CHIRAL EFT AND LATTICE

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Overview

Chiral Perturbation Theory

A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about M_ρ.

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Goldstone Bosons

Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(3)_L \times SU(3)_R$ broken spontaneously to $SU(3)_V$
- 8 generators broken $\Longrightarrow$ 8 massless degrees of freedom and interaction vanishes at zero momentum

Chiral Perturbation Theories

- Which chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$, for $N_f = 2, 3, \ldots$ and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite T
- Which interactions to include beyond the strong one
- Which particles included as non Goldstone Bosons
- Here: restrict to standard ChPT but with extensions for the lattice

Lagrangians: Lowest order

$U(\phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix} .$$

LO Lagrangian: $\mathcal{L}_2 = \frac{F_0^2}{4}\{\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle\}\,,$

$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$,
left and right external currents: $r(l)_\mu = v_\mu + (-) a_\mu$

Scalar and pseudoscalar external densities: $\chi = 2B_0(s + ip)$ quark masses via scalar density: $s = \mathcal{M} + \cdots$

$\langle A \rangle = Tr_F\,(A)$

Lagrangians: structure (mesons, strong)

	2 flavour		3 flavour		PQChPT/N_f flavour	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	51*+4	C_i^r	90+4	K_i^r	112+3
p^8	x_i^r	452+23	X_i^r	1233+21	Y_i^r	1840+22

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

p^8: JB, Hermansson-Truedsson, Wang 1810.06834

- L_i LEC = Low Energy Constants = ChPT parameters
- H_i: contact terms: value depends on definition of currents/densities
- Finite volume: no new LECs
- Other effects: (many) new LECs

How to find a Lagrangian

- Find building blocks with an easy transformation law
- u_μ, $\chi_\pm$, $f_\pm^{\mu\nu}$
- all transform simply under the compensating transformation $h \in SU(N_f)_V$
- and covariant derivatives of them (uses Γ_μ)
- make all invariant traces up to the order you want
- Combine into C, P and hermitian conjugation invariant combinations
- relatively easy to handle

How to find a minimal Lagrangian

- integration-by-parts: construct all total derivatives
- field redefinitions: construct all terms constaining the LO EOM
- Bianchi of $\Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}$,
- the identities $f_{-\mu\nu} - \nabla_\nu u_\mu + \nabla_\mu u_\nu = 0$ and $[\nabla_\mu, \nabla_\nu] X = [\Gamma_{\mu\nu}, X]$,
- the Cayley-Hamilton theorem (for $N_f = 2, 3$).
- 9740 monomials and 12444 different relations plus Cayley-Hamilton
- positive intrinsic parity: Schouten identity not needed (p^8)
- Contact terms: many simplifications not applicable
- Remember $\tilde{\chi}^\dagger \equiv \det(\chi)\chi^{-1}$

Methods/Checks/results

- generate all terms and relations using FORM
- generate from the output back identities etc using PYTHON
- Final diagonalization done using C++ with GMP
- reproduce all known p^6 results (Dipl. thesis J. Weber)
- have two independent sets of programs
- Determined for no external fields, s, p-only, v, a-only, all
- Basis with all terms as supplementary material (57 pages)

Lagrangian: all external fields set to zero

	N_f	$N_f = 3$	$N_f = 2$
p^2	1	1	1
p^4	4	3	2
p^6	19	11	5
p^8	135	56	16

	#mesons	N_f	$N_f = 3$	$N_f = 2$
p^2	4	1	1	1
p^4	4	4	3	2
p^6	4	4	3	2
	6	15	8	3
p^8	4	6	5	3
	6	60	31	9
	8	69	20	4

Fits with what can be determined from meson-meson scattering

Two flavour ChPT: mass and decay constant

- First step towards finding out why hard-pion ChPT does not work at three-loops
- Lowest order: Gell-Mann, Oakes, Renner (1968)
- Chiral logarithm Langacker, Pagels (1973)
- Full NLO (and properly starting ChPT) Gasser-Leutwyler (1984)
- NNLO Buergi (1996), JB, Colangelo, Ecker, Gasser, Sainio (1996)
- NNNLO JB, Hermansson-Truedsson (1710.01901)

Methods used

- LO and Chiral logs: current algebra
- NLO: Feynman diagrams (by hand) and direct expansion of functional integral (with REDUCE)
- NNLO: Feynman diagrams
  (with a little help from FORM)
- NNLO: Feynman diagrams purely with FORM
- Main stumbling block: integrals
  - Reduction to master integrals with REDUZE Studerus (2009)
  - Master Integrals known
    Laporta-Remiddi (1996); Melnikov, van Ritbergen (2001)
- Lots of book-keeping: FORM
- Checks:
  - All nonlocal divergences must cancel
  - Use different parametrizations of the Lagrangian
  - Agree with known leading log result JB, Carloni (2009)

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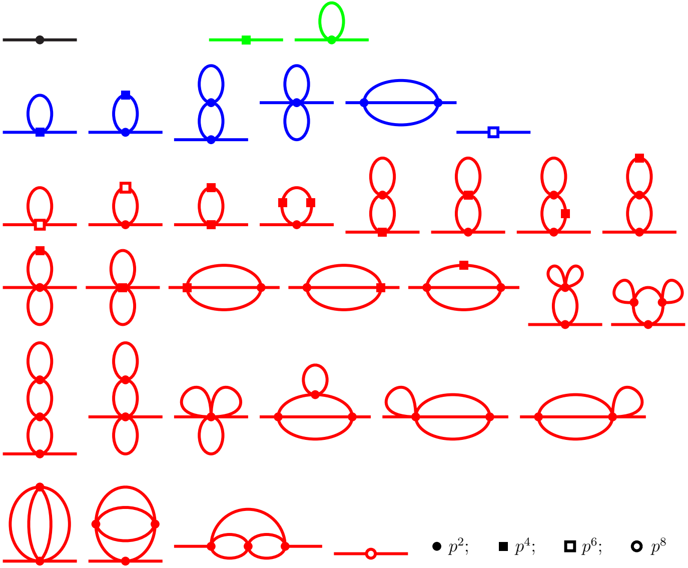
Diagrams



Results: LO or x-expansion/physical or ξ-expansion

- $x = \frac{M^2}{16\pi^2 F^2}, \qquad L_x = \log\frac{M^2}{\mu^2}, \qquad M^2 = 2B\hat{m}$

$$\frac{M_\pi^2}{M^2} = 1 + x\left(a_{11}^M L_x + a_{10}^M\right) + x^2\left(a_{22}^M L_x^2 + a_{21}^M L_x + a_{20}^M\right) + x^3\left(a_{33}^M L_x^3 + a_{32}^M L_x^2 + a_{31}^M L_x + a_{30}^M\right) + \cdots$$

$$\frac{F_\pi}{F} = 1 + x\left(a_{11}^F L_x + a_{10}^F\right) + x^2\left(a_{22}^F L_x^2 + a_{21}^F L_x + a_{20}^F\right) + x^3\left(a_{33}^F L_x^3 + a_{32}^F L_x^2 + a_{31}^F L_x + a_{30}^F\right) + \cdots$$

