THE CHIRAL LAGRANGIAN AT ORDER p^8

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Effective Theories of Quantum Phases of Matter – Nordita, Stockholm – 23 May 2019

Overview

- Karol told you on Tuesday about freedom in the amplitudes starting from amplitude methods
- I will do it from the Lagrangian perspective
- JB, Nils Hermansson-Truedsson, Si Wang, JHEP01(2019)102 [1810.06834]
- 2,13,115,1862,...
- 1,4,19,132,...

Chiral Perturbation Theory

Exploring the consequences of
the chiral symmetry of QCD
and its spontaneous breaking
using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
On The Foundations Of Chiral Perturbation Theory,
Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

For references to lectures see:
http://www.thep.lu.se/∼bijnens/chpt.html

Chiral Perturbation Theory

A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about M_ρ.

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Chiral Symmetry

Chiral Symmetry

QCD: N_f light quarks: equal mass: interchange: $SU(N_f)_V$

But $$\mathcal{L}_{QCD} = \sum_{q=u,d,\dots} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

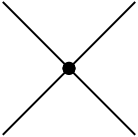
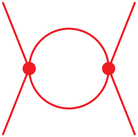
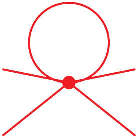
So if $m_q = 0$ then $SU(N_f)_L \times SU(N_f)_R$.

Spontaneous breakdown

- $\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(N_f)_L \times SU(N_f)_R$ broken spontaneously to $SU(N_f)_V$
- $N_f(N_f - 1)$ generators broken $\Longrightarrow$ $N_f(N_f - 1)$ massless degrees of freedom and interaction vanishes at zero momentum

Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg powercounting

rules		one loop example	
	p^2		$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$
	$1/p^2$		$(p^2)\,(1/p^2)\,p^4 = p^4$
$\int d^4p$	p^4		

Chiral Perturbation Theories

- Which chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$, for $N_f = 2, 3, \ldots$ and extensions to (partially) quenched
- Or beyond QCD
- Space-time symmetry: Continuum or broken on the lattice: Wilson, staggered, mixed action
- Volume: Infinite, finite in space, finite T
- Which interactions to include beyond the strong one
- Which external fields to include
- Which particles included as non Goldstone Bosons
- My general belief: if it involves soft pions (or soft K, η) some version of ChPT exists

External field method

- Problem: Ward identities for fields that transform nonlinearly
- Solution: Gasser, Leutwyler 84,85: use external field method and generate Green functions of QCD currents/densities from those
- with $q^T = (u\ d\ s \cdots)$:

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \overline{q} i \gamma^\mu \left(D_\mu - i v_\mu - i a_\mu \gamma_5 \right) q - \overline{q} s q + \overline{q} i \gamma_5 p q$$

- v_μ, a_μ, s, p are $N_f \times N_f$ matrices: the external fields
- Chiral symmetry made local $g_L, g_R \in SU(N_f)_L \times SU(N_f)_R$

$$q_{L,R} \longrightarrow g_{L,R} q_{L,R}$$

$$s + ip \longrightarrow g_R (s + ip) g_L^\dagger$$

$$\ell_\mu \equiv v_\mu - a_\mu \longrightarrow g_L \ell_\mu g_L^\dagger - i \partial_\mu g_L g_L^\dagger$$

$$r_\mu \equiv v_\mu + a_\mu \longrightarrow g_R r_\mu g_R^\dagger - i \partial_\mu g_R g_R^\dagger$$

External field method

- Define Green functions of QCD currents by functional derivatives w.r.t. the external fields of

  $Z_{QCD}(v_\mu, a_\mu, s, p) = \int [dq d\overline{q} dG] \exp\left(i \int d^4x \mathcal{L}_{QCD}\right)$
- Put in photons in v_μ, quark masses in s,... by comparing with the Lagrangian withose parts included
- If dealing with other operators: add more external fields (spurions)
- Now write theory with the Goldstone bosons ϕ^a:

  $Z_{ChPT}(v_\mu, a_\mu, s, p) = \int [d\phi^a] \exp\left(i \int d^4x \mathcal{L}_{ChPT}\right)$
- $\mathcal{L}_{ChPT}$ has the same (chiral) symmetries as $\mathcal{L}_{QCD}$
- Finally (proof follows from all singularities at low energies included this way, the remainder can be Taylor expanded)

  $Z_{QCD}(v_\mu, a_\mu, s, p) \approx Z_{ChPT}(v_\mu, a_\mu, s, p)$

Parametrizing G/H

- Goldstone Boson manifold for $G \longrightarrow H$ is G/H.
- G generators: X^a, T^a: $X^a|0\rangle \neq 0$, $T^a|0\rangle = 0$
- Goldstone boson states: $g \in G$ $g|0>$ but many equivalent
- (Callan), Coleman, Wess, Zumino 1969
- Coset: g_1, g_2 equivalent if $g_1 = g_2 h$ $h \in H$
- Parametrize G/H by $\tilde{u} = \exp\left(i\phi^a X^a/(\sqrt{2}F)\right)$
- Transformation: $u \longrightarrow g\tilde{u}$
- $g\tilde{u}$ not of the form $\tilde{u}'$
- Compensating $\tilde{h}(\tilde{u}, g)$ exists with
  $\tilde{u} \longrightarrow \tilde{u}' = \exp\left(i\phi^{a\prime} X^a/(\sqrt{2}F)\right) = g\tilde{u}\tilde{h}^\dagger(\tilde{u}, g)$

