Results: LO or $x$-expansion/physical or $\xi$-expansion

\[ x = \frac{M^2}{16\pi^2 F^2}, \quad L_x = \log \frac{M^2}{\mu^2}, \quad M^2 = 2B\hat{m} \]

\[ \frac{M^2_\pi}{M^2} = 1 + x \left( a_{11}^M L_x + a_{10}^M \right) + x^2 \left( a_{22}^M L_x^2 + a_{21}^M L_x + a_{20}^M \right) + x^3 \left( a_{33}^M L_x^3 + a_{32}^M L_x^2 + a_{31}^M L_x + a_{30}^M \right) + \cdots \]

\[ \frac{F_\pi}{F} = 1 + x \left( a_{11}^F L_x + a_{10}^F \right) + x^2 \left( a_{22}^F L_x^2 + a_{21}^F L_x + a_{20}^F \right) + x^3 \left( a_{33}^F L_x^3 + a_{32}^F L_x^2 + a_{31}^F L_x + a_{30}^F \right) + \cdots \]

\[ \xi = \frac{M^2_\pi}{16\pi^2 F^2_\pi}, \quad L_\pi = \log \frac{M^2_\pi}{\mu^2} \]

\[ \frac{M^2}{M^2_\pi} = 1 + \xi \left( b_{11}^M L_\pi + b_{10}^M \right) + \xi^2 \left( b_{22}^M L_\pi^2 + b_{21}^M L_\pi + b_{20}^M \right) + \xi^3 \left( b_{33}^M L_\pi^3 + b_{32}^M L_\pi^2 + b_{31}^M L_\pi + b_{30}^M \right) + \cdots \]

\[ \frac{F}{F_\pi} = 1 + \xi \left( b_{11}^F L_\pi + b_{10}^F \right) + \xi^2 \left( b_{22}^F L_\pi^2 + b_{21}^F L_\pi + b_{20}^F \right) + \xi^3 \left( b_{33}^F L_\pi^3 + b_{32}^F L_\pi^2 + b_{31}^F L_\pi + b_{30}^F \right) + \cdots \]
### Results

\[
\tilde{l}_i = 16\pi^2 l_i, \quad \tilde{c}_i = (16\pi^2)^2 c_i
\]

<table>
<thead>
<tr>
<th>$a^M_{11}$</th>
<th>$\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^M_{10}$</td>
<td>$2\tilde{l}_3$</td>
</tr>
<tr>
<td>$a^M_{22}$</td>
<td>$\frac{17}{8}$</td>
</tr>
<tr>
<td>$a^M_{21}$</td>
<td>$-3\tilde{l}_3 - 8\tilde{l}_2 - 14\tilde{l}_1 - \frac{49}{12}$</td>
</tr>
<tr>
<td>$a^M_{20}$</td>
<td>$64\tilde{c}<em>{18} + 32\tilde{c}</em>{17} + 96\tilde{c}<em>{11} + 48\tilde{c}</em>{10} - 16\tilde{c}_9 - 32\tilde{c}_8 - 16\tilde{c}_7$ $- 32\tilde{c}_6 + 1\tilde{l}_3 + 2\tilde{l}_2 + \tilde{l}_1 + \frac{193}{96}$</td>
</tr>
<tr>
<td>$a^M_{33}$</td>
<td>$\frac{103}{24}$</td>
</tr>
<tr>
<td>$a^M_{32}$</td>
<td>$\frac{23}{2}l_3 - 11l_2 - 38l_1 - \frac{91}{24}$</td>
</tr>
<tr>
<td>$a^M_{31}$</td>
<td>$-416\tilde{c}<em>{18} - 208\tilde{c}</em>{17} - 32\tilde{c}<em>{16} + 96\tilde{c}</em>{14} + 8\tilde{c}<em>{13} - 48\tilde{c}</em>{12}$ $- 384\tilde{c}<em>{11} - 192\tilde{c}</em>{10} + 72\tilde{c}_9 + 144\tilde{c}_8 + 72\tilde{c}_7 + 64\tilde{c}_6 - 8\tilde{c}_5$ $- 56\tilde{c}_4 + 16\tilde{c}_3 + 32\tilde{c}_2 - 96\tilde{c}_1 - 8\tilde{l}_3 - 48\tilde{l}_3\tilde{l}_2 - 84\tilde{l}_3\tilde{l}_1$ $- \frac{88}{3}\tilde{l}_3 - \frac{231}{10}\tilde{l}_2 - \frac{69}{5}\tilde{l}_1 - \frac{74971}{8640}$</td>
</tr>
<tr>
<td>$a^M_{30}$</td>
<td>contains free $p^8$ LECs (and a lot more terms)</td>
</tr>
</tbody>
</table>
Results: comments