- $\xi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}, \qquad L_\pi = \log\frac{M_\pi^2}{\mu^2}$
- $$\frac{M^2}{M_\pi^2} = 1 + \xi\left(b_{11}^M L_\pi + b_{10}^M\right) + \xi^2\left(b_{22}^M L_\pi^2 + b_{21}^M L_\pi + b_{20}^M\right) + \xi^3\left(b_{33}^M L_\pi^3 + b_{32}^M L_\pi^2 + b_{31}^M L_\pi + b_{30}^M\right) + \cdots$$
- $$\frac{F}{F_\pi} = 1 + \xi\left(b_{11}^F L_\pi + b_{10}^F\right) + \xi^2\left(b_{22}^F L_\pi^2 + b_{21}^F L_\pi + b_{20}^F\right) + \xi^3\left(b_{33}^F L_\pi^3 + b_{32}^F L_\pi^2 + b_{31}^F L_\pi + b_{30}^F\right) + \cdots$$

Results $\tilde{l}_i = 16\pi^2 l_i^r$, $\tilde{c}_i = (16\pi^2)^2 c_i^r$

a_{11}^M	$\frac{1}{2}$
a_{10}^M	$2\tilde{l}_3$
a_{22}^M	$\frac{17}{8}$
a_{21}^M	$-3\tilde{l}_3 - 8\tilde{l}_2 - 14\tilde{l}_1 - \frac{49}{12}$
a_{20}^M	$64\tilde{c}_{18} + 32\tilde{c}_{17} + 96\tilde{c}_{11} + 48\tilde{c}_{10} - 16\tilde{c}_9 - 32\tilde{c}_8 - 16\tilde{c}_7 -32\tilde{c}_6 + \tilde{l}_3 + 2\tilde{l}_2 + \tilde{l}_1 + \frac{193}{96}$
a_{33}^M	$\frac{103}{24}$
a_{32}^M	$\frac{23}{2}\tilde{l}_3 - 11\tilde{l}_2 - 38\tilde{l}_1 - \frac{91}{24}$
a_{31}^M	$-416\tilde{c}_{18} - 208\tilde{c}_{17} - 32\tilde{c}_{16} + 96\tilde{c}_{14} + 8\tilde{c}_{13} - 48\tilde{c}_{12} -384\tilde{c}_{11} - 192\tilde{c}_{10} + 72\tilde{c}_9 + 144\tilde{c}_8 + 72\tilde{c}_7 + 64\tilde{c}_6 - 8\tilde{c}_5 -56\tilde{c}_4 + 16\tilde{c}_3 + 32\tilde{c}_2 - 96\tilde{c}_1 - 8\tilde{l}_3^2 - 48\tilde{l}_3\tilde{l}_2 - 84\tilde{l}_3\tilde{l}_1 -\frac{88}{3}\tilde{l}_3 - \frac{231}{10}\tilde{l}_2 - \frac{69}{5}\tilde{l}_1 - \frac{74971}{8640}$
a_{30}^M	contains free p^8 LECs (and a lot more terms)

Results: comments

- Similar tables for a_i^F, b_i^M, b_i^F
- Coefficients depend on scale μ, but whole expression is μ-independent
- Can be rewritten in terms of scales in the logarithm rather than in terms of LECs à la FLAG
- Leading log: a number
- NLL: depends on l_i^r
- NNLL: depends on c_i^r
- For the mass all needed c_i^r can be had from mass, decay-constant and $\pi\pi$ parameters fitted to two-loop or p^6 (i.e. $r_M, r_F, r_1, \ldots, r_6$).
- For decay need one more

Results: numerics preliminary

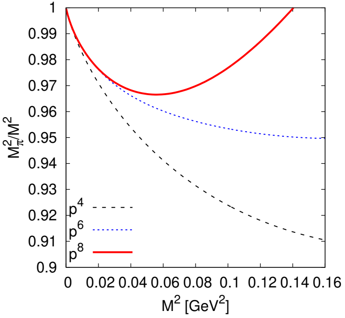
ij	a^M_{ij}	b^M_{ij}	a^F_{ij}	b^F_{ij}
10	$+0.00282$	-0.00282	$+1.09436$	-1.09436
11	$+0.5$	-0.5	-1	$+1$
20	$+1.65296$	-1.65771	-0.04734	-1.15001
21	$+2.4573$	-3.29038	-1.90577	$+4.13885$
22	$+2.125$	-0.625	-1.25	-0.25
30	$+0.39527$	-6.7854	-244.499	242.236
31	-3.75977	$+4.32719$	-19.0601	32.1315
32	$+17.1476$	$+0.62039$	-9.39462	-6.77511
33	$+4.29167$	$+5.14583$	-3.45833	-0.41666

Note the large coefficients in the decay constant

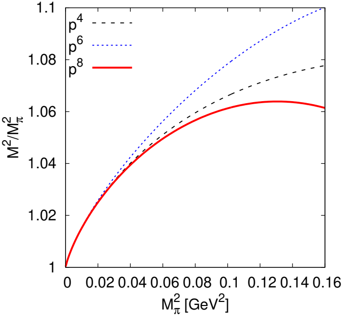
Pion mass

$F_\pi = 92.2$ MeV, $F = F_\pi/1.037$, $\bar{l}_1 = -0.4$, $\bar{l}_2 = 4.3$, $\bar{l}_3 = 3.41$, $\bar{l}_4 = 4.51$,

r_i from JB et al 1997, other $c_i^r = 0$, $\mu = 0.77$ GeV



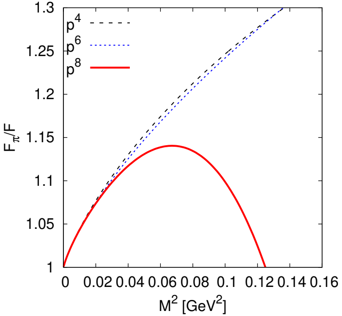
x-expansion (F-fixed)



ξ-expansion (F_π fixed)

ξ-expansion converges notably better

Pion decay constant



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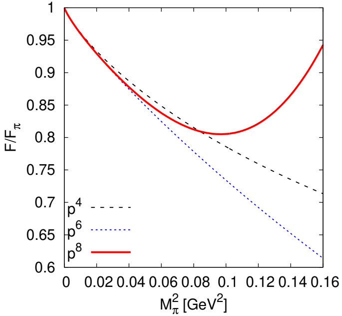


ξ-expansion (F_π fixed)

- ξ-expansion converges better
- Large p^8 due to the "240" in a^F_{30} and b^F_{30}

Masses/decay constants: older results

- Two-flavour

  One-loop: J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

  Two-loop:

  U. Burgi, Phys. Lett. B **377** (1996) 147 [hep-ph/9602421], Nucl. Phys. B **479** (1996) 392; [hep-ph/9602429]

  JB, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Phys.Lett.B**374** (1996)210 [hep-ph/9511397], Nucl.Phys.B**508**(1997)263 [hep-ph/9707291].

