Invitation:

Electromagnetism is about solving Maxwell's equations

\[
\begin{align*}
    \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0} \rho \\
    \nabla \cdot \vec{B} &= 0 \\
    \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
    \nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{J}
\end{align*}
\]

(Gauss's law)

(Faraday's law)

(Ampère's law with Maxwell's correction)

where \( \rho \) and \( \vec{J} \) fulfill the continuity equation

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}
\]

together with the Lorentz force law

\[
\vec{F} = q (\vec{E} + \vec{v} \times \vec{B})
\]

which tells us how charges move.

Note that in general the e.m. fields \( \vec{E} \) and \( \vec{B} \) as well as the density \( \rho \) and current \( \vec{J} \) are functions of both the position \( \vec{r} \) and time \( t \) - i.e. they are fields. By convention we do not write out this dependence.

We will solve Maxwell's equations using symmetry arguments as well as direct calculations using vector analysis.
Chapter 1 in the book by Griffiths contains most of the mathematics we will need. In this course we will follow the tradition and start with considering the case when there is no time-dependence
\[
\frac{\partial}{\partial t} (E, B, \rho, \mathbf{J}) = 0
\]
and start by exploring electrostatics followed by magnetostatics. Only then will we include the time-dependence.

Finally, before starting we should also make it clear that you have already encountered many of the ideas and concepts of the course in earlier courses (Physics 1) - the difference is that now we will use much more rigorous and mathematical tools that will allow us to solve much more complicated problems.
Mathematical tools

In order to proceed we need to develop some mathematical tools called vector calculus - how to take derivatives of vector fields and how these can be integrated. In other words the meaning of $\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$.

Let's start with a brief reminder about vectors which we will denote $\vec{a}, \vec{b}, \vec{c}$ etc. Note that they in principle could be vector fields, such as the electric and magnetic fields, which depend on the space-time point $(x,y,z,t)$.

Consider two vectors $\vec{a}$ and $\vec{b}$ with an angle $\Theta$ in between the directions.

![Diagram](image)

**Scalar product:**

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \Theta = ab \cos \Theta$$

**Cross product:**

$$\vec{a} \times \vec{b} = ab \sin \Theta \hat{n}$$

In component form (with $\vec{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$)

$$\vec{a} \cdot \vec{b} = (a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) = a_x b_x + a_y b_y + a_z b_z$$

(note: $\vec{a}^2 = a_x^2 + a_y^2 + a_z^2 = |\vec{a}|^2 = a^2$)

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y) \hat{x} + (a_z b_x - a_x b_z) \hat{y} + (a_x b_y - a_y b_x) \hat{z}$$

or $\vec{a} \times \vec{b} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
a_x & a_y & a_z \\
b_x & b_y & b_z
\end{vmatrix}$
Special vectors:
- position \( \mathbf{r} = x \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \)
- infinitesimal displacement \( d\mathbf{\ell} = dx \mathbf{\hat{x}} + dy \mathbf{\hat{y}} + dz \mathbf{\hat{z}} \)
- radial unit vector \( \mathbf{r} = \frac{\mathbf{\ell}}{|\mathbf{\ell}|} = \frac{\mathbf{\ell}}{r} \)
  \( r = \sqrt{x^2 + y^2 + z^2} \)

Differential calculus

Gradient of a scalar function \( f = f(x,y,z) = f(\mathbf{r}) \)
\( \nabla f = \frac{\partial f}{\partial x} \mathbf{\hat{x}} + \frac{\partial f}{\partial y} \mathbf{\hat{y}} + \frac{\partial f}{\partial z} \mathbf{\hat{z}} \)
gives change under infinitesimal displacement \( d\mathbf{\ell} \)
\( df = (\nabla f) \cdot d\mathbf{\ell} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \)

Ex.
\( \nabla \mathbf{r} = \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{\hat{x}} + \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{\hat{y}} + \frac{1}{2} \frac{2z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{\hat{z}} \)
\( \frac{\partial}{\partial r} \mathbf{r} = \frac{\mathbf{\hat{r}}}{r} \)

Think of \( \nabla = \mathbf{\hat{x}} \frac{\partial}{\partial x} + \mathbf{\hat{y}} \frac{\partial}{\partial y} + \mathbf{\hat{z}} \frac{\partial}{\partial z} \) as an operator which can act on both scalar and vector functions.
- gradient \( \nabla f = \frac{\partial f}{\partial x} \mathbf{\hat{x}} + \frac{\partial f}{\partial y} \mathbf{\hat{y}} + \frac{\partial f}{\partial z} \mathbf{\hat{z}} \)
- divergence \( \nabla \cdot \mathbf{a} = (\mathbf{\hat{x}} \frac{\partial}{\partial x} + \mathbf{\hat{y}} \frac{\partial}{\partial y} + \mathbf{\hat{z}} \frac{\partial}{\partial z}) \cdot (a_x \mathbf{\hat{x}} + a_y \mathbf{\hat{y}} + a_z \mathbf{\hat{z}}) \)
  \( = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \)
- curl \( \nabla \times \mathbf{a} = \begin{vmatrix} \mathbf{\hat{x}} & \mathbf{\hat{y}} & \mathbf{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \)
\[
\hat{x} \left( \frac{\partial}{\partial y} a_y - \frac{\partial}{\partial z} a_z \right) + \hat{y} \left( \frac{\partial}{\partial z} a_z - \frac{\partial}{\partial x} a_x \right) + \hat{z} \left( \frac{\partial}{\partial x} a_x - \frac{\partial}{\partial y} a_y \right)
\]