- Similar tables for $a^F_i$, $b^M_i$, $b^F_i$
- Coefficients depend on scale $\mu$, but whole expression is $\mu$-independent
- Can be rewritten in terms of scales in the logarithm rather than in terms of LECs à la FLAG
- Leading log: a number
- NLL: depends on $l^r_i$
- NNLL: depends on $c^r_i$
- For the mass all needed $c^r_i$ can be had from mass, decay-constant and $\pi\pi$ parameters fitted to two-loop or $p^6$ (i.e. $r_M, r_F, r_1, \ldots, r_6$).
- For decay need one more
### Results: numerics preliminary

<table>
<thead>
<tr>
<th>$ij$</th>
<th>$a_{ij}^M$</th>
<th>$b_{ij}^M$</th>
<th>$a_{ij}^F$</th>
<th>$b_{ij}^F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>+0.00282</td>
<td>−0.00282</td>
<td>+1.09436</td>
<td>−1.09436</td>
</tr>
<tr>
<td>11</td>
<td>+0.5</td>
<td>−0.5</td>
<td>−1</td>
<td>+1</td>
</tr>
<tr>
<td>20</td>
<td>+1.65296</td>
<td>−1.65771</td>
<td>−0.04734</td>
<td>−1.15001</td>
</tr>
<tr>
<td>21</td>
<td>+2.4573</td>
<td>−3.29038</td>
<td>−1.90577</td>
<td>+4.13885</td>
</tr>
<tr>
<td>22</td>
<td>+2.125</td>
<td>−0.625</td>
<td>−1.25</td>
<td>−0.25</td>
</tr>
<tr>
<td>30</td>
<td>+0.39527</td>
<td>−6.7854</td>
<td>−244.499</td>
<td>242.236</td>
</tr>
<tr>
<td>31</td>
<td>−3.75977</td>
<td>+4.32719</td>
<td>−19.0601</td>
<td>32.1315</td>
</tr>
<tr>
<td>32</td>
<td>+17.1476</td>
<td>+0.62039</td>
<td>−9.39462</td>
<td>−6.77511</td>
</tr>
<tr>
<td>33</td>
<td>+4.29167</td>
<td>+5.14583</td>
<td>−3.45833</td>
<td>−0.41666</td>
</tr>
</tbody>
</table>

Note the large coefficients in the decay constant
Pion mass

\[ F_\pi = 92.2 \text{ MeV}, \ F = F_\pi / 1.037, \ \bar{t}_1 = -0.4, \ \bar{t}_2 = 4.3, \ \bar{t}_3 = 3.41, \ \bar{t}_4 = 4.51, \]

\[ r_i \text{ from JB et al 1997, other } c_i^r = 0, \ \mu = 0.77 \text{ GeV} \]

\[ \xi \text{-expansion converges notably better} \]
Pion decay constant

- x-expansion ($F$-fixed)
- $\xi$-expansion ($F_\pi$ fixed)

- $\xi$-expansion converges better
- Large $p^8$ due to the “240” in $a_{30}^F$ and $b_{30}^F$
Masses/decay constants: older results

- **Two-flavour**
  - Two-loop:
  - Three loops: just talked about

- **Three-flavour**
  - Two-loop:
  - Added: isospin breaking, finite volume, partial quenching all at two loops
Masses/decay constants: underlying idea

- A lot more is now known about the sun(set)(rise)integrals
  \[
  p \rightarrow q + r - p, m_3 \\
  q, m_1 \\
  p \\
  r, m_2 \\
  p \\
  = H(m_1, m_2, m_3, p^2)
  \]

- Reduction to master integrals

- Many integrals with two scales known analytically

- Expansions around certain points much better known

- Used for three-flavour result for \( m_\pi \):

- Do the same for: \( F_\pi, m_K, F_K, m_\eta, F_\eta \)
Masses/decay constants: papers

- **Summarize existing sunset knowledge and make it easier to use (mathematica notebooks as supplementary material)**
  B. Ananthanarayan, JB, S. Ghosh and A. Hebbar,

- **$F_\pi$:**
  only $H(m_\eta, m_K, m_K, m_\pi^2)$ not analytically known, expand in $m_\pi^2$, expansion coefficients analytically known
  B. Ananthanarayan, JB and S. Ghosh,

- **$F_K/F_\pi$:**
  $H(m_\pi, m_K, m_\eta, m_K^2)$: needs Mellin-Barnes to expand in $m_\pi$
  B. Ananthanarayan, JB, S. Friot and S. Ghosh,

- **$m_K, F_K, m_\eta, F_\eta$:** B. Ananthanarayan, JB, S. Friot and S. Ghosh,
How to best fit to lattice data?