  Three loops: just talked about

- Three-flavour

  One-loop: J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

  Two-loop:

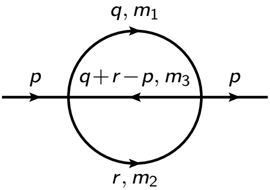
  E.Golowich, J.Kambor, Phys.Rev.D**58**(1998)036004 [hep-ph/9710214]($\pi\eta$);

  G.Amoros, JB, P.Talavera, Nucl.Phys.B**568**(2000)319 [hep-ph/9907264].

- Added: isospin breaking, finite volume, partial quenching all at two loops

Masses/decay constants: underlying idea

- A lot more is now known about the sun(set)(rise)integrals



$= H(m_1, m_2, m_3, p^2)$

- Reduction to master integrals

  O. V. Tarasov, Nucl. Phys. B **502** (1997) 455 [hep-ph/9703319].

- Many integrals with two scales known analytically
- Expansions around certain points much better known
- Used for three-flavour result for m_π:

  R. Kaiser, JHEP **0709** (2007) 065 [arXiv:0707.2277 [hep-ph]].

- Do the same for: F_π, m_K, F_K, m_η, F_η

Masses/decay constants: papers

- Summarize existing sunset knowledge and make it easier to use (mathematica notebooks as supplementary material)
  B. Ananthanarayan, JB, S. Ghosh and A. Hebbar,
  Eur. Phys. J. A **52** (2016) 374 [arXiv:1608.02386 [hep-ph]]
- F_π:
  only $H(m_\eta, m_K, m_K, m_\pi^2)$ not analytically known, expand in m_π^2, expansion coefficients analytically known
  B. Ananthanarayan, JB and S. Ghosh,
  Eur. Phys. J. C **77** (2017) 497 [arXiv:1703.00141 [hep-ph]].
- F_K/F_π:
  $H(m_\pi, m_K, m_\eta, m_K^2)$: needs Mellin-Barnes to expand in m_π
  B. Ananthanarayan, JB, S. Friot and S. Ghosh,
  Phys. Rev. D **97** (2018) 091502 [arXiv:1711.11328 [hep-ph]].
- m_K, F_K, m_η, F_η: B. Ananthanarayan, JB, S. Friot and S. Ghosh,
  Phys. Rev. D **97** (2018) 114004 [arXiv:1804.06072 [hep-ph]].

How to best fit to lattice data?

Two standard options for expanding(two flavours):

in "quark masses" (lowest order M, F), $x = M^2/(16\pi^2 F^2)$

$$m_\pi^2 = M^2 \left\{ 1 + x \left(\frac{1}{2} \log \frac{M^2}{\mu^2} + l_M^r \right) + x^2 \left(\frac{17}{8} \log^2 \frac{M^2}{\mu^2} + c_{1M}^r \log \frac{M^2}{\mu^2} + c_{2M}^r \right) + \cdots \right\}$$

in physical masses (m_π, F_π), $\xi = m_\pi^2/(16\pi^2 F_\pi^2)$

$$M^2 = m_\pi^2 \left\{ 1 + \xi \left(-\frac{1}{2} \log \frac{m_\pi^2}{\mu^2} + \tilde{l}_M^r \right) + \xi^2 \left(-\frac{5}{8} \log^2 \frac{m_\pi^2}{\mu^2} + \tilde{c}_{1M}^r \log \frac{m_\pi^2}{\mu^2} + \tilde{c}_{2M}^r \right) + \cdots \right\}$$

- only logarithms of mass present
- split is unique
- few constants to fit
- same for decay constants
- Simplified: lattice data on m_π,F_π $\Longrightarrow$ fit
- ξ approach usually easier

How to best fit to lattice data?

- Lattice data: m_π, F_π, m_K, F_K, usually not m_η
- ξ expansion not so obvious
- Hybrid solution: physical m_π, m_K, F_π
  GMO for *eta*: $m^2_{\eta g} = (4m^2_K - m^2_\pi)/3$ physical m_π, m_K
- $\xi_\pi = m^2_\pi/(16\pi^2 F^2_\pi)$, $\xi_K = m^2_K/(16\pi^2 F^2_\pi)$,
  $\lambda_\pi = \log(m^2_\pi/\mu^2)$, $\lambda_K = \log(m^2_K/\mu^2)$, $\lambda_\eta = \log(m^2_{\eta g}/\mu^2)$
- All the difficult stuff (per observable) in one function $F_I(m^2_\pi/m^2_K)$ which is evaluated by expanding in m^2_π/m^2_K
- (Small) remaining ambiguity: expand λ_η in m^2_π then can move stuff around and $\lambda_\pi - \lambda_K = \log(m^2_\pi/m^2_K)$

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How to best fit to lattice data?

$$\frac{F_K}{F_\pi} = 1 + 4(4\pi)^2 L_5^r(\xi_K - \xi_\pi) + \frac{5}{8}\xi_\pi\lambda_\pi - \frac{1}{4}\xi_K\lambda_K + \left(\frac{1}{8}\xi_\pi - \frac{1}{2}\xi_K\right)\lambda_\eta$$
$$+\xi_K^2 F\left(\frac{m_\pi^2}{m_K^2}\right) + \hat{K}_1^r\lambda_\pi^2 + \hat{K}_2^r\lambda_\pi\lambda_K + \hat{K}_3^r\lambda_\pi\lambda_\eta + \hat{K}_4^r\lambda_K^2$$
$$+\hat{K}_5^r\lambda_K\lambda_\eta + \hat{K}_6^r\lambda_\eta^2\xi_K^2 + \hat{C}_1\lambda_\pi + \hat{C}_2\lambda_K + \hat{C}_3\lambda_\eta + \hat{C}_4$$

- $\hat{K}_i, \hat{C}_i = a_1\xi_K^2 + a_2\xi_K\xi_\pi + a_3\xi_\pi^2$
- blue: p^4
- red: p^6 fully determined by m_π, m_K, F_π
- green: p^6, also dependent on L_i^r
- dark blue: p^6, also dependent on L_i^r, C_i^r
- Numerically fast expressions for all F, a_i in the papers

