



# SHORT-DISTANCE CONTRIBUTION TO HLbL

Johan Bijmens  
Lund University



Vetenskapsrådet

[bijmens@thep.lu.se](mailto:bijmens@thep.lu.se)  
<http://thep.lu.se/~bijmens>

# Why do we do this?

The muon  $a_\mu = \frac{g_\mu - 2}{2}$  will be measured more precisely

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Introduction

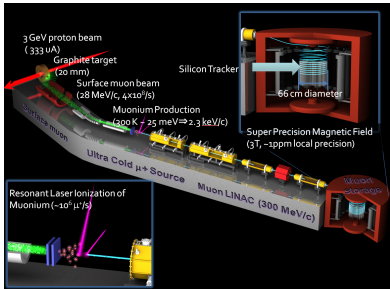
General  
properties

Quarkloop

Short-  
distance:  
naive

Short-  
distance:  
correct

Numerical  
results



J-PARC



Fermilab



# Why do we do this?

- Experiment dominated by BNL, FNAL error down by four
- Theory taken from PDG2018
- $a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{Had}}$
- $a_{\mu}^{\text{Had}} = a_{\mu}^{\text{LO-HVP}} + a_{\mu}^{\text{HO-HVP}} + a_{\mu}^{\text{HLbL}}$
- Impressive agreement with  $g_{\mu}$  to  $2 \times 10^{-9}$

Part	value	errors	units
$a_{\mu}^{\text{EXP}}$ :	116 592 091.x	(54)(33)	$\times 10^{-11}$
$a_{\mu}^{\text{SM}}$ :	116 591 823.x	(1)(34)(26)	$\times 10^{-11}$
$\Delta a_{\mu}$ :	268.x	(63)(43)	$\times 10^{-11}$
$a_{\mu}^{\text{QED}}$ :	116 584 719.0	(0.1)	$\times 10^{-11}$
$a_{\mu}^{\text{EW}}$ :	153.6	(1.0)	$\times 10^{-11}$
$a_{\mu}^{\text{LO-HVP}}$ :	6 931.x	(33)(7)	$\times 10^{-11}$
$a_{\mu}^{\text{HO-HVP}}$ :	-86.3	(0.9)	$\times 10^{-11}$
$a_{\mu}^{\text{HLbL}}$ :	105.x	(26)	$\times 10^{-11}$

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# Hadronic contributions



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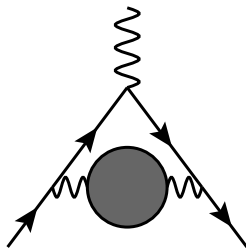
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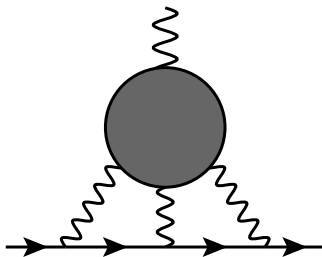
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LO-HVP



HLbL

- The blobs are hadronic contributions
- I will present some results for HLbL short-distance

JB, N. Hermansson-Truedsson, A. Rodriguez-Sanchez, arxiv:1908.03331

# HLbL: the main object to calculate



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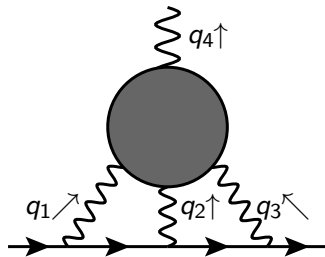
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- Muon line and photons: well known
- The blob: **fill in with hadrons/QCD**
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks

# General properties



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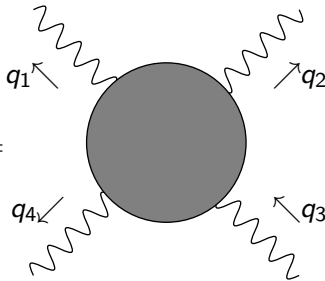
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$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) =$$



Actually we really need  $\left. \frac{\delta \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}} \right|_{q_4=0}$



# General properties

$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$ :

- In general 138 Lorentz structures (136 in 4 dimensions)
- Using  $q_{1\mu}\Pi^{\mu\nu\lambda\sigma} = q_{2\nu}\Pi^{\mu\nu\lambda\sigma} = q_{3\lambda}\Pi^{\mu\nu\lambda\sigma} = q_{4\sigma}\Pi^{\mu\nu\lambda\sigma} = 0$   
43 (41) gauge invariant structures
- 41 helicity amplitudes
- Bose symmetry relates some of them
- Compare HVP: one function, one variable
- General calculation from experiment via dispersion relations: recent progress  
Colangelo, Hoferichter, Kubis, Procura, Stoffer, ...
- Well defined separation between different contributions
- Theory initiative: paper under preparation
- **One remaining problem: intermediate- and short-distances**