Two standard options for expanding (two flavours):

in “quark masses” (lowest order $M$, $F$), $x = M^2/(16\pi^2 F^2)$

$$m^2_\pi = M^2 \left\{ 1 + x \left( \frac{1}{2} \log \frac{M^2}{\mu^2} + l_M^r \right) + x^2 \left( \frac{17}{8} \log^2 \frac{M^2}{\mu^2} + c_{1M}^r \log \frac{M^2}{\mu^2} + c_{2M}^r \right) + \cdots \right\}$$

in physical masses ($m_\pi$, $F_\pi$), $\xi = m^2_\pi/(16\pi^2 F^2_\pi)$

$$M^2 = m^2_\pi \left\{ 1 + \xi \left( -\frac{1}{2} \log \frac{m^2_\pi}{\mu^2} + \tilde{l}_M^r \right) + \xi^2 \left( -\frac{5}{8} \log^2 \frac{m^2_\pi}{\mu^2} + \tilde{c}_{1M}^r \log \frac{m^2_\pi}{\mu^2} + \tilde{c}_{2M}^r \right) + \cdots \right\}$$

- only logarithms of mass present
- split is unique
- few constants to fit
- same for decay constants
- Simplified: lattice data on $m_\pi, F_\pi \implies$ fit
- $\xi$ approach usually easier
How to best fit to lattice data?

- Lattice data: $m_\pi, F_\pi, m_K, F_K$, usually not $m_\eta$
- $\xi$ expansion not so obvious
- **Hybrid solution**: physical $m_\pi, m_K, F_\pi$
  
  GMO for *eta*: $m_{\eta g}^2 = (4m_K^2 - m_\pi^2)/3$  
  physical $m_\pi, m_K$
  
  $\xi_\pi = m_\pi^2/(16\pi^2 F_\pi^2)$, $\xi_K = m_K^2/(16\pi^2 F_\pi^2)$,

  $\lambda_\pi = \log(m_\pi^2/\mu^2)$, $\lambda_K = \log(m_K^2/\mu^2)$, $\lambda_\eta = \log(m_{\eta g}^2/\mu^2)$

- All the difficult stuff (per observable) in one function

  $F_I(m_\pi^2/m_K^2)$ which is evaluated by expanding in $m_\pi^2/m_K^2$

- (Small) remaining ambiguity: expand $\lambda_\eta$ in $m_\pi^2$ then can move stuff around and $\lambda_\pi - \lambda_K = \log(m_\pi^2/m_K^2)$
How to best fit to lattice data?

- Lattice data: \( m_\pi, F_\pi, m_K, F_K \), usually not \( m_\eta \)
- \( \xi \) expansion not so obvious
- Hybrid solution: physical \( m_\pi, m_K, F_\pi \)
  GMO for \( \eta \): \( m_{\eta g}^2 = (4m_K^2 - m_\pi^2)/3 \) physical \( m_\pi, m_K \)
- \( \xi_\pi = m_\pi^2/(16\pi^2F_\pi^2) \), \( \xi_K = m_K^2/(16\pi^2F_\pi^2) \),
  \( \lambda_\pi = \log(m_\pi^2/\mu^2) \), \( \lambda_K = \log(m_K^2/\mu^2) \), \( \lambda_\eta = \log(m_{\eta g}^2/\mu^2) \)
- All the difficult stuff (per observable) in one function \( F_I(m_\pi^2/m_K^2) \) which is evaluated by expanding in \( m_\pi^2/m_K^2 \)
- (Small) remaining ambiguity: expand \( \lambda_\eta \) in \( m_\pi^2 \) then can move stuff around and \( \lambda_\pi - \lambda_K = \log(m_\pi^2/m_K^2) \)
How to best fit to lattice data?

\[
\frac{F_K}{F_\pi} = 1 + 4(4\pi)^2L_5^r (\xi_K - \xi_\pi) + \frac{5}{8}\xi_\pi\lambda_\pi - \frac{1}{4}\xi_K\lambda_K + \left(\frac{1}{8}\xi_\pi - \frac{1}{2}\xi_K\right)\lambda_\eta
\]

\[
+\xi_K^2F_1\left(\frac{m_\pi^2}{m_K^2}\right) + \hat{K}_1^r\lambda_\pi^2 + \hat{K}_2^r\lambda_\pi\lambda_K + \hat{K}_3^r\lambda_\pi\lambda_\eta + \hat{K}_4^r\lambda_K^2
\]

\[
+\hat{K}_5^r\lambda_K\lambda_\eta + \hat{K}_6^r\lambda_\eta^2\xi_K + \hat{C}_1\lambda_\pi + \hat{C}_2\lambda_K + \hat{C}_3\lambda_\eta + \hat{C}_4
\]