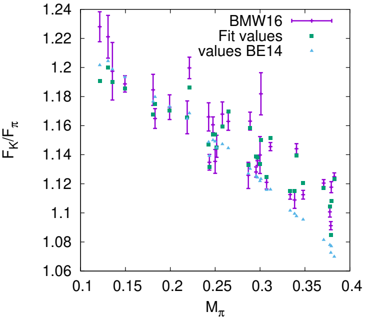
An illustrative lattice fit

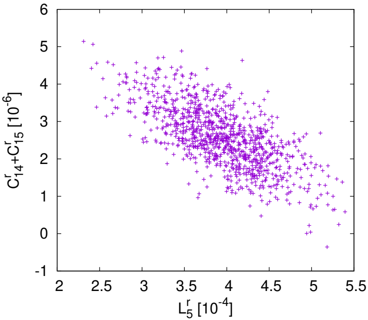
- BMW16: Phys. Rev. D95 (2017) 054513[1601.05998]
- Ignore lattice artefacts (not too bad fit so maybe small)
- Use BE14 and $F_\pi = 92.2$ MeV fixed
  JB and G. Ecker, Ann. Rev. Nucl. Part. Sci. **64** (2014) 149 [1405.6488].
- Fit gave:

$$L_5^r = (3.92 \pm 0.55)\ 10^{-4}$$
$$C_{14}^r + C_{15}^r = (2.59 \pm 0.63)\ 10^{-6}$$
$$C_{15}^r + 2C_{17}^r = (6.10 \pm 1.41)\ 10^{-6}$$

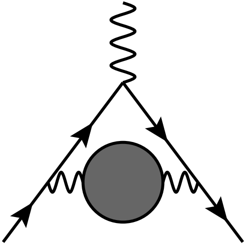
- Agree well with G. Ecker, P. Masjuan and H. Neufeld, Phys. Lett. B **692** (2010) 184 [1004.3422]., Eur. Phys. J. C **74** (2014) 2748 [1310.8452]. (7.5,1.7,6.0) in the same units
- Not so well with BE14 (10.1,−4.0,−5.0)
- Shows the importance of including lattice data in chiral fits

Some results from the fit

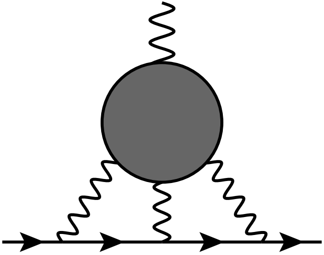




Hadronic contributions



HVP



HLbL

- The blobs are hadronic contributions
- I will present some results that are useful for HVP
- There are higher order contributions of both types: I will stick to LO-HVP

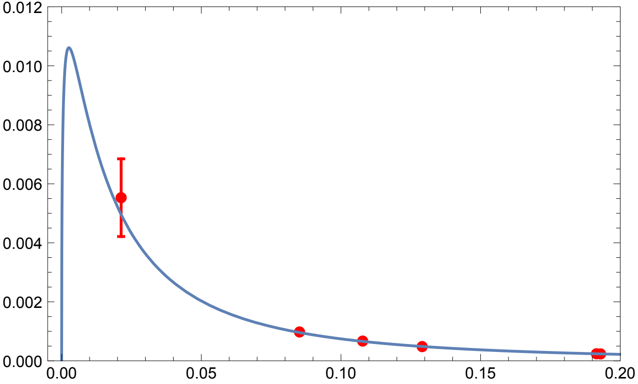
Conventions

- Main object: $\Pi^{\mu\nu}_{ab}(q) = i \int d^4x e^{iq\cdot x} \langle 0| T(j^\mu_a(x) j^{\nu\dagger}_b(0)|0\rangle$
- $j^\mu_{EM} = (2/3) j^\mu_U - (1/3) j^\mu_D - (1/3) j^\mu_S \qquad j^\mu_Q = \overline{q}\gamma^\mu q$
- Continuum/infinite volume:
  $\Pi^{\mu\nu}_{EM}(q) = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \Pi_{EM}(q^2)$
- Known positive weight functions v, w and $Q^2 = -q^2$:
  - $a_\mu = \int_{\text{threshold}}^\infty dq^2 w(q^2) \frac{1}{\pi} \mathrm{Im}\Pi_{EM}(q^2)$
  - $a_\mu = \int_0^\infty dQ^2 v(Q^2) \left(-\Pi(Q^2) + \Pi(0)\right)$
- Dispersion relation:

$$\Pi(q^2) = \Pi(0) + \frac{q^2}{\pi} \int_{\text{threshold}}^\infty ds \frac{1}{s(s-q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s)$$

Two-point: Why

Muon: $a_\mu = (g-2)/2$ and $a_\mu^{\mathrm{LO,HVP}} = \int_0^\infty dQ^2 f\left(Q^2\right)\tilde{\Pi}\left(Q^2\right)$



plot: $f\left(Q^2\right)\tilde{\Pi}\left(Q^2\right)$ with $Q^2 = -q^2$ in GeV^2

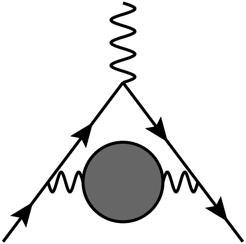
Figure and data: Aubin, Blum, Chau, Golterman, Peris, Tu, Phys. Rev. D93 (2016) 054508 [arXiv:1512.07555]

To ChPT or not to ChPT

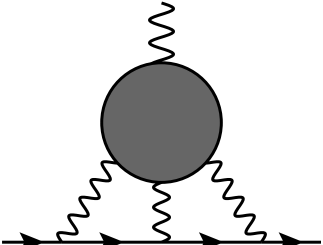
- ChPT = Effective field theory describing the lowest order pseudo-scalar representation
- or the (pseudo) Goldstone bosons from spontaneous breaking of chiral symmetry.
- Describes pions, kaons and etas at low energies
- It's an effective field theory: new parameters or LECs at each new order
- Recent review of LECs:

  JB, Ecker,Ann.Rev.Nucl.Part.Sci. 64 (2014) 149 [arXiv:1405.6488]
- a_μ is a very low-energy quantity, why not just calculate it in ChPT?

To ChPT or not to ChPT



HVP



HLbL

- Fill the blobs with pions and kaons
- Lowest order for both HVP and HLbL:
  pure pion loop (or scalar QED): well defined answer
- NLO: the blob is nicely finite
  but not after the muon/photon integrations
- Needs a counterterm (NLO LEC) that is the muon $g - 2$

To ChPT or not to ChPT

- So need more than ChPT
- Experiment
- Dispersion relations
- lattice QCD
- ChPT can be used to put constraints, help understanding results and estimate not evaluated parts,. . .

This talk

- HVP: electromagnetic corrections in scalar QED
  - JB, P. Boyle, J. Harrison, N. Hermansson Truedsson, A, Jüttner, T. Janowski, A. Portelli
  - Infinite volume
  - Finite volume in QED_L
- HVP (Chiral perturbation theory (ChPT))
  - Our old result from 2000 and chiral corrections
  - Finite volume corrections/Partially quenched

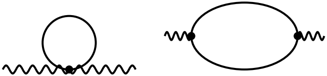
    JB, J. Relefors, "Vector two-point functions in finite volume using partially quenched chiral perturbation theory at two loops," JHEP **1712** (2017) 114 [arXiv:1710.04479 [hep-lat]].