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- Formalism of Colangelo et al., JHEP 1704 (2017) 161 [1702.07347]

- $\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1,54} \hat{T}^{\mu\nu\lambda\sigma} \hat{\Pi}_i$

- $Q_i^2 = -q_i^2$

- $Q_3^2 = Q_1^2 + Q_2^2 + 2\tau Q_1 Q_3$

- $a_\mu = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1,12} \hat{T}_i \bar{\Pi}_i$

- The 12  $\bar{\Pi}_i$  are related to 6  $\hat{\Pi}_i$  with  $q_4 \rightarrow 0$ .

$$\bar{\Pi}_1 = \hat{\Pi}_1, \quad \bar{\Pi}_2 = C_{23} [\hat{\Pi}_1], \quad \bar{\Pi}_3 = \hat{\Pi}_4, \quad \bar{\Pi}_4 = C_{23} [\hat{\Pi}_4],$$

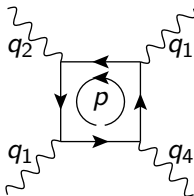
$$\bar{\Pi}_5 = \hat{\Pi}_7, \quad \bar{\Pi}_6 = C_{12} [C_{13} [\hat{\Pi}_7]], \quad \bar{\Pi}_7 = C_{23} [\hat{\Pi}_7],$$

$$\bar{\Pi}_8 = C_{13} [\hat{\Pi}_{17}], \quad \bar{\Pi}_9 = \hat{\Pi}_{17}, \quad \bar{\Pi}_{10} = \hat{\Pi}_{39},$$

$$\bar{\Pi}_{11} = -C_{23} [\hat{\Pi}_{54}], \quad \bar{\Pi}_{12} = \hat{\Pi}_{54},$$



- Use (constituent) quark loop
- Used for full estimates since the beginning (1970s)
- Used for short-distance estimates with mass as a cut-off  
JB, Pallante, Prades, 1996



- We recalculated:
- In agreement with quarkloop formulae from  
Hoferichter, Stoffer, private communication
- In agreement with known numerics

# Charm loop



- Kühn et al., hep-ph/0301151, Phys.Rev. D68 (2003) 033018
- $a_\mu = \left(\frac{\alpha}{\pi}\right)^3 N_c e_q^4 \left[ \frac{m_\mu^2}{M^2} \left( \frac{3}{3} \zeta_3 - \frac{19}{16} \right) + \dots \right]$
- Up to  $m_\mu^{10}/M^{10}$  in paper
- $m_c = 1.27$  GeV
- $a_\mu^{\text{HLbLc}} = (3.165 - 0.0786 - 0.00033 + \dots) \times 10^{-11}$   
 $= 3.1(1) \times 10^{-11}$
- $m_b = 4.18$  GeV
- $a_\mu^{\text{HLbLb}} = 1.8 \times 10^{-13}$

# Quarkloop: $u, d, s$

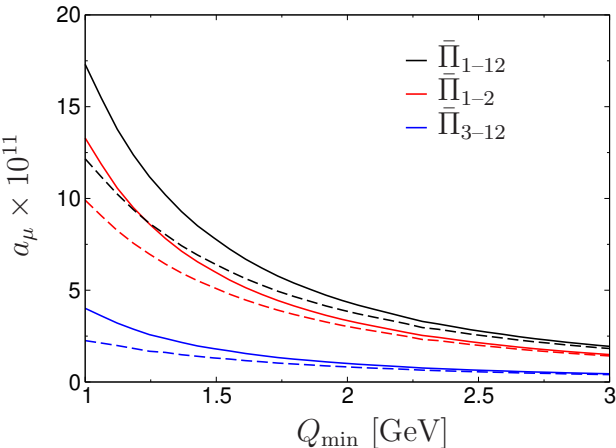


figure: Hoferichter

$$Q_1, Q_2, Q_3 > Q_{\min}$$

$M_Q = 0$ : full  
 $M_Q = 0.3$  GeV  
 dashed

$$a_{\mu}^{\text{HLbLQ}} = 54 \times 10^{-11}$$

- $M_Q$  provides an infrared cut-off,  $M_Q \rightarrow 0$  divergent
- About  $12 \times 10^{-11}$  from above 1 GeV for  $M_Q = 0.3$  GeV
- About  $17 \times 10^{-11}$  from above 1 GeV for  $M_Q = 0$