- \(\hat{K}_i, \hat{C}_i = a_1\xi_K^2 + a_2\xi_K\xi_\pi + a_3\xi_\pi^2\)
- blue: \(p^4\)
- red: \(p^6\) fully determined by \(m_\pi, m_K, F_\pi\)
- green: \(p^6\), also dependent on \(L_i^r\)
- dark blue: \(p^6\), also dependent on \(L_i^r, C_i^r\)
- Numerically fast expressions for all \(F, a_i\) in the papers
An illustrative lattice fit

- Ignore lattice artefacts (not too bad fit so maybe small)
- Use BE14 and $F_\pi = 92.2$ MeV fixed
  

- Fit gave:
  
  $L_5^r = (3.92 \pm 0.55) \times 10^{-4}$
  
  $C_{14}^r + C_{15}^r = (2.59 \pm 0.63) \times 10^{-6}$
  
  $C_{15}^r + 2C_{17}^r = (6.10 \pm 1.41) \times 10^{-6}$


- Not so well with BE14 (10.1,−4.0,−5.0)

- Shows the importance of including lattice data in chiral fits
Some results from the fit
Hadronic contributions

- The blobs are hadronic contributions
- I will present some results that are useful for HVP
- There are higher order contributions of both types: I will stick to LO-HVP
Conventions

- Main object: \[ \Pi_{ab}^{\mu\nu}(q) = i \int d^4xe^{iq \cdot x} \langle 0 | T(j_a^\mu(x)j_b^{\nu\dagger}(0) | 0 \rangle \]

- \[ j_{EM}^\mu = (\frac{2}{3})j_{U}^\mu - (\frac{1}{3})j_{D}^\mu - (\frac{1}{3})j_{S}^\mu \quad j_{Q}^\mu = \overline{q} \gamma^\mu q \]

- Continuum/infinite volume:
  \[ \Pi_{EM}^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{EM}(q^2) \]

- Known positive weight functions \( v, w \) and \( Q^2 = -q^2 \):
  - \[ a_\mu = \int_{\text{threshold}}^{\infty} d q^2 w(q^2) \frac{1}{\pi} \text{Im} \Pi_{EM}(q^2) \]
  - \[ a_\mu = \int_{0}^{\infty} d Q^2 v(Q^2) (-\Pi(Q^2) + \Pi(0)) \]

- Dispersion relation:
  \[ \Pi(q^2) = \Pi(0) + \frac{q^2}{\pi} \int_{\text{threshold}}^{\infty} ds \frac{1}{s(s-q^2)} \frac{1}{\pi} \text{Im} \Pi(s) \]
Two-point: Why

Muon: \( a_\mu = (g - 2)/2 \) and \( a_\mu^{\text{LO,HVP}} = \int_0^\infty dQ^2 f(Q^2) \tilde{\Pi}(Q^2) \)

plot: \( f(Q^2) \tilde{\Pi}(Q^2) \) with \( Q^2 = -q^2 \) in GeV\(^2\)

Figure and data: Aubin, Blum, Chau, Golterman, Peris, Tu, Phys. Rev. D93 (2016) 054508 [arXiv:1512.07555]
To ChPT or not to ChPT

- **ChPT =** Effective field theory describing the lowest order pseudo-scalar representation
- or the (pseudo) Goldstone bosons from spontaneous breaking of chiral symmetry.
- Describes pions, kaons and etas at low energies
- It’s an effective field theory: new parameters or LECs at each new order
- Recent review of LECs:
- $a_{\mu}$ is a very low-energy quantity, why not just calculate it in ChPT?
To ChPT or not to ChPT

- Fill the blobs with pions and kaons
- Lowest order for both HVP and HLbL: pure pion loop (or scalar QED): well defined answer
- NLO: the blob is nicely finite but not after the muon/photon integrations
- Needs a counterterm (NLO LEC) that is the muon $g - 2$
To ChPT or not to ChPT

- So need more than ChPT
- Experiment
- Dispersion relations
- lattice QCD
- ChPT can be used to put constraints, help understanding results and estimate not evaluated parts,...
This talk