Scalar QED

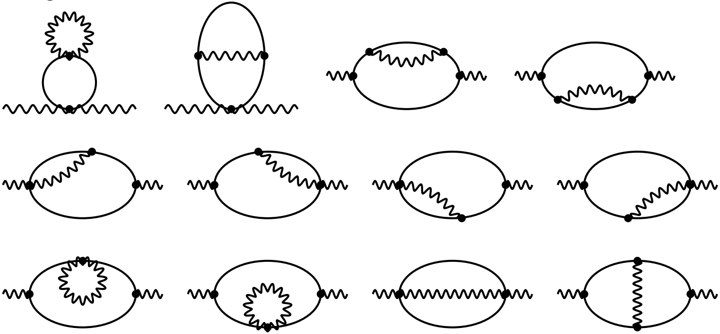
- Pion loops: finite volume effects suppressed by $e^{-m_\pi L}$ (if off-shell)
- Photon loops have suppression only by powers of $1/L$
- Dynamical photons: large finite volume effects possible
- Scalar QED in usual $\overline{MS}$ ($\mu^2_{ChPT} = \mu^2_{\overline{MS}} e$, $e = 2.71\ldots$)
- $\mathcal{L} = (\partial_\mu \Phi^* + ieA_\mu \Phi^*)(\partial_\mu \Phi - ieA_\mu \Phi) - m_0^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ ($-\lambda(\phi^*\phi)^2$ not needed)
- JB, P. Boyle, J. Harrison, N. Hermansson Truedsson, A, Jüttner, T. Janowski, A. Portelli i.e. Lund, Edinburgh, Southampton

Infinite volume

- Lowest order:



- NLO:



contributions from counterterms and 

Infinite volume

- QED known since long ago (real and imaginary part):
  G. Källén, A. Sabry, Dan. Mat. Medd 29 (1955) 1
  R. Barbieri, E. Remiddi, Nuovo Cimento 13A(1973)99
- Scalar QED can be done using the same methods but we did not find any published results
- Imaginary part follows from $e^+e^- \to \pi^+\pi^-(\gamma)$ via
  $$\sigma(s)(e^+e^- \to (\gamma) \to \text{ hadrons}) = \frac{16\pi^3\alpha}{s}\frac{1}{\pi}\mathrm{Im}\Pi(s)$$

Infinite volume

- Use Laporta algorithm to reduce to master integrals done via the package REDUZE2
- Find all the needed integrals, e.g. in
  S.P. Martin, Phys. Rev. D68 (2003) 075002 [hep-ph/0307101]
- Most complicated function needed is Li_3
- You need to go to an on-shell scheme or explicitly rewrite in physical mass. (otherwise you get a negative imaginary part to take care of threshold shifting)
- $m^2 = m_0^2 + \frac{\alpha}{4\pi} m_0^2 \left(7 - 3\log\frac{m_0^2}{\mu^2}\right)$

Results

$$\sigma^2 = 1 - \frac{4m^2}{p^2}$$

$$\Pi^{\text{1-loop}} = \frac{4}{3}A(m^2) + \frac{1}{3}\sigma^2 B(m^2,p^2) + \frac{1}{16\pi^2}\left(\frac{2}{9} - \frac{4m^2}{3p^2}\right)$$

$$\Pi^{\delta m^2} = -\delta m^2 \frac{2}{p^2}\left(\frac{1}{m^2}A(m^2) - B(m^2,p^2) - \frac{1}{16\pi^2}\right)$$

$$\Pi^{disc} = \left(\Pi^{one-loop}(p^2)\right)^2$$

$$\begin{aligned}\Pi^{\text{2-loop}} = & \frac{1}{(16\pi^2)^2}\left(\frac{10}{3} - \frac{8m^2}{p^2}\right) + \frac{A(m^2)}{16\pi^2}\frac{22}{3p^2} + \frac{10}{3m^2p^2}A(m^2)^2 \\ & + \left(\frac{8}{3m^2} - \frac{26}{3p^2}\right)A(m^2)B(m^2,p^2) + \frac{8\sigma^2}{48\pi^2}B(m^2,p^2) \\ & - \frac{8}{3}\left(\frac{1}{p^2}S(m^2,p^2) + \sigma^2 T(m^2,p^2) - m^2\sigma^2 V(m^2,p^2)\right) \\ & + \left(-\frac{4}{3} + \frac{8m^2}{3p^2}\right)B(m^2,p^2)^2 - \frac{2\sigma^2}{3}\left(p^2 - \frac{2m^2}{p^2}\right)M(m^2,p^2)\end{aligned}$$

Integrals

The integrals used are the $\overline{MS}$ subtracted parts of

$$A(m^2) = \frac{1}{i}\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2 - m^2}$$

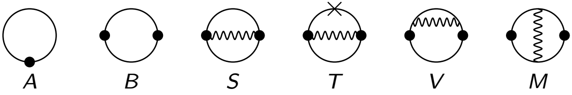
$$B(m^2, p^2)) = \frac{1}{i}\int \frac{d^d l}{(2\pi)^d}\frac{1}{(l^2 - m^2)\left((l-p)^2 - m^2\right)}$$

$$S(m^2, p^2) = \frac{1}{i^2}\int \frac{d^d l}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}\frac{1}{k^2(l^2 - m^2)\left((k+l-p)^2 - m^2\right)}$$

$$T(m^2, p^2) = \frac{1}{i^2}\int \frac{d^d l}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}\frac{1}{k^2(l^2 - m^2)^2\left((k+l-p)^2 - m^2\right)}$$

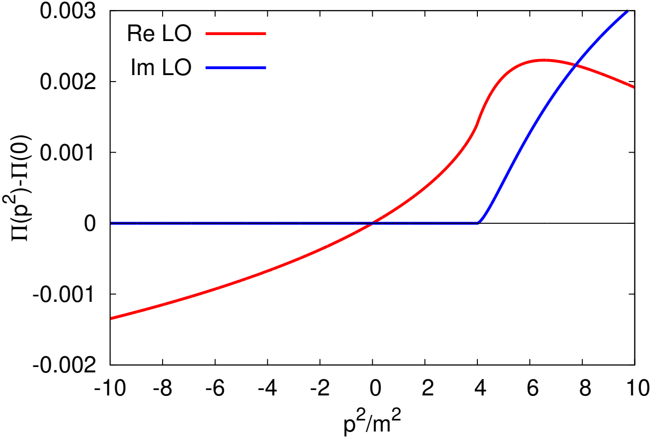
$$V(m^2, p^2) = \frac{1}{i^2}\int \frac{d^d l}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}\frac{1}{k^2(l^2 - m^2)^2\left((k+l)^2 - m^2\right)\left((l-p)^2 - m^2\right)}$$

$$M(m^2, p^2) = \frac{1}{i^2}\int \frac{d^d l}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}\frac{1}{k^2(l^2 - m^2)\left((k+l)^2 - m^2\right)\left((l-p)^2 - m^2\right)\left((k+l-p)^2 - m^2\right)}$$