(beware: \(\nabla\) behaves in many ways as a vector, but remember that it is a differential operator)

**Geometrical Interpretation**

- **Gradient**: points in direction of maximal change, magnitude gives rate of change \(\nabla f = 0\) at local max or min

- **Divergence**: \(\nabla \cdot \vec{a} \to 0\) where vector-fcn \(\vec{a}\) has source or sink

- **Curl**: measure of how much a vector fcn rotates

**Ex.**

\[
\vec{a} = \hat{r}
\]

\[
\nabla \cdot \vec{r} = 3
\]

\[
\nabla \times \vec{r} = \vec{0}
\]

\[
\vec{a} = \hat{z}
\]

\[
\nabla \cdot \vec{a} = 1
\]

\[
\nabla \times \vec{a} = \vec{0}
\]

\[
\vec{a} = xy
\]

\[
\nabla \cdot \vec{a} = 0
\]

\[
\nabla \times \vec{a} = \hat{z}
\]
Product rules for scalar \((f,g)\) and vector \((\vec{a}, \vec{b})\) forms:

\[
\nabla (fg) = f(\nabla g) + g(\nabla f)
\]

\[
\nabla (\vec{a} \cdot \vec{b}) = \vec{a} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{a}) + (\vec{a} \cdot \nabla) \vec{b} + (\vec{b} \cdot \nabla) \vec{a}
\]

\[
\nabla \cdot (f \vec{a}) = f (\nabla \cdot \vec{a}) + (\nabla f) \cdot \vec{a}
\]

\[
\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})
\]

\[
\nabla \times (f \vec{a}) = f (\nabla \times \vec{a}) - \vec{a} \times (\nabla f)
\]

\[
= f (\nabla \times \vec{a}) + (\nabla f) \times \vec{a}
\]

\[
\nabla \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla) \vec{a} - (\vec{a} \cdot \nabla) \vec{b} + \vec{a} (\nabla \cdot \vec{b}) - \vec{b} (\nabla \cdot \vec{a})
\]

where \(\vec{a} \cdot \nabla\) is a scalar operator

\[
(a \hat{x} + b \hat{y} + c \hat{z}) \cdot (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) =
\]

\[
= a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}
\]

so that

\[
(\vec{a} \cdot \nabla) \vec{b} = (a_\hat{x} \frac{\partial}{\partial x} b_\hat{x} + a_\hat{y} \frac{\partial}{\partial y} b_\hat{y} + a_\hat{z} \frac{\partial}{\partial z} b_\hat{z}) \hat{\xi} + \ldots
\]

Ex. (probl 1.22 b)

\[
\]

\[
(\hat{r} \cdot \nabla ) \hat{r} = \left( (\frac{x}{r} \hat{x} + \frac{y}{r} \hat{y} + \frac{z}{r} \hat{z}) \right) \cdot \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right)
\]

\[
\]

\[
= \frac{1}{r} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{x \hat{x} + y \hat{y} + z \hat{z}}{\sqrt{x^2 + y^2 + z^2}}
\]

consider \(x\)-component

\[
[\hat{r} \cdot \nabla] \hat{r}]_x = \frac{1}{r} \left( x \frac{\partial}{\partial x} \frac{1}{r} + x^2 \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \right) + \ldots
\]
\[ + y \times \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) + 2 \times \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \]

\[ = \frac{1}{r} \left( \frac{x}{r} + x^2 \left( -\frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{3/2}} \right) + y \times \left( -\frac{1}{2} \frac{2y}{(x^2 + y^2 + z^2)^{3/2}} \right) + \right) \]

\[ = \frac{1}{r} \left( \frac{x}{r} - x \frac{x^2}{r^3} - x \frac{y^2}{r^3} - x \frac{z^2}{r^3} \right) = \frac{x}{r^2} (1 - \frac{x^2 + y^2 + z^2}{r^2}) = 0 \]

The same holds for \( y \) and \( z \)-components

\[ \therefore \left( \hat{\nabla} \cdot \vec{\nabla} \right) \hat{r} = 0 \]

**Second derivatives**

\[ \nabla \cdot (\vec{\nabla} f) = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) \cdot (\frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}) = \]

\[ = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f \equiv \nabla^2 f \]

\( \text{Laplacean} \)

Also \( \nabla^2 \vec{a} = (\nabla^2 a_x) \hat{x} + (\nabla^2 a_y) \hat{