- HVP: electromagnetic corrections in scalar QED
  - JB, P. Boyle, J. Harrison, N. Hermansson Truedsson, A, Jüttner, T. Janowski, A. Portelli
  - Infinite volume
  - Finite volume in QED\(_L\)
- HVP (Chiral perturbation theory (ChPT))
  - Our old result from 2000 and chiral corrections
  - Finite volume corrections/Partially quenched
    
Scalar QED

- Pion loops: finite volume effects suppressed by $e^{-m_\pi L}$ (if off-shell)
- Photon loops have suppression only by powers of $1/L$
- Dynamical photons: large finite volume effects possible
- Scalar QED in usual $\overline{MS}$
  \[ \mu_{\text{ChPT}}^2 = \mu_{\overline{MS}}^2 e, \quad e = 2.71 \ldots \]
- \[ \mathcal{L} = (\partial_\mu \Phi^* + i e A_\mu \Phi^*) (\partial_\mu \Phi - i e A_\mu \Phi) - m_0^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]
  \[ (-\lambda (\phi^* \phi)^2 \text{ not needed}) \]
- JB, P. Boyle, J. Harrison, N. Hermansson Truedsson, A, Jüttner, T. Janowski, A. Portelli
  i.e. Lund, Edinburgh, Southampton
Infinite volume

- Lowest order:

- NLO:

contributions from counterterms and \(\cdots\)
QED known since long ago (real and imaginary part): G. Källén, A. Sabry, Dan. Mat. Medd 29 (1955) 1

Scalar QED can be done using the same methods but we did not find any published results

Imaginary part follows from $e^+ e^- \rightarrow \pi^+ \pi^- (\gamma)$ via

$$\sigma(s)(e^+ e^- \rightarrow (\gamma) \rightarrow \text{hadrons}) = \frac{16\pi^3\alpha}{s}\frac{1}{\pi}\text{Im}\Pi(s)$$
Infinite volume

- Use Laporta algorithm to reduce to master integrals done via the package \texttt{Reduze2}
- Most complicated function needed is \( \text{Li}_3 \)
- You need to go to an on-shell scheme or explicitly rewrite in physical mass. (otherwise you get a negative imaginary part to take care of threshold shifting)

\[ m^2 = m_0^2 + \frac{\alpha}{4\pi} m_0^2 \left( 7 - 3 \log \frac{m_0^2}{\mu^2} \right) \]
\[
\sigma^2 = 1 - \frac{4m^2}{p^2}
\]

\[
\Pi^{1\text{-loop}} = \frac{4}{3} A(m^2) + \frac{1}{3} \sigma^2 B(m^2, p^2) + \frac{1}{16\pi^2} \left( \frac{2}{9} - \frac{4m^2}{3p^2} \right)
\]

\[
\Pi^{\delta m^2} = -\delta m^2 \frac{2}{p^2} \left( \frac{1}{m^2} A(m^2) - B(m^2, p^2) - \frac{1}{16\pi^2} \right)
\]

\[
\Pi^{\text{disc}} = \left( \Pi^{\text{one-loop}}(p^2) \right)^2
\]

\[
\Pi^{2\text{-loop}} = \frac{1}{(16\pi^2)^2} \left( \frac{10}{3} - \frac{8m^2}{p^2} \right) + \frac{A(m^2)}{16\pi^2} \frac{22}{3p^2} + \frac{10}{3m^2p^2} A(m^2)^2
\]

\[
+ \left( \frac{8}{3m^2} - \frac{26}{3p^2} \right) A(m^2) B(m^2, p^2) + \frac{8\sigma^2}{48\pi^2} B(m^2, p^2)
\]

\[
- \frac{8}{3} \left( \frac{1}{p^2} S(m^2, p^2) + \sigma^2 T(m^2, p^2) - m^2 \sigma^2 V(m^2, p^2) \right)
\]

\[
+ \left( -\frac{4}{3} + \frac{8m^2}{3p^2} \right) B(m^2, p^2)^2 - \frac{2\sigma^2}{3} \left( p^2 - \frac{2m^2}{p^2} \right) M(m^2, p^2)
\]
Integrals

The integrals used are the $\overline{MS}$ subtracted parts of

\begin{align*}
A(m^2) &= \frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2} \\
B(m^2, p^2) &= \frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m^2)((l - p)^2 - m^2)} \\
S(m^2, p^2) &= \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2 - m^2)((k + l - p)^2 - m^2)} \\
T(m^2, p^2) &= \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2 - m^2)^2((k + l - p)^2 - m^2)} \\
V(m^2, p^2) &= \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2 - m^2)^2((k + l)^2 - m^2)((l - p)^2 - m^2)} \\
M(m^2, p^2) &= \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2 - m^2)((k + l)^2 - m^2)((l - p)^2 - m^2)((k + l - p)^2 - m^2)}
\end{align*}
Numerical results: lowest order