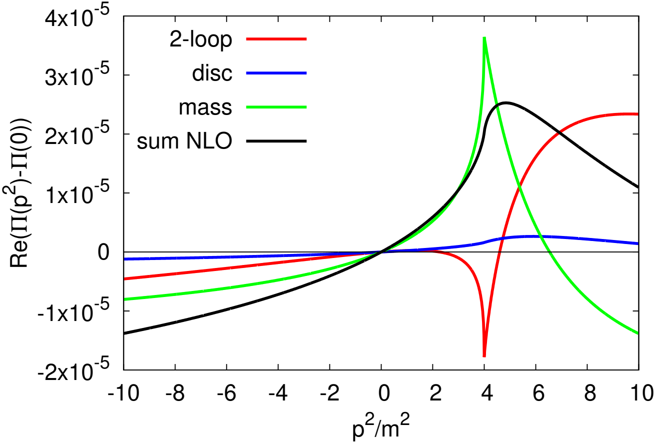
Numerical results: lowest order

$m = 139.5$ MeV, $\mu = 500$ MeV, $e = 0.303$



Numerical results: electromagnetic correction

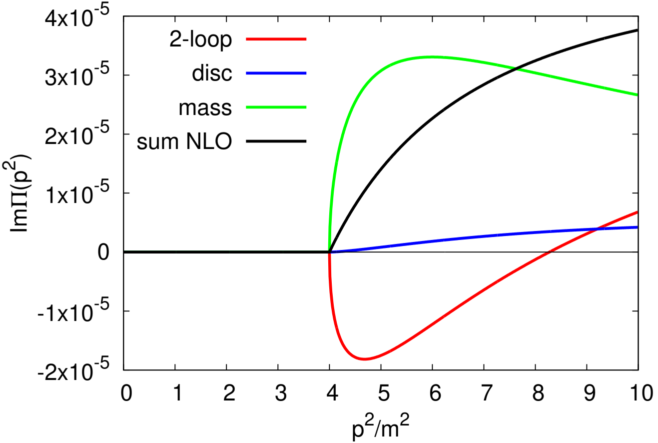
$m = 139.5$ MeV, $\mu = 500$ MeV, $e = 0.303$



About 1% of lowest order

Numerical results: electromagnetic correction

$m = 139.5$ MeV, $\mu = 500$ MeV, $e = 0.303$



Finite volume

- Finite volume for photons started in earnest with
  M. Hayakawa, S. Uno, Prog. Theor. Phys. **120**(2008)413 [0804.2044]
- Get the explicit $1/L$ behaviour from one-loop integrals:
  Z. Davoudi, M. Savage, Phys. Rev. D90(2014)054503 [1402,6471]
  S. Borsyani et al, Science 347(2015)1452 [1406.4088]
- Problem for photon: $\int d^dk \frac{1}{k^2} \to \int dk^0 \sum_{\vec{k}} \frac{1}{(k^0)^2 - \vec{k}^2}$
  Singularity no longer has the k^{d-1} to soften it
- Solution QED$_L$: drop all modes with $\vec{k} = 0$ Hayakawa-Uno
- We extend the arguments of Davoudi-Savage to two-loop order, use QED$_L$ and a lattice with infinite time extension
- We only calculate corrections suppressed by $1/L$ and powers, not exponentially suppressed contributions
- Below threshold so mesons (pions) are off-shell

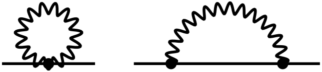
Integrals at finite volume

- $S = \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2-m^2)((k+l-p)^2-m^2))}$
- do l^0, k^0 integrals via contour integration
- $\vec{k} = \frac{2\pi}{L}\vec{n}$ and expand in $1/L$
- Write the $\vec{k}$ part as

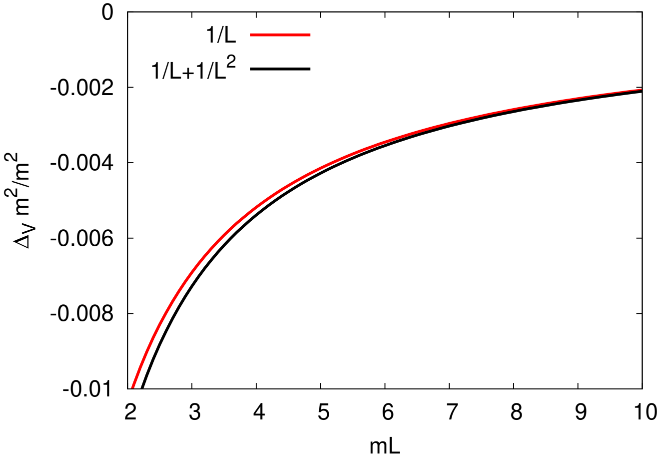
$$\frac{1}{L^{d-1}} \sum_{\vec{n}\neq\vec{0}} = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} + \left[\frac{1}{L^{d-1}} \sum_{\vec{n}\neq\vec{0}} - \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \right]$$

- In the first term resum the series in $1/L$: infinite volume contribution
- Call the quantity in brackets $\left(1/L^{d-1}\right) \widehat{\sum}_{\vec{n}}$
- Define $c_m = \widehat{\sum}_{\vec{n}} \frac{1}{|\vec{n}|^m}$
- These are known numerically

Example: mass

- 
- $\frac{\Delta_V m^2}{m^2} = e^2\left(\frac{4c_2}{16\pi^2 mL} + \frac{2c_1}{16\pi^2 m^2 L^2} + \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)\right)$
- Agrees with earlier results
- c_2 shows up since BOTH propagators can be 'on-shell'
- For $\Pi_{\mu\nu}$ below threshold only photon line goes 'on-shell'
  $\Longrightarrow$ corrections start only at $1/L^2$

Numerical results mass



Infinite volume is $\dfrac{\delta m^2}{m^2} = 0.00852$

So very large finite volume corrections to electromagnetic part

Twopoint function

- Do the k^0, l^0 integral
- expand for large L as explained earlier
- Rest can be expressed in terms of
$$Z_{ij}(m^2,p^2) = \int \frac{d^{d-1}l}{(2\pi)^{d-1}} \frac{1}{(\vec{l}^2+m^2)^{i/2}(4\vec{l}^2+4m^2-p^2)^j}$$
- these correspond to one-loop integrals with masses
- $\Omega_{ij} \equiv Z_{ij}(m^2,p^2)m^{i+2j-d+1}$
- Calculate in center of mass frame: $p = (p^0,\vec{0})$
- QED_L: no disconnected contribution
- $t_{\mu\nu}$ spatial part of $g_{\mu\nu}$
- $\widetilde{\Pi}((p^0)^2) \equiv \frac{-1}{3p^2} t_{\mu\nu}\left(\Pi^{\mu\nu}(p) - \Pi^{\mu\nu}(p=0)\right)$
- Infinite volume: $\widetilde{\Pi}(p^2) = \Pi(p^2)$