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- Is it a first term in a systematic OPE?
- OPE has been used as constraints on specific contributions
  - $\pi^0 \gamma^* \gamma^*$  asymptotic behaviour
  - Constraints on many other hadronic formfactors
  - $q_1^2 \approx q_2^2 \gg q_3^2$  Melnikov, Vainshtein 2003
  - These are discussed in the next two talks

# Short-distance: first attempt



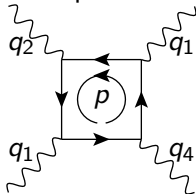
$$\Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle T (j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)) \rangle$$

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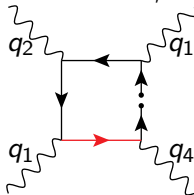
- Usual OPE:  $x, y, z$  all small
- First term in the expansion is the quark-loop  
no problem with  $\partial/\partial q_4^\rho$  and  $q_4 \rightarrow 0$

$p$  in loop  $\Rightarrow$  no singular propagators:



- Next term problems: no loop momentum;

$q_4 \rightarrow 0$  propagator diverges:



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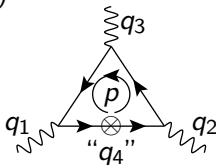
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## Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, 1984
- For the  $q_4$ -leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge:  $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$   
whole calculation is immediately with  $q_4 = 0$ .
- First term is exactly the usual quark loop (even including quark masses)



# Quarkloop results: massless case



$$\hat{\Pi}_i^{\text{ql}} = \sum_q \frac{N_c e_q^4}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy I_i(x, y)$$

$$D = x(1-x-y)Q_1^2 + y(1-x-y)Q_2^2 + xyQ_3^2$$

$$I_1 = \frac{16}{D^2} \left( y^2 - 3y^3 + 2y^4 - 4xy^2 + 4xy^3 - x^2y + 2x^2y^2 \right. \\ \left. - \frac{Q_1^2}{Q_3^2} (y+x)(1-x-y)^2 (-1+2x+2y) \right)$$

$$I_7 = \frac{64}{D^3} xy(-1+2x)(x+y)(1-x-y)^2$$

$$I_{39} = \frac{64}{D^3} xy(1-x-y)(y-y^2+x-3xy+2xy^2-x^2+2x^2y)$$

$$I_{54} = \frac{16}{D^2} \left( \frac{1}{Q_1^2} y^2(1-y)(1-2y) - \frac{1}{Q_2^2} x^2(1-x)(1-2x) \right)$$

$I_4$  and  $I_{17}$  similar but slightly longer expressions

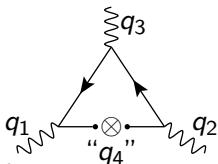


## Short-distance: next term(s)

- Do the usual QCD sum rule expansion in terms of vacuum condensates
- There are new condensates, induced by the constant magnetic field:  $\langle \bar{q}\sigma_{\alpha\beta}q \rangle \equiv e_q F_{\alpha\beta} X_q$
- Lattice QCD [Bali et al., arXiv:1206.4205](#)  
 $X_u = 40.7 \pm 1.3 \text{ MeV}$ ,  
 $X_d = 39.4 \pm 1.4 \text{ MeV}$ ,  
 $X_s = 53.0 \pm 7.2 \text{ MeV}$
- Could have started at order  $1/Q$ , only starts at  $1/Q^2$  via  $m_q X_q$  corrections to the leading quark-loop result
- $X_q$  and  $m_q$  are very small, only a very small correction
- Next order: very many condensates contribute, work in progress



- Result derived from:



- $N_c = 3$  and one quark

$$\hat{\Pi}_1 = m_q X_q e_q^4 \frac{-4(Q_1^2 + Q_2^2 - Q_3^2)}{Q_1^2 Q_2^2 Q_3^4} \quad \hat{\Pi}_7 = 0$$

$$\hat{\Pi}_4 = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^2} \quad \hat{\Pi}_{17} = m_q X_q e_q^4 \frac{8}{Q_1^2 Q_2^2 Q_3^4}$$

$$\hat{\Pi}_{54} = m_q X_q e_q^4 \frac{-4(Q_1^2 - Q_2^2)}{Q_1^4 Q_2^4 Q_3^2} \quad \hat{\Pi}_{39} = 0$$



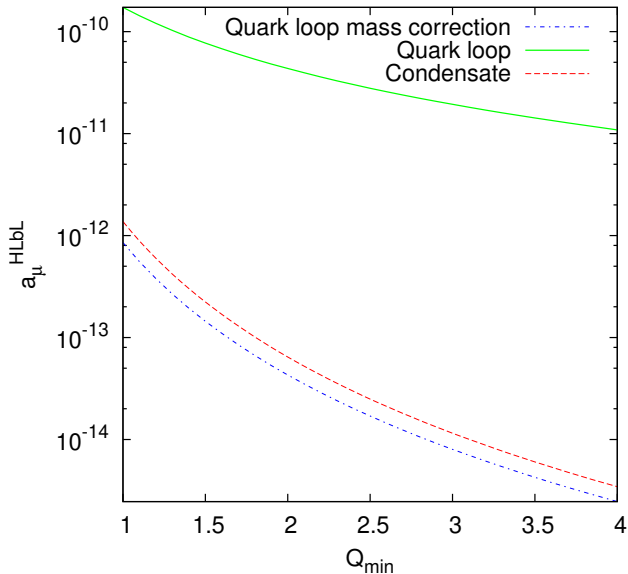
# Short-distance: numerical results

- preliminary
- $Q_1, Q_2, Q_3 \geq Q_{\min}$
- $m_u = m_d = m_s = 0$  for quark-loop
- $m_u = m_d = 5$  MeV and  $m_s = 100$  MeV for  $m_q X_q$

$Q_{\min}$	quarkloop	$m_u X_u + m_d X_d$	$m_s X_s$
1 GeV	$17.3 \times 10^{-11}$	$5.40 \times 10^{-13}$	$8.29 \times 10^{-13}$
2 GeV	$4.35 \times 10^{-11}$	$3.40 \times 10^{-14}$	$5.22 \times 10^{-14}$

- Above 1 GeV still 15% of total value of HLbL
- Quarkloop goes roughly as  $1/Q_{\min}^2$
- $m_q X_q$  goes roughly as  $1/Q_{\min}^4$
- Naive suppression is  $m_q X_q / Q_{\min}^2 \sim 2 \times 10^{-3}$
- Observed is roughly that

# Numerical results



Short-distance:  $1/Q_{\min}^2$ 

- Can we understand scaling with  $Q_{\min}$ ?

$$\bullet a_{\mu} = \frac{2\alpha^3}{3\pi^2} \int_0^{\infty} dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1,12} \hat{T}_i \bar{\Pi}_i$$

- Do  $Q_i \rightarrow \lambda Q_i$

- overall factor goes as  $\lambda^8$

- Quark loop has no scale thus  $\hat{\Pi}_i$  scale with their dimension  
 $\hat{\Pi}_1, \hat{\Pi}_4 \sim \lambda^{-4}$ ,  $\hat{\Pi}_7, \hat{\Pi}_{17}, \hat{\Pi}_{39}, \hat{\Pi}_{54} \sim \lambda^{-6}$

$$\bullet \implies \bar{\Pi}_{1,\dots,4} \sim \lambda^{-4} \quad \bar{\Pi}_{5,\dots,12} \sim \lambda^{-6}$$

- Expand the  $T_i$  for  $Q_i \gg m_{\mu}$ :  $T_1 \sim m_{\mu}^4$ ,  $T_{i \neq 1} \sim m_{\mu}^2$   
 $T_1 \sim \lambda^{-8}$ ,  $T_{2,3,4} \sim \lambda^{-6}$ ,  $T_{5,\dots,12} \sim \lambda^{-4}$

- Put all together: quark-loop scales as  $a_{\mu}^{\text{SD ql}} \sim \lambda^{-2}$

- $m_q X_q$  adds an overall factor  $\implies a_{\mu}^{\text{SDX}_q} \sim \lambda^{-4}$

# Short-distance: Conclusions



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- We have shown that the quarkloop really is the first term of a proper OPE expansion for the HLbL
- We have calculated the next term which is suppressed by quark masses and a small  $X_q$ : negligible
- The next term contains both the usual vacuum and a large number of induced condensates but will not be suppressed by small quark masses
- Why do this: matching of the sum over hadronic contributions to the expected short distance domain
- Finding the onset of the asymptotic domain