\( m = 139.5 \text{ MeV}, \mu = 500 \text{ MeV}, e = 0.303 \)

\[
\frac{\Pi(p^2)}{\Pi(0)} = \frac{p^2}{m^2}
\]

- **Re LO**
- **Im LO**

\[
\begin{array}{c}
\text{p}^2/m^2 \\
-10 & -8 & -6 & -4 & 0 & 2 & 4 & 6 & 8 & 10
\end{array}
\]

\[
\begin{array}{c}
\text{Re LO} \\
\text{Im LO}
\end{array}
\]

\[
\begin{array}{c}
0.000 & 0.001 & 0.002 & 0.003
\end{array}
\]

\[
\begin{array}{c}
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{array}
\]

- To ChPT or not to ChPT
- sQED
- HVP: chiral corrections
- Conclusions
Numerical results: electromagnetic correction

\[ m = 139.5 \text{ MeV}, \mu = 500 \text{ MeV}, e = 0.303 \]

\[
\text{Re}(\Pi(p^2)-\Pi(0))
\]

\[ \frac{p^2}{m^2} \]

About 1% of lowest order
Numerical results: electromagnetic correction

$m = 139.5$ MeV, $\mu = 500$ MeV, $e = 0.303$
Finite volume

- Finite volume for photons started in earnest with
- Get the explicit $1/L$ behaviour from one-loop integrals:
- Problem for photon: $\int d^d k \frac{1}{k^2} \rightarrow \int dk^0 \sum_k \frac{1}{(k^0)^2 - \vec{k}^2}$
  Singularity no longer has the $k^{d-1}$ to soften it
- Solution QED$_L$: drop all modes with $\vec{k} = 0$ Hayakawa-Uno
- We extend the arguments of Davoudi-Savage to two-loop order, use QED$_L$ and a lattice with infinite time extension
- We only calculate corrections suppressed by $1/L$ and powers, not exponentially suppressed contributions
- Below threshold so mesons (pions) are off-shell
Integrals at finite volume

- \( S = \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2-m^2)((k+l-p)^2-m^2)} \)

- do \( l^0, k^0 \) integrals via contour integration

- \( \vec{k} = \frac{2\pi}{L} \vec{n} \) and expand in \( 1/L \)

- Write the \( \vec{k} \) part as

\[
\frac{1}{L^{d-1}} \sum_{\vec{n} \neq \vec{0}} \frac{d^{d-1}k}{(2\pi)^{d-1}} + \left[ \frac{1}{L^{d-1}} \sum_{\vec{n} \neq \vec{0}} - \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \right]
\]

- In the first term resum the series in \( 1/L \): infinite volume contribution

- Call the quantity in brackets \( (1/L^{d-1}) \sum \vec{n} \)

- Define \( c_m = \sum |\vec{n}|^m \)

- These are known numerically
Example: mass

\[ \Delta \nu \frac{m^2}{m^2} = e^2 \left( \frac{4c_2}{16\pi^2 mL} + \frac{2c_1}{16\pi^2 m^2 L^2} + \mathcal{O} \left( \frac{1}{L^4}, e^{-mL} \right) \right) \]

- Agrees with earlier results
- \( c_2 \) shows up since BOTH propagators can be 'on-shell'
- For \( \Pi_{\mu\nu} \) below threshold only photon line goes 'on-shell'
  \[ \implies \text{corrections start only at } 1/L^2 \]
Numerical results mass

Infinite volume is $\frac{\delta m^2}{m^2} = 0.00852$

So very large finite volume corrections to electromagnetic part
Twopoint function

- Do the $k^0$, $l^0$ integral
- expand for large $L$ as explained earlier
- Rest can be expressed in terms of
  \[ Z_{ij}(m^2, p^2) = \int \frac{d^{d-1}l}{(2\pi)^{d-1}} \frac{1}{(l^2 + m^2)^{i/2}(4l^2 + 4m^2 - p^2)^{j}} \]
- these correspond to one-loop integrals with masses
- $\Omega_{ij} \equiv Z_{ij}(m^2, p^2)m^{i+2j-d+1}$
- Calculate in center of mass frame: $p = (p^0, \mathbf{0})$
- $\text{QED}_L$: no disconnected contribution
- $t_{\mu\nu}$ spatial part of $g_{\mu\nu}$
- $\tilde{\Pi}((p^0)^2) \equiv -\frac{1}{3p^2} t_{\mu\nu} (\Pi^{\mu\nu}(p) - \Pi^{\mu\nu}(p = 0))$
- Infinite volume: $\tilde{\Pi}(p^2) = \Pi(p^2)$
Results (preliminary)