Results (preliminary)

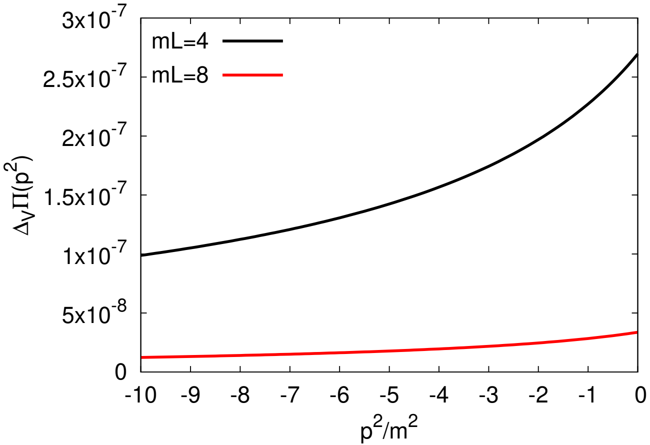
$$\widetilde{\Pi}(p^2) = \frac{c_1}{\pi m^2 L^2}\left(\frac{16}{3}\Omega_{-1,3} - \frac{1}{3}\Omega_{1,2} - \frac{32}{3}\Omega_{1,3} - \frac{2}{3}\Omega_{3,2} + \frac{16}{3}\Omega_{3,3} - \frac{1}{8}\Omega_{5,1} + \Omega_{5,2}\right)$$
$$+ \frac{c_0}{m^3 L^3}\left(- \frac{128}{3}\Omega_{-2,4} + \frac{256}{3}\Omega_{0,4} - \frac{5}{3}\Omega_{2,2} + \frac{8}{3}\Omega_{2,3} - \frac{128}{3}\Omega_{2,4} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} - \frac{8}{3}\Omega_{4,3}\right) + \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)$$

Simplify using relations of the Ω_{ij} to

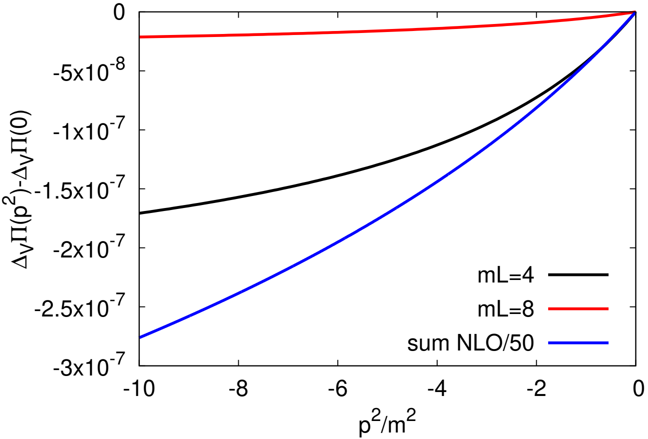
$$\widetilde{\Pi}(p^2) = + \frac{c_0}{m^3 L^3}\left(- \frac{16}{3}\Omega_{0,3} - \frac{5}{3}\Omega_{2,2} + \frac{40}{9}\Omega_{2,3} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} + \frac{8}{9}\Omega_{4,3}\right) + \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)$$

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect

Numerical results Π



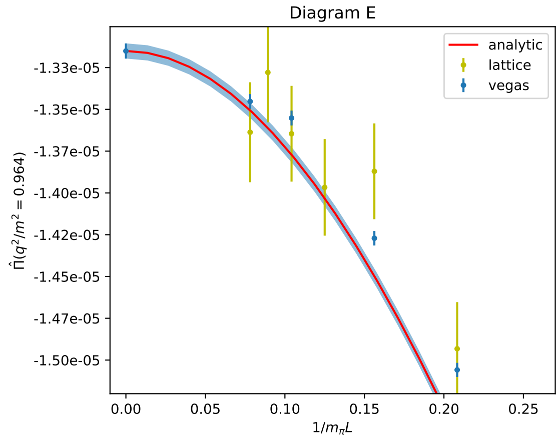
Numerical results Π

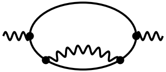


- Finite volume corrections to the electromagnetic contribution are small: below 2% for $mL = 4$
- only photon 'on-shell' and only dipole effect $\Longrightarrow 1/L^3$

Lattice

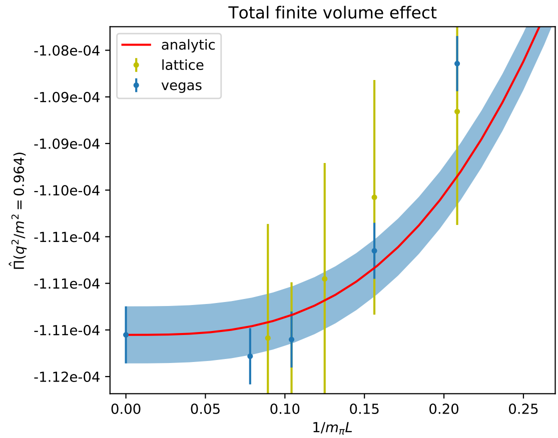
- put scalar QED on a lattice (purely bosonic so runs fast)
- the different diagrams correspond to different contractions
- show you as an example the exchange diagram
- calculate the same thing in lattice perturbation theory (vegas)
- compare L dependence with our analytical results





Note: usual lattice conventions so opposite signs of rest of talk

Lattice



Note: usual lattice conventions so opposite signs of rest of talk

Chiral corrections

Amoros, JB, Talavera, Nucl. Phys. B568(2000)319 [hep-ph/9907264]

JB, Relefors, JHEP 1611(2016) 086 [1609.01573]

Two-flavour case (and $I=1$ part only)

$$\Pi(q^2) = \Pi^{(4)}(q^2) + \Pi^{(6)}(q^2)$$
$$\Pi^{(4)}(q^2) = -8\mathcal{G}(m_\pi^2, q^2) + 16h_2^r$$
$$F_\pi^2\Pi^{(6)}(q^2) = 16q^2\mathcal{G}(m_\pi^2, q^2)^2 + 16q^2 l_6^r\mathcal{G}(m_\pi^2, q^2) - 8(2l_5^r - l_6^r)\overline{A}(m_\pi^2) - 32m_\pi^2 c_{34}^r - 8q^2 c_{56}^r$$

$\mathcal{G}(m_\pi^2, q^2)$ has a complicated m_π^2 dependence: not easy to get simple m_π^2 corrections to a_μ