Parametrizing G/H

- External fields V_μ (a G matrix) $V_\mu \rightarrow g V_\mu g^\dagger - i\partial_\mu g g^\dagger$
- Define:
  $\tilde{u}^\dagger \left(\partial - iV_\mu\right) \tilde{u} = -(i/2)\tilde{u}^a_\mu X^a + \tilde{\Gamma}^a_\mu T^a = -(i/2)\tilde{u}_\mu + \tilde{\Gamma}_\mu$
  $\tilde{u}_\mu \longrightarrow \tilde{h}\tilde{u}_\mu\tilde{h}^\dagger, \quad \tilde{\Gamma}_\mu \longrightarrow \tilde{h}\tilde{\Gamma}_\mu\tilde{h}^\dagger - \partial_\mu\tilde{h}\tilde{h}^\dagger$
- Covariant derivatives if $\psi \longrightarrow h\psi$
  $\nabla_\mu\psi \equiv \left(\partial_\mu + \tilde{\Gamma}_\mu\right)\psi \longrightarrow h\nabla_\mu\psi$
- If you know G transformation, e.g. $F \longrightarrow gF$ construct a H transforming object via $\tilde{F} \equiv \tilde{u}^\dagger F \longrightarrow h\tilde{F}$
- Use $\tilde{u}_\mu, \nabla_\mu, V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i\left(V_\mu V_\nu - V_\nu V_\mu\right), \tilde{F}$ to construct Lagrangians: i.e. the building blocks

Building blocks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

- $g = \begin{pmatrix} g_R & \\ & g_L \end{pmatrix}, \quad \tilde{h} = \begin{pmatrix} h & \\ & h \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u & \\ & u^\dagger \end{pmatrix}$

  $V_\mu = \begin{pmatrix} r_\mu & \\ & \ell_\mu \end{pmatrix}, \quad \tilde{u}_\mu = \begin{pmatrix} u_\mu & \\ & -u_\mu \end{pmatrix}, \quad \tilde{\Gamma}_\mu = \begin{pmatrix} \Gamma_\mu & \\ & \Gamma_\mu \end{pmatrix}$
- Or in terms of the earlier QCD notation:

$$u = \exp i\phi/(\sqrt{2}F) \longrightarrow g_R u h^\dagger = h u g_L^\dagger$$

$$u_\mu = i\left(u^\dagger(\partial - i r_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\right) \longrightarrow h u_\mu h^\dagger$$

$$\Gamma_\mu = \frac{1}{2}\left(u^\dagger(\partial - i r_\mu)u + u(\partial_\mu - i\ell_\mu)u\dagger\right) \longrightarrow h\Gamma_\mu h^\dagger - \partial_\mu h h^\dagger$$

Building blocks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

- $\nabla_\mu \Psi = \partial_\mu \Psi + \Gamma_\mu \Psi$ for $\Psi \to h\Psi$
- $\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]$ for $X \to hXh^\dagger$
- $\chi \equiv 2B_0(s + ip) \longrightarrow g_R \chi g_L^\dagger$
- $F_{L\mu\nu} = \partial_\mu \ell_\nu - \partial_\nu \ell_\mu - i\,[\ell_\mu, \ell_\nu] \longrightarrow g_L F_{L\mu\nu} g_L^\dagger$
- $F_{R\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i\,[r_\mu, r_\nu] \longrightarrow g_R F_{R\mu\nu} g_R^\dagger$
- $\chi_\pm \equiv u^\dagger \chi u^\dagger \pm u \chi^\dagger u$
- $f_{\pm\mu\nu} = u F_{L\mu\nu} u^\dagger \pm u^\dagger F_{R\mu\nu} u$
- Final building blocks all go as $X \longrightarrow hXh^\dagger$:
  Order p^1: u_μ, ∇_μ; order p^2 $\chi_\pm, f_{\pm\mu\nu}$
- $\langle u_\mu \rangle = \langle f_{\pm\mu\nu} \rangle = 0$
- Other choices, purely left-handed,... are possible

Building blocks

- Transformations under discrete symmetries

	P	C	h.c.
u_μ	$-\varepsilon(\mu)u_\mu$	u_μ^T	u_μ
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$
$f_{\pm\mu\nu}$	$\pm\varepsilon(\mu)\varepsilon(\nu)f_{\pm\mu\nu}$	$\mp f_{\pm\mu\nu}^T$	$f_{\pm\mu\nu}$

$\varepsilon(0) = -\varepsilon(i = 1, 2, 3) = 1.$

- Is this more general?
  - Clearly works for $G \times G \to G$
  - Valid also for $SU(2N)/SO(2N)$ and $SU(2N)/Sp(2N)$
  - Lagrangian clearly complete for those cases but might not be minimal due to additional matrix relations
  - We used $\chi_\pm$ and $f_{\pm\mu\nu}$: do not always exist in this form
  - $\langle X^a \rangle = 0$ and N_f is the size of the matrices upper limit to number of terms

Lagrangians: Lowest order

- $N_f = 2$ and add for $N_f = 3$

$$\phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix} .$$

- p^0: no building block exists
- LO or p^2: $\langle u_\mu u^\mu \rangle, \langle \nabla^\mu u_\mu \rangle, \langle \chi_+ \rangle, \langle \chi_- \rangle$,
  use P and $\langle u_\mu \rangle = 0$
  $\mathcal{L}_2 = \frac{F_0^2}{4} \left[\langle u_\mu u^\mu \rangle + \langle \chi_+ \rangle \right]$
- Usually in terms of $U = u^2 \longrightarrow g_R U g_L^\dagger$ and
  $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$,
- $\mathcal{L}_2 = \frac{F_0^2}{4} \left[\langle D_\mu U D^\mu U^\dagger \rangle + \langle \chi U^\dagger + U \chi^\dagger \rangle \right]$

Lagrangians: Lagrangian structure

	2 flavour		3 flavour		PQChPT/N_f flavour	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

- L_i LEC = Low Energy Constants = ChPT parameters
- H_i: contact terms: value depends on definition of currents/densities
- Finite volume: no new LECs
- Other effects: (many) new LECs
- Many extensions classified: $\varepsilon_{\mu\nu\alpha\beta}$, weak decays,. . .