\[ \tilde{\Pi}(p^2) = \frac{c_1}{\pi m^2 L^2} \left( \frac{16}{3} \Omega_{-1,3} - \frac{1}{3} \Omega_{1,2} - \frac{32}{3} \Omega_{1,3} - \frac{2}{3} \Omega_{3,2} + \frac{16}{3} \Omega_{3,3} - \frac{1}{8} \Omega_{5,1} + \Omega_{5,2} \right) \]
\[ + \frac{c_0}{m^3 L^3} \left( - \frac{128}{3} \Omega_{-2,4} + \frac{256}{3} \Omega_{0,4} - \frac{5}{3} \Omega_{2,2} + \frac{8}{3} \Omega_{2,3} - \frac{128}{3} \Omega_{2,4} \right) \]
\[ - \frac{3}{8} \Omega_{4,1} + \frac{7}{6} \Omega_{4,2} - \frac{8}{3} \Omega_{4,3} \right) + \mathcal{O} \left( \frac{1}{L^4}, e^{-mL} \right) \]

Simplify using relations of the \( \Omega_{ij} \) to

\[ \tilde{\Pi}(p^2) = + \frac{c_0}{m^3 L^3} \left( - \frac{16}{3} \Omega_{0,3} - \frac{5}{3} \Omega_{2,2} + \frac{40}{9} \Omega_{2,3} - \frac{3}{8} \Omega_{4,1} + \frac{7}{6} \Omega_{4,2} + \frac{8}{9} \Omega_{4,3} \right) \]
\[ + \mathcal{O} \left( \frac{1}{L^4}, e^{-mL} \right) \]

the 1/L^2 cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect
Numerical results $\Pi$

![Graph showing numerical results $\Pi$ for $m_L=4$ and $m_L=8$.](image)

$.5x10^{-8}$

$.1x10^{-7}$

$.1.5x10^{-7}$

$.2x10^{-7}$

$.2.5x10^{-7}$

$.3x10^{-7}$

$\Delta V\Pi(p^2)$

$p^2/m^2$

$m_L=4$

$m_L=8$

Conclusions
Numerical results $\Pi$

- Finite volume corrections to the electromagnetic contribution are small: below 2% for $mL = 4$
- only photon 'on-shell' and only dipole effect $\implies 1/L^3$
Lattice

- put scalar QED on a lattice (purely bosonic so runs fast)
- the different diagrams correspond to different contractions
- show you as an example the exchange diagram
- calculate the same thing in lattice perturbation theory (vegas)
- compare $L$ dependence with our analytical results
Note: usual lattice conventions so opposite signs of rest of talk
Note: usual lattice conventions so opposite signs of rest of talk
Chiral corrections

JB, Relefors, JHEP 1611(2016) 086 [1609.01573]

Two-flavour case (and $I = 1$ part only)

$$\Pi(q^2) = \Pi^{(4)}(q^2) + \Pi^{(6)}(q^2)$$

$$\Pi^{(4)}(q^2) = -8G(m_\pi^2, q^2) + 16h_2^r$$

$$F_\pi^2\Pi^{(6)}(q^2) = 16q^2G(m_\pi^2, q^2)^2 + 16q^2l_6^rG(m_\pi^2, q^2)$$

$$- 8(2l_5^r - l_6^r)\overline{A}(m_\pi^2) - 32m_\pi^2c_{34}^r - 8q^2c_{56}^r$$

$G(m_\pi^2, q^2)$ has a complicated $m_\pi^2$ dependence: not easy to get simple $m_\pi^2$ corrections to $a_\mu$


$g - 2$ workshops J-PARC, November 2016, FNAL, June 2017, lattice 2017
Chiral corrections

\( G(m_\pi^2, q^2) \) has a simple \( q^2 \) expansion (misprint in paper):

\[
G(m^2, q^2) = \frac{1}{16\pi^2} \left( \frac{1}{12} + \frac{1}{12} \log \frac{m^2}{\mu^2} - \frac{q^2}{120m^2} - \frac{q^4}{1680m^4} + \ldots \right)
\]