Golterman, Maltman, Peris, Phys. Rev D95(2017)074509 [1701.08685]

$g-2$ workshops J-PARC, November 2016, FNAL, June 2017, lattice 2017

Chiral corrections

$\mathcal{G}(m_\pi^2, q^2)$ has a simple q^2 expansion (misprint in paper):

$$\mathcal{G}(m^2, q^2) = \frac{1}{16\pi^2}\left(\frac{1}{12} + \frac{1}{12}\log\frac{m^2}{\mu^2} - \frac{q^2}{120m^2} - \frac{q^4}{1680m^4} + \ldots\right)$$

Moments have a well defined m_π^2 expansion ($I = 1$ case)

$$\Pi(q^2) = \Pi(0) + \Pi_1 q^2 - \Pi_2 q^4 \ldots$$

$$\Pi_1 = \frac{1}{16\pi^2}\frac{1}{15m_\pi^2} + \frac{1}{F_\pi^2}\left[-8c_{56}^r + \frac{l_6^r}{16\pi^2}\frac{4}{3}\left(1 + \log\frac{m_\pi^2}{\mu^2}\right) + \frac{1}{(16\pi^2)^2}\left(\frac{49}{405} + \frac{2}{9}\log\frac{m_\pi^2}{\mu^2} + \frac{1}{9}\log^2\frac{m_\pi^2}{\mu^2}\right)\right]$$

$$\Pi_2 = \frac{1}{16\pi^2}\frac{-1}{210m_\pi^4} + \frac{1}{16\pi^2 F_\pi^2 m_\pi^2}\left[\frac{2}{15}l_6^r + \frac{1}{16\pi^2}\left(\frac{23}{945} + \frac{1}{45}\log\frac{m_\pi^2}{\mu^2}\right)\right]$$

Other cases: plug $\mathcal{G}$ expansion in formulas of section 4 of

JB, Relefors, JHEP 1611(2016) 086 [1609.01573]

Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i/L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta_q^i} q(x^i)$
- Then momenta are $p^i = \theta^i/L + 2\pi n^i/L$. Allows to map out momentum space on the lattice much better

  Bedaque,...
- Small note:
  - Beware what people call momentum: is θ^i/L included or not?
  - Reason: a colour singlet gauge transformation $G_\mu^S \to G_\mu^S - \partial_\mu \epsilon(x)$, $q(x) \to e^{i\epsilon(x)} q(x)$, $\epsilon(x) = -\theta_q^i x^i/L$
  - Boundary condition Twisted $\Leftrightarrow$ constant background field+periodic

Twisted boundary conditions: Drawbacks

Drawbacks:

- Box: Rotation invariance $\rightarrow$ cubic invariance
- Twisting: reduces symmetry further
- Can only be done for the connected part

Consequences:

- $m^2(\vec{p}^2) = E^2 - \vec{p}^2$ is not constant
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum

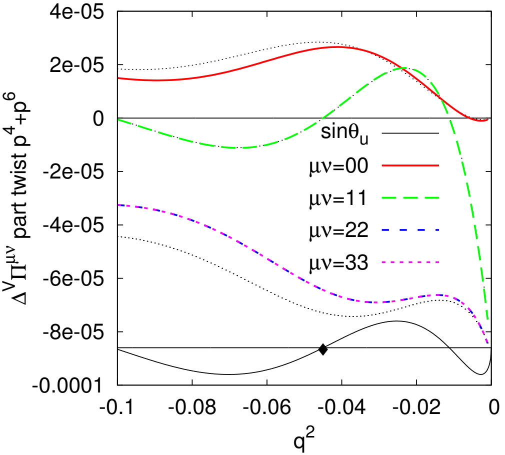
Two-point function: twisted boundary conditions

JB, Relefors, JHEP 05 (2014) 015 [arXiv:1402.1385]

JB, Relefors, JHEP 1712(2017)114 [1710.04479]

- $\int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0$
- $\langle \bar{u}\gamma^\mu u \rangle \neq 0$
- $j^\mu_{\pi^+} = \bar{d}\gamma^\mu u$
  satisfies $\partial_\mu \left\langle T(j^\mu_{\pi^+}(x) j^{\nu\dagger}_{\pi^+}(0)) \right\rangle = \delta^{(4)}(x) \left\langle \bar{d}\gamma^\nu d - \bar{u}\gamma^\nu u \right\rangle$
- $\Pi^{\mu\nu}_a(q) \equiv i \int d^4 x e^{iq\cdot x} \left\langle T(j^\mu_a(x) j^{\nu\dagger}_a(0)) \right\rangle$
  Satisfies WT identity. $q_\mu \Pi^{\mu\nu}_{\pi^+} = \left\langle \bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d \right\rangle$
- ChPT at one-loop satisfies this
  see also Aubin et al, Phys.Rev. D88 (2013) 7, 074505 [arXiv:1307.4701]
- two-loop in partially quenched satisfies the WT identity (as it should)

Two-point: partially twisted, with two-loop



$q = \left(0, \sqrt{-q^2}, 0, 0\right)$

$\Pi^{22} = \Pi^{33}$

$\vec{\theta}_u = L\, q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

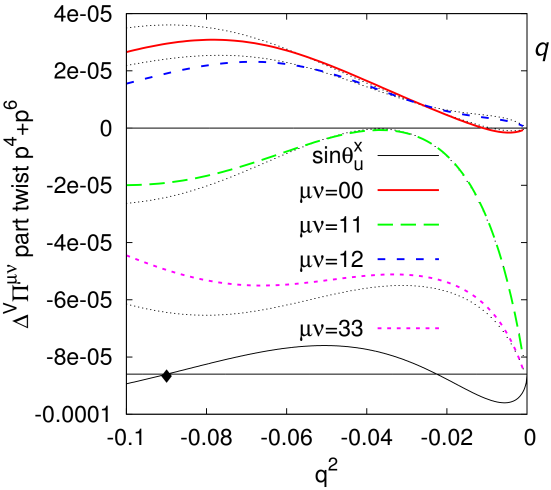
$$-q^2 \Pi^{(1)}_{\rm VMD} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2}$$
$$\approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction is at the % level

Correction from two loop is reasonable (thin lines are p^4)



$$q = \left(0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0\right)$$

$\Pi^{11} = \Pi^{22}$

$\vec{\theta}_u = L\, q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction is at the % level

Two loop correction again reasonable (thin lines are p^4)

Conclusions

Showed you results for:

- p^8 ChPT: Lagrangian
- p^8 ChPT: mass and decay constant for $N_f = 2$
- p^6 ChPT: can we fit lattice in a better way
- Finite volume corrections to the electromagnetic contribution as estimated in scalar QED are small
- Connected: twisting and finite volume at two-loops
  The corrections are sizable for present lattices (few%) but two-loop corrections were normal size