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms
- includes Isospin and the eightfold way $(SU(3)_V)$
- Unitarity included perturbatively

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2} \log \frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$$M^2 = 2B\hat{m}$$

P, C Hermitian conjugate (H)

- X_i: a building block
- P: enforce by having an even number of parity-odd blocks (we assume no $\epsilon_{\mu\nu\alpha\beta}$)
- C and H relate the same building blocks

$$C\left(\langle X_1 \dots X_n\rangle\right) = \pm\langle X_1^T \dots X_n^T\rangle = \pm\langle X_n \dots X_1\rangle\,,$$
$$\left(\langle X_1 \dots X_n\rangle\right)^\dagger = \langle X_n^\dagger \dots X_1^\dagger\rangle = \pm\langle X_n \dots X_1\rangle\,,$$

- $\mathcal{O}_i \longrightarrow \lambda_\pm^C \, \lambda_\pm^{\mathrm{h.c.}} \, \mathcal{O}_j$ look at $(\pm,\pm)$
- $i = j$

$$(+,+): \quad \mathcal{O}_i = \mathcal{O}_i^+ \qquad (-,+): \quad \mathcal{O}_i = \mathcal{O}_i^-\,,$$
$$(+,-): \quad \mathcal{O}_i = i\mathcal{O}_i^+ \qquad (-,-): \quad \mathcal{O}_i = i\mathcal{O}_i^-\,.$$

P, C Hermitian conjugate (H)

- $j \neq i$

$$
\begin{aligned}
(+,+):\quad \mathcal{O}_i &= \frac{\mathcal{O}_i^+ + i\mathcal{O}_i^-}{2}, \quad \mathcal{O}_j = \frac{\mathcal{O}_i^+ - i\mathcal{O}_i^-}{2},\\
(-,+):\quad \mathcal{O}_i &= \frac{i\mathcal{O}_i^+ + \mathcal{O}_i^-}{2}, \quad \mathcal{O}_j = \frac{-i\mathcal{O}_i^+ + \mathcal{O}_i^-}{2},\\
(+,-):\quad \mathcal{O}_i &= \frac{i\mathcal{O}_i^+ + \mathcal{O}_i^-}{2}, \quad \mathcal{O}_j = \frac{i\mathcal{O}_i^+ - \mathcal{O}_i^-}{2},\\
(-,-):\quad \mathcal{O}_i &= \frac{\mathcal{O}_i^+ + i\mathcal{O}_i^-}{2}, \quad \mathcal{O}_j = \frac{-\mathcal{O}_i^+ + i\mathcal{O}_i^-}{2}.
\end{aligned}
$$

- The final Lagrangian should only contain the monomials $\mathcal{O}_i^+$.

Constraints

- p^2, p^4, p^6 were done essentially by hand
- p^8 way too many terms for that
- use FORM
- Cyclicity: use cyclic functions
- Check all ways of relabelling indices explicitly
  use Python to rewrite FORM output back into FORM commands
- Easier to work with single term operators for all relations: rewriting in $\mathcal{O}_i^+$ done at the end

Contact terms

- Building blocks: $\chi, \chi^\dagger, F_{L\mu\nu}, F_{R\mu\nu}$
- all under same simple group lost
- Covariant derivatives:

$$
\begin{aligned}
D_\mu \chi &= \partial_\mu \chi - i r_\mu \chi + i \chi \ell_\mu \\
D_\rho F_{L\mu\nu} &= \partial_\rho F_{L\mu\nu} - i \ell_\rho \chi + i F_{L\mu\nu} \ell_\rho \\
D_\rho F_{R\mu\nu} &= \partial_\rho F_{R\mu\nu} - i r_\rho \chi + i F_{R\mu\nu} r_\rho
\end{aligned}
$$

- P, C, H more tricky as well
- If finite N_f: (Kaplan-Manohar)
  new operator of order p^{2N_f-2}, not singular for $\chi \to 0$
  $\tilde\chi \equiv \left(\det(\chi)\chi^{-1}\right)^\dagger \longrightarrow g_R \tilde\chi g_L^\dagger$.

$$
N_f = 2: \qquad \tilde\chi = \begin{pmatrix} x_{22}^* & -x_{21}^* \\ -x_{12}^* & x_{11}^* \end{pmatrix},
$$

$$
N_f = 3: \qquad \tilde\chi = \begin{pmatrix} x_{22}^* x_{33}^* - x_{23}^* x_{32}^* & x_{31}^* x_{23}^* - x_{21}^* x_{33}^* & x_{21}^* x_{32}^* - x_{31}^* x_{22}^* \\ x_{32}^* x_{13}^* - x_{12}^* x_{33}^* & x_{11}^* x_{33}^* - x_{31}^* x_{13}^* & x_{31}^* x_{12}^* - x_{11}^* x_{32}^* \\ x_{12}^* x_{23}^* - x_{22}^* x_{13}^* & x_{21}^* x_{13}^* - x_{11}^* x_{23}^* & x_{11}^* x_{22}^* - x_{12}^* x_{21}^* \end{pmatrix}
$$

Sources of relations

- Partial integration/total derivatives
- Terms that can be removed by LO EOM/field redefinitions
- "Commuting of partial derivatives"
- Bianchi identity
- Cayley Hamilton (for finite N_f)
- Schouten identity

Partial integration/Total derivatives

- Partial integration can lead to very different looking terms
- Main problem: how to make sure we have all of them
- Solution: each partial derivative relation corresponds to a total derivative
- Classify all invariant monomials as before but now with one free Lorentz index
- Take ∂^μ of those and it gives all partial integration relations
- Example:

$$
\begin{aligned}
0 &= \partial^\mu \langle \nabla_\mu u_\nu u^\nu u_\alpha u_\alpha \rangle \\
&= \langle \nabla^\mu \nabla_\mu u_\nu u^\nu u_\alpha u_\alpha \rangle + \langle \nabla_\mu u_\nu \nabla^\mu u^\nu u_\alpha u_\alpha \rangle \\
&\quad + \langle \nabla_\mu u_\nu u^\nu \nabla^\mu u_\alpha u_\alpha \rangle + \langle \nabla_\mu u_\nu u^\nu u_\alpha \nabla^\mu u_\alpha \rangle
\end{aligned}
$$

- Having all relations allows for many simplifications later

Field redefinitions – LO equations of motion

- The S-matrix does not change under a field redefinition: $\phi = \phi' + F(\phi')$ with $F(x \to 0) \to 0$ fast enough.
- In the functional integral: "just" a change of variables
- For classifying a Lagrangian: equivalent to removing "equation of motion terms"
- Simple explanation in the one-flavour case
- Works also if symmetries present
- Need a concept of power-counting or otherwise ordering

Field redefinitions – LO equations of motion

Use g to indicate orders

$$\mathcal{L} = \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V_0(\phi)\right) + g\left[\left(\partial^2\phi + V_0'(\phi)\right)V_{1EOM}(\phi,\partial\phi) + V_1(\phi,\partial\phi)\right] + \mathcal{O}(g^2)$$

Now define $\phi = \phi' + gV_{1EOM}(\phi',\partial\phi')$

$$\mathcal{L} = \left(\frac{1}{2}\partial^\mu\phi'\partial_\mu\phi' - V_0(\phi')\right) + g\left(\partial^\mu\phi\partial_\mu(V_0(\phi') - V_0'(\phi')\right)V_{1EOM}(\phi',\partial\phi') + g\left(\partial^2\phi' + V_0'(\phi')\right)V_{1EOM}(\phi',\partial\phi') + gV_1(\phi',\partial\phi') + \mathcal{O}(g^2)$$

Note: $\mathcal{O}(g^2)$ changed

After partial integration: EOM terms at $\mathcal{O}(g)$ cancel but changes at higher orders:

using EOM OK for classifying terms, not for doing calculations

Field redefinitions vs EOM

- Do order by order (include g^2 changes from step 1)
  $\phi' = \phi'' + g^2 V_{2EOM}(\phi'', \partial\phi'')$
- More than one field also works: use
  $\phi^a = \phi^{a\prime} + g V_1^a(\{\phi^{b\prime}, \partial\phi^{b\prime}\})$
- What about symmetries?
  - $\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_1 + g^2\mathcal{L}_2 + \cdots$
  - Each $\mathcal{L}_i$ is invariant under the symmetry
  - EOM^a is derived by $\phi^a \to \phi^a + \delta\phi^a$ where $\delta\phi^a$ must be compatible with the symmetry
  - $\mathcal{L}_0 \to \mathcal{L}_0 + (\delta\phi^a \mathrm{EOM}^a)$ means that $\delta\phi^a \mathrm{EOM}^a$ is invariant under the symmetry
  - EOM terms in $\mathcal{L}_i$ are invariant under the symmetry and of the form EOM$^a V_i^a(\{\phi^b, \partial\phi^b\})$
  - $\Longrightarrow \phi^a = \phi' + g^i V_i^a(\{\phi^{b\prime}, \partial\phi^{b\prime}\})$ are field transformations compatible with the symmetry that remove the EOM terms

Field redefinitions – LO equations of motion

- So we use
  $\nabla^\mu u_\mu - \frac{i}{2}\left(\chi_- - \frac{1}{N_f}\langle\chi_-\rangle\right) = 0\,.$
- take all operators, look for $\nabla^\mu u_\mu$ and replace by above to get a relation
- Example
  operator: $\langle\chi_+ u^\rho \chi_+ \nabla_\rho \nabla^\mu u_\mu\rangle$
  relation: $0 = \langle\chi_+ u^\rho \chi_+ \nabla_\rho \nabla^\mu u_\mu\rangle - \frac{i}{2}\langle\chi_+ u^\rho \chi_+ \nabla_\rho \chi_-\rangle + \frac{i}{2N_f}\langle\chi_+ u^\rho \chi_+\rangle\langle\nabla_\rho \chi_-\rangle$
- Since we have all operators, all partial integrations and all "commuting": only need to do it for "naked" $\nabla^\mu u_\mu$.

"Commuting of partial derivatives"/Bianchi

- Commuting:
  - $f_{-\mu\nu} - \nabla_\nu u_\mu + \nabla_\mu u_\nu = 0$
  - $[\nabla_\mu, \nabla_\nu] X = [\Gamma_{\mu\nu}, X]$
  - $\Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}$
- Bianchi
  - $B_{\mu\nu\rho} \equiv \nabla_\mu \Gamma_{\nu\rho} + \nabla_\nu \Gamma_{\rho\mu} + \nabla_\rho \Gamma_{\mu\nu} = 0\,.$
    $$B_{\mu\nu\rho} = \frac{1}{4}\Big([u_\rho, f_{-\mu\nu}] + [u_\mu, f_{-\nu\rho}] + [u_\nu, f_{-\rho\mu}]\Big) - \frac{i}{2}\Big(\nabla_\rho f_{+\mu\nu} + \nabla_\mu f_{+\nu\rho} + \nabla_\nu f_{+\rho\mu}\Big)$$
  - Generate all terms including $B_{\mu\nu\lambda}$
  - $\nabla_\rho B_{\mu\nu\lambda}$ not needed: we have all p.i. relations
  - $D_\mu F_{L\nu\rho} + D_\mu F_{L\nu\rho} + D_\mu F_{L\nu\rho} = 0$ (and $L \leftrightarrow R$) follow from "Commuting" and Bianchi for $\Gamma_{\mu\nu}$

Cayley-Hamilton relations

- Any matrix A satistfies its own characteristic polynomial:
  $p(\lambda) \equiv \det(\lambda I - A)\,, \qquad \text{and} \qquad p(A) = 0\,.$
- Expand in $1/\lambda$ using $p(\lambda) = \lambda^n \exp\{\mathrm{tr}\,[\ln(I - A/\lambda)]\}$
  The expansion has no terms negative in λ and stops at λ^n
- This leads to the relations

$$
\begin{aligned}
n = 1 &: A - I\langle A\rangle = 0 \\
n = 2 &: A^2 - A\langle A\rangle - \frac{1}{2}I\langle A^2\rangle + \frac{1}{2}I\langle A\rangle^2 = 0 \\
n = 3 &: A^3 - A^2\langle A\rangle - \frac{1}{2}A\langle A^2\rangle + \frac{1}{2}A\langle A\rangle^2 - \frac{1}{3}I\langle A^3\rangle \\
&\quad + \frac{1}{2}I\langle A\rangle\langle A^2\rangle - \frac{1}{6}I\langle A\rangle^3 = 0
\end{aligned}
$$

- The last terms with I always are related to $\det(A)$.

Cayley-Hamilton relations

- Make more useful by $A = B + C + \cdots$
- $n = 2$: $A = B + C$, only keep terms with B and C:
  $\{B, C\} = B\langle C\rangle + C\langle B\rangle + \langle BC\rangle - \langle B\rangle\langle C\rangle$.
- $n = 3$ and $\langle B\rangle = \langle C\rangle = \langle D\rangle = \langle E\rangle = 0$ use $A = B + C + D$, multiply by E and take trace

$$\sum_{6\ perm} \langle BCDE\rangle = \sum_{3\ perm} \langle BC\rangle\langle DE\rangle .$$

- Simply taking the trace of the relations does not give any new results, that is satisfied automatically.
- When implementing: use $B, C, \ldots$ as building blocks and products of building blocks

Schouten identity

- No fully antisymmetric tensor with five indices in four dimensions
- We have assumed no $\epsilon_{\mu\nu\alpha\beta}$: only relevant when have 5 different indices, so just from p^{10}
- It's never clear whether more relations exist: only full criterion I know: can you determine all LECs from explicit Green functions (or experiment)

What to keep

Preferentially keep/remove (as much as possible):

1. Keep maximal number of independent contact terms
2. Remove terms that vanish when external fields vanish
3. Remove terms with covariant derivatives in favour of those involving external fields.
4. Remove terms that contribute to processes with a low number of mesons, count occurrences of u_μ, χ_- and $f_{-\mu\nu}$.
5. Scalar-pseudoscalar external fields are placed before those with only vector-axial-vector external fields.
6. Keep terms with lower number of flavour traces. This is to make the large N_c counting of the monomials explicit, only leading in N_c is equivalent to keeping only single trace monomials.

Still leaves a choice to be made which to keep

Technically

- FORM for the main part: generating all terms and relations.
- equivalent terms produced by FORM, using Python rewritten back into FORM.
- Identical relations removed using FORM
- Final number of independent relations done with Gaussian elimination (sparse matrix methods) with `gmp` exact arithmetic
- Main restriction: memory size, not CPU time (but grows very fast with order)
- Example general N_f: 9740 C-even monomials, 12444 different relations of which 7878 lineraly independent: 1862 terms left

Results: all

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	2	0	2	0	2	0
p^4	13	2	12	2	10	3
p^6	115	3	94	4	56	4
p^8	1862	22	1254	21	475	23

Table: Number of monomials in the minimal basis for the case with all external fields included. Also listed is how many of them are contact terms. Our results agree with the known ones for p^2, p^4, p^6.

Results: only vector-axial-vector

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	1	0	1	0	1	0
p^4	7	1	6	1	5	1
p^6	59	2	44	2	27	2
p^8	963	15	591	13	238	11

Table: Number of monomials in the minimal basis for the case with no scalar or pseudoscalar external fields included. Also listed is how many of them are contact terms.

Note: tested using Green functions at p^6

P. Ruiz-Femenía and M. Zahiri-Abyaneh, 1507.00269

Results: scalar/pseudo-scalar external fields

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	2	0	2	0	2	0
p^4	10	1	9	1	7	1
p^6	62	1	48	2	27	2
p^8	538	3	328	4	122	6

Table: Number of monomials in the minimal basis for the case with no vector or axial-vector external fields included. Also listed is how many of them are contact terms.

Results: No external fields

	N_f	$N_f = 3$	$N_f = 2$
p^2	1	1	1
p^4	4	3	2
p^6	19	11	5
p^8	135	56	16

Table: Number of monomials in the minimal basis for the case with no external fields included. There are no contact terms in this case.

Results: no external fields; by meson number

	#mesons	N_f	$N_f = 3$	$N_f = 2$
p^2	4	1	1	1
p^4	4	4	3	2
p^6	4	4	3	2
	6	15	8	3
p^8	4	6	5	3
	6	60	31	9
	8	69	20	4

Table: Number of monomials in the minimal basis for the case with no external fields included that produce vertices starting at the given number of mesons.

Meson-meson scattering

- $M(s,t,u) = \left\langle \phi^c(p_c)\,\phi^d(p_d)\,\middle|\,\phi^a(p_a)\,\phi^b(p_b)\,\right\rangle$:

$$
\begin{aligned}
M(s,t,u) =& \left[\langle X^aX^bX^cX^d\rangle + \langle X^aX^dX^cX^b\rangle\right]B(s,t,u)\\
&+\left[\langle X^aX^cX^dX^b\rangle + \langle X^aX^bX^dX^c\rangle\right]B(t,u,s)\\
&+\left[\langle X^aX^dX^bX^c\rangle + \langle X^aX^cX^bX^d\rangle\right]B(u,s,t)\\
&+\delta^{ab}\delta^{cd}C(s,t,u)+\delta^{ac}\delta^{bd}C(t,u,s)+\delta^{ad}\delta^{bc}C(u,s,t)\,,
\end{aligned}
$$

- $s = (p_a+p_b)^2,\; t = (p_a+p_c)^2,\; u = (p_a+p_d)^2$
- X^a are $SU(N_f)$ generators normalized to 1.
- $C(s,t,u) = C(s,u,t)$
- $B(s,t,u) = B(u,t,s)$
- $N_f = 2$: B not needed $(C(s,t,u) \to A(s,t,u))$
- $N_f = 3$: Fully symmetric combination of B not needed

Meson-meson scattering

- Expand in s, t, u:

$$
\begin{aligned}
C(s,t,u) =& \alpha_0 + \alpha_2 s + \alpha_{41} s^2 + \alpha_{42}(t-u)^2 \\
& + \alpha_{61} s^3 + \alpha_{62} s (t-u)^2 \\
& + \alpha_{81} s^4 + \alpha_{82} s^2 (t-u)^2 + \alpha_{83}(t-u)^4 \,, \\
B(s,t,u) =& \beta_0 + \beta_2 t + \beta_{41} t^2 + \beta_{42}(s-u)^2 \\
& + \beta_{61} t^3 + \beta_{62} t (s-u)^2 \\
& + \beta_{81} t^4 + \beta_{82} t^2 (s-u)^2 + \beta_{83}(s-u)^4 \,.
\end{aligned}
$$

- Chiral invariance requires $\alpha_0 = \beta_0 = 0$.
- The number of parameters agree

Conclusions of constructing p^8

- Constructed the p^8 even intrinsic parity Lagrangian
- Fully automatized except for counterterms
- Compared with all known results
- By itself not very useful (but fun and interesting) but interplay with amplitude methods allows to check both

QCDlike and/or technicolor theories

- One can also have different symmetry breaking patterns from underlying fermions
- Three generic cases
  - $SU(N) \times SU(N)/SU(N)$
  - $SU(2N)/SO(2N)$
  - $SU(2N)/Sp(2N)$
- Many one-loop results existed especially for the first case (several discovered only after we published our work)
- Equal mass case pushed to two loops JB, Lu, 2009-11

N_F fermions in a representation of the gauge group

- complex (QCD):
  - $q^T = (q_1 \; q_2 \ldots \; q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$
    $q_L \to g_L q_L$ and $g_R \to g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$
  - $g_L = g_R$ then $\Sigma_{ij} \to \Sigma_{ij} \Longrightarrow$ conserved $H = SU(N_F)_V$:
- Real (e.g. adjoint): $\hat{q}^T = (q_{R1} \; \ldots \; q_{RN_F} \; \tilde{q}_{R1} \; \ldots \; \tilde{q}_{RN_F})$
  - $\tilde{q}_{Ri} \equiv C\overline{q}^T_{Li}$ goes under gauge group as q_{Ri}
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - $\langle \overline{q}_j q_i \rangle$ is really $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto J_{Sij}$ $J_S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$
  - Conserved if $gJ_S g^T = J_S \Longrightarrow H = SO(2N_F)$

N_F fermions in a representation of the gauge group

- complex (QCD):
  - $q^T = (q_1\ q_2 \ldots\ q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R$
    $q_L \to g_L q_L$ and $g_R \to g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$
  - $g_L = g_R$ then $\Sigma_{ij} \to \Sigma_{ij} \Longrightarrow$ conserved $H = SU(N_F)_V$:
- Real (e.g. adjoint): $\hat{q}^T = (q_{R1}\ \ldots\ q_{RN_F}\ \tilde{q}_{R1}\ \ldots\ \tilde{q}_{RN_F})$
  - $\tilde{q}_{Ri} \equiv C\overline{q}_{Li}^T$ goes under gauge group as q_{Ri}
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - $\langle \overline{q}_j q_i \rangle$ is really $\langle (\hat{q}_j)^T C \hat{q}_i \rangle \propto J_{Sij}\ J_S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$
  - Conserved if $g J_S g^T = J_S \Longrightarrow H = SO(2N_F)$

N_F fermions in a representation of the gauge group

- complex (QCD): $q^T = (q_1 \; q_2 \ldots \; q_{N_F})$
  - Global $G = SU(N_F)_L \times SU(N_F)_R \quad q_L \to g_L q_L$ and $g_R \to g_R q_R$
  - Vacuum condensate $\Sigma_{ij} = \langle \overline{q}_j q_i \rangle \propto \delta_{ij}$
  - Conserved $H = SU(N_F)_V$: $g_L = g_R$ then $\Sigma_{ij} \to \Sigma_{ij}$
- Pseudoreal (e.g. two-colours):
  $\hat{q}^T = (q_{R1} \; \ldots \; q_{RN_F} \; \tilde{q}_{R1} \; \ldots \; \tilde{q}_{RN_F})$
  - $\tilde{q}_{R\alpha i} \equiv \epsilon_{\alpha\beta} C \bar{q}^T_{L\beta i}$ goes under gauge group as $q_{R\alpha i}$
  - some Goldstone bosons have baryonnumber
  - Global $G = SU(2N_F)$ and $\hat{q} \to g\hat{q}$
  - $\langle \overline{q}_j q_i \rangle$ is really $\epsilon_{\alpha\beta}\langle (\hat{q}_{\alpha j})^T C \hat{q}_{\beta i} \rangle \propto J_{Aij}$ $J_A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$
  - Conserved if $g J_A g^T = J_A \Longrightarrow H = Sp(2N_F)$

Lagrangians

JB, Lu, arXiv:0910.5424: 3 cases similar with $u = \exp\left(\frac{i}{\sqrt{2}F}\phi^a X^a\right)$

But the matrices X^a are:

- Complex or $SU(N) \times SU(N)/SU(N)$:
  all $SU(N)$ generators
- Real or $SU(2N)/SO(2N)$:
  $SU(2N)$ generators with $X^a J_S = J_S X^{aT}$
- Pseudoreal or $SU(2N)/Sp(2N)$:
  $SU(2N)$ generators with $X^a J_A = J_A X^{aT}$
- Note that the latter are not the usual ways of parametrizing $SO(2N)$ and $Sp(2N)$ matrices

Lagrangians

- $u \to g_R u h^\dagger \equiv h u g_L^\dagger$ for complex
- $u \to g u h^\dagger$ for real, pseudoreal
- h is in the conserved part of the group for all cases
- $u_\mu = i\left(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger\right) \to h u_\mu h^\dagger$
- external fields can also be included.
- a generalized mass term $\chi_\pm \to h \chi_\pm h^\dagger$ can be defined with $\chi_\pm = u^\dagger \tilde{\chi} u^\dagger \pm u \tilde{\chi} u$
- $\mathcal{L}_{LO} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle$
- $$\begin{aligned}\mathcal{L}_4 &= L_0\langle u^\mu u^\nu u_\mu u_\nu\rangle + L_1\langle u^\mu u_\mu\rangle\langle u^\nu u_\nu\rangle + L_2\langle u^\mu u^\nu\rangle\langle u_\mu u_\nu\rangle + L_3\langle u^\mu u_\mu u^\nu u_\nu\rangle \\ &+ L_4\langle u^\mu u_\mu\rangle\langle\chi_+\rangle + L_5\langle u^\mu u_\mu \chi_+\rangle + L_6\langle\chi_+\rangle^2 + L_7\langle\chi_-\rangle^2 + \frac{1}{2}L_8\langle\chi_+^2 + \chi_-^2\rangle \\ &- iL_9\langle f_{+\mu\nu} u^\mu u^\nu\rangle + \frac{1}{4}L_{10}\langle f_+^2 - f_-^2\rangle + H_1\langle \ell_{\mu\nu}\ell^{\mu\nu} + r_{\mu\nu} r^{\mu\nu}\rangle + H_2\langle\chi\chi^\dagger\rangle\,.\end{aligned}$$

The main useful formulae

Calculating for equal mass case goes through using:

$$
\begin{aligned}
\text{Complex}: \quad & \left\langle X^a A X^a B \right\rangle = \langle A \rangle \langle B \rangle - \frac{1}{N_F} \langle AB \rangle \,, \\
& \left\langle X^a A \right\rangle \left\langle X^a B \right\rangle = \langle AB \rangle - \frac{1}{N_F} \langle A \rangle \langle B \rangle \,. \\
\text{Real}: \quad & \left\langle X^a A X^a B \right\rangle = \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \left\langle A J_S B^T J_S \right\rangle - \frac{1}{2N_F} \langle AB \rangle \,, \\
& \left\langle X^a A \right\rangle \left\langle X^a B \right\rangle = \frac{1}{2} \langle AB \rangle + \frac{1}{2} \left\langle A J_S B^T J_S \right\rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle \,. \\
\text{Pseudoreal}: \quad & \left\langle X^a A X^a B \right\rangle = \frac{1}{2} \langle A \rangle \langle B \rangle + \frac{1}{2} \left\langle A J_A B^T J_A \right\rangle - \frac{1}{2N_F} \langle AB \rangle \,, \\
& \left\langle X^a A \right\rangle \left\langle X^a B \right\rangle = \frac{1}{2} \langle AB \rangle - \frac{1}{2} \left\langle A J_A B^T J_A \right\rangle - \frac{1}{2N_F} \langle A \rangle \langle B \rangle
\end{aligned}
$$

So can do the calculations for all cases

Divergences etc

i	QCD	Adjoint	2-colour
0	$N_F/48$	$(N_F+4)/48$	$(N_F-4)/48$
1	$1/16$	$1/32$	$1/32$
2	$1/8$	$1/16$	$1/16$
3	$N_F/24$	$(N_F-2)/24$	$(N_F+2)/24$
4	$1/8$	$1/16$	$1/16$
5	$N_F/8$	$N_F/8$	$N_F/8$
6	$(N_F^2+2)/(16N_F^2)$	$(N_F^2+1)/(32N_F^2)$	$(N_F^2+1)/(32N_F^2)$
7	0	0	0
8	$(N_F^2-4)/(16N_F)$	$(N_F^2+N_F-2)/(16N_F)$	$(N_F^2-N_F-2)/(16N_F)$
9	$N_F/12$	$(N_F+1)/2$	$(N_F-1)/2$
10	$-N_F/12$	$-(N_F+1)/2$	$-(N_F-1)/2$
1'	$-N_F/24$	$-(N_F+1)/4$	$-(N_F+1)/4$
2'	$(N_F^2-4)/(8N_F)$	$(N_F^2+N_F-2)/(8N_F)$	$(N_F^2-N_F-2)/(8N_F)$

Vacuum expectation value

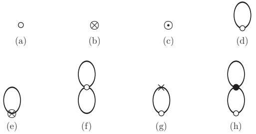
All cases: $\langle \overline{q}q\rangle_{\rm LO} \equiv \sum_{i=1,N_F} \langle \overline{q}_{Ri} q_{Li} + \overline{q}_{Li} q_{Ri}\rangle_{\rm LO} = -N_F B_0 F^2$

$M^2 = 2B_0\hat{m}$ and $\overline{A}(M^2) = -\frac{M^2}{16\pi^2}\log\frac{M^2}{\mu^2}$.

$\langle \overline{q}q\rangle = \langle \overline{q}q\rangle_{\rm LO} + \langle \overline{q}q\rangle_{\rm NLO} + \langle \overline{q}q\rangle_{\rm NNLO}$.

$$\langle \overline{q}q\rangle_{\rm NLO} = \langle \overline{q}q\rangle_{\rm LO}\left(a_V\frac{\overline{A}(M^2)}{F^2} + b_V\frac{M^2}{F^2}\right),$$

$$\langle \overline{q}q\rangle_{\rm NNLO} = \langle \overline{q}q\rangle_{\rm LO}\left(c_V\frac{\overline{A}(M^2)^2}{F^4} + \frac{M^2\overline{A}(M^2)}{F^4}\left(d_V + \frac{e_V}{16\pi^2}\right) + \frac{M^4}{F^4}\left(f_V + \frac{g_V}{16\pi^2}\right)\right).$$



Vacuum expectation value

	QCD	
a_V	$n - \frac{1}{p}$	
b_V	$16nL_6^r + 8L_8^r + 4H_2^r$	
c_V	$\frac{3}{2}\left(-1 + \frac{1}{n^2}\right)$	
d_V	$-24\left(n^2 - 1\right)\left(L_A + \frac{1}{n}L_B\right)$	$L_A = L_4^r - 2L_6^r$
e_V	$1 - \frac{1}{n^2}$	$L_B = L_5^r - 2L_8^r$
f_V	$48\left(K_{25}^r + nK_{26}^r + n^2K_{27}^r\right)$	
g_V	$8\left(n^2 - 1\right)\left(L_A + \frac{1}{n}L_B\right)$	
	Adjoint	2-colour
a_V	$n + \frac{1}{2} - \frac{1}{2n}$	$n - \frac{1}{2} - \frac{1}{2n}$
b_V	$32nL_6^r + 8L_8^r + 4H_2^r$	$32nL_6^r + 8L_8^r + 4H_2^r$
c_V	$\frac{3}{8}\left(-1 + \frac{1}{n^2} - \frac{2}{n} + 2n\right)$	$\frac{3}{8}\left(-1 + \frac{1}{n^2} + \frac{2}{n} - 2n\right)$
d_V	$-12\left(2n^2 + n - 1\right)\left(2L_A + \frac{1}{n}L_B\right)$	$-12\left(2n^2 - n - 1\right)\left(2L_A + \frac{1}{n}L_B\right)$
e_V	$\frac{1}{4}\left(1 - \frac{1}{n^2} + \frac{2}{n} - 2n\right)$	$\frac{1}{4}\left(1 - \frac{1}{n^2} - \frac{2}{n} + 2n\right)$
f_V	r_{VA}^r	r_{VT}^r
g_V	$4\left(2n^2 + n - 1\right)\left(2L_A + \frac{1}{n}L_B\right)$	$4\left(2n^2 - n - 1\right)\left(2L_A + \frac{1}{n}L_B\right)$

Note: relations in the large n limit.

$\phi\phi \to \phi\phi$

- $\pi\pi$ scattering
  - Amplitude in terms of $A(s,t,u)$

    $M_{\pi\pi}(s,t,u) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t)\,.$
  - Three intermediate states $I = 0,1,2$
- Our three cases
  - Two amplitudes needed $B(s,t,u)$ and $C(s,t,u)$

$$
\begin{aligned}
M(s,t,u) \quad = \quad & \left[\left\langle X^aX^bX^cX^d\right\rangle + \left\langle X^aX^dX^cX^b\right\rangle\right] B(s,t,u) \\
& + \left[\left\langle X^aX^cX^dX^b\right\rangle + \left\langle X^aX^bX^dX^c\right\rangle\right] B(t,u,s) \\
& + \left[\left\langle X^aX^dX^bX^c\right\rangle + \left\langle X^aX^cX^bX^d\right\rangle\right] B(u,s,t) \\
& + \delta^{ab}\delta^{cd}C(s,t,u) + \delta^{ac}\delta^{bd}C(t,u,s) + \delta^{ad}\delta^{bc}C(u,s,t)\,.
\end{aligned}
$$

$B(s,t,u) = B(u,t,s) \qquad C(s,t,u) = C(s,u,t)\,.$

  - 7, 6 and 6 possible intermediate states
- All formulas similar length to $\pi\pi$ cases but there are so many of them

$\phi\phi \to \phi\phi$

- calculate all the diagrams
- Do all integrals, renormalize,...
- Construct states for all the presentations and their projection operators
- Get the amplitudes for all intermediate states
- Get all scattering lengths
- All formulas similar length to $\pi\pi$ cases but there are so many of them
- JB, Lu, arXiv:1102.0172:
  - Very long appendix part
  - References for the Young diagrams, tensor algebra we did ourselves but probably exists (e.g. Cvitanovic group theory book)

$\phi\phi \to \phi\phi$

Some curious large $N_F = n$ relations

Leading in n:

$$
\begin{aligned}
a_0^I|_{\text{complex}} &= a_0^I|_{\text{real}} = a_0^I|_{\text{pseudoreal}} =_{LO} \frac{x_2}{\pi}\,\frac{n}{8}\,,\\
a_0^S|_{\text{complex}} &= a_0^S|_{\text{real}} = a_0^A|_{\text{pseudoreal}} =_{LO} \frac{x_2}{\pi}\,\frac{n}{16}\,,\\
a_1^A|_{\text{complex}} &= a_1^A|_{\text{real}} = a_1^S|_{\text{pseudoreal}} =_{LO} \frac{x_2}{\pi}\,\frac{n}{48}\,,
\end{aligned}
$$

Subleading:

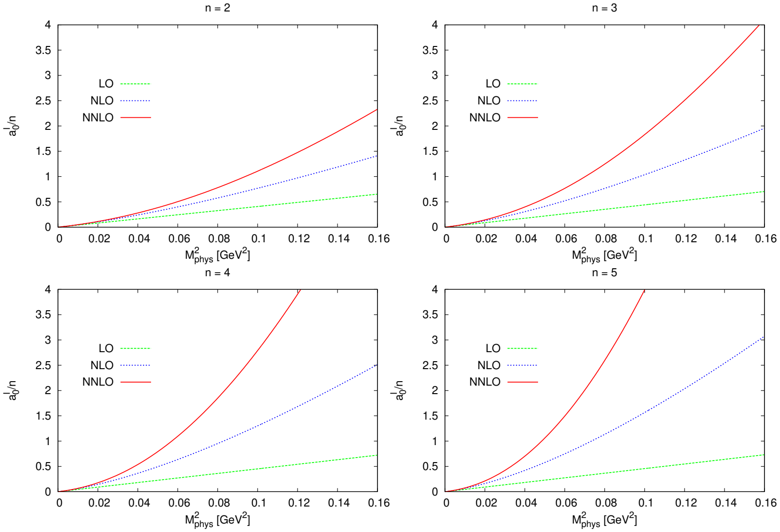
$$
\begin{aligned}
a_0^{SS}|_{\text{complex}} &= a_0^{FS}|_{\text{real}} = 2a_0^{MS}|_{\text{pseudoreal}} =_{LO} \frac{x_2}{\pi}\,\frac{-1}{16}\,,\\
a_0^{AA}|_{\text{complex}} &= 2a_0^{MS}|_{\text{real}} = a_0^{FA}|_{\text{pseudoreal}} =_{LO} \frac{x_2}{\pi}\,\frac{1}{16}\,.
\end{aligned}
$$

Subsubleading:

$$
a_1^{SA}|_{\text{complex}} = a_1^{AS}|_{\text{complex}} = 2a_1^{MA}|_{\text{real}} = 2a_1^{MA}|_{\text{pseudoreal}} =_{LO} 0\,.
$$

At NNLO here violated by an $L_4^r L_6^r$ term

$\phi\phi \to \phi\phi$: a_0^I/n



Conclusions for "Beyond QCD"

Calculations done:

- M^2_{phys}
- F_{phys}
- Meson-meson scattering
- Equal mass case: allows to get fully analytical result just as for 2-flavour ChPT
- Two-point functions relevant for S-parameter

To remember:

- Different symmetry patterns can appear for different gaugegroups and fermion representations
- Nonperturbative: lattice needs extrapolation formulae

Conclusions

- see previous page
- Constructing higher order Lagrangians is possible but not entirely trivial, especially to get a minimal one.