Moments have a well defined \( m_\pi^2 \) expansion (\( l = 1 \) case)

\[
\Pi(q^2) = \Pi(0) + \Pi_1 q^2 - \Pi_2 q^4 \ldots
\]

\[
\Pi_1 = \frac{1}{16\pi^2} \frac{1}{15m_\pi^2} + \frac{1}{F_\pi^2} \left[ - 8c_5^r + \frac{l_6^r}{16\pi^2} \frac{4}{3} \left( 1 + \log \frac{m_\pi^2}{\mu^2} \right) \right.
\]

\[
+ \frac{1}{(16\pi^2)^2} \left( \frac{49}{405} + \frac{2}{9} \log \frac{m_\pi^2}{\mu^2} + \frac{1}{9} \log^2 \frac{m_\pi^2}{\mu^2} \right) \]

\[
\Pi_2 = \frac{1}{16\pi^2} \frac{-1}{210m_\pi^4} + \frac{1}{16\pi^2 F_\pi^2 m_\pi^2} \left[ \frac{2}{15} l_6^r + \frac{1}{16\pi^2} \left( \frac{23}{945} + \frac{1}{45} \log \frac{m_\pi^2}{\mu^2} \right) \right]
\]

Other cases: plug \( G \) expansion in formulas of section 4 of JB, Relefors, JHEP 1611(2016) 086 [1609.01573]
Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i / L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta^i} q(x^i)$
- Then momenta are $p^i = \theta^i / L + 2\pi n^i / L$. Allows to map out momentum space on the lattice much better

Bedaque, ...

- Small note:
  - Beware what people call momentum: is $\theta^i / L$ included or not?
  - Reason: a colour singlet gauge transformation $G^S_\mu \rightarrow G^S_\mu - \partial_\mu \epsilon(x), \ q(x) \rightarrow e^{i\epsilon(x)} q(x), \ \epsilon(x) = -\theta^i x^i / L$
  - Boundary condition
    Twisted $\Leftrightarrow$ constant background field + periodic
Twisted boundary conditions: Drawbacks

Drawbacks:
- Box: Rotation invariance $\rightarrow$ cubic invariance
- Twisting: reduces symmetry further
- Can only be done for the connected part

Consequences:
- $m^2(p^2) = E^2 - \vec{p}^2$ is not constant
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum
Two-point function: twisted boundary conditions

\[ \int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0 \]

\[ \langle \bar{u} \gamma^\mu u \rangle \neq 0 \]

\[ j_{\pi^+}^\mu = \bar{d} \gamma^\mu u \]

satisfies \( \partial_\mu \langle T (j_{\pi^+}^\mu (x) j_{\pi^+}^{\nu \dagger} (0)) \rangle = \delta^{(4)}(x) \langle \bar{d} \gamma^\nu d - \bar{u} \gamma^\nu u \rangle \)

\[ \Pi_{\alpha}^{\mu \nu} (q) \equiv i \int d^4 x e^{i q \cdot x} \langle T (j_{\alpha}^\mu (x) j_{\alpha}^{\nu \dagger} (0)) \rangle \]

Satisfies WT identity. \( q_\mu \Pi_{\pi^+}^{\mu \nu} = \langle \bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d \rangle \)

ChPT at one-loop satisfies this


two-loop in partially quenched satisfies the WT identity (as it should)
Two-point: partially twisted, with two-loop

\[ q = \left(0, \sqrt{-q^2}, 0, 0\right) \]

\[ \Pi^{22} = \Pi^{33} \]

\[ \vec{\theta}_u = L q \]

\[ m_{\pi_0}L = 4 \]

\[ m_{\pi_0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi^{(1)}_{\text{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5e-3 \cdot \frac{q^2}{0.1} \]

diamond: periodic

Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Correction is at the % level
Correction from two loop is reasonable (thin lines are \( p^4 \))
Two-point: partially twisted, two-loop

\[ q = \left( 0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0 \right) \]

\[ \Pi^{11} = \Pi^{22} \]
\[ \vec{\theta}_u = L q \]
\[ m_{\pi_0} L = 4 \]
\[ m_{\pi_0} = 0.135 \text{ GeV} \]

\[ -q^2 \Pi_{\text{VMD}}^{(1)} = \frac{-4q^2 F^2_{\pi}}{M_V^2 - q^2} \approx 5e^{-3} \cdot \frac{q^2}{0.1} \]

diamond: periodic
Note: \( \Pi^{\mu\nu}(0) \neq 0 \)

Correction is at the % level
Two loop correction again reasonable (thin lines are \( p^4 \))
Conclusions

Showed you results for:

- $p^8$ ChPT: Lagrangian
- $p^8$ ChPT: mass and decay constant for $N_f = 2$
- $p^6$ ChPT: can we fit lattice in a better way
- Finite volume corrections to the electromagnetic contribution as estimated in scalar QED are small
- Connected: twisting and finite volume at two-loops
  - The corrections are sizable for present lattices (few%) but two-loop corrections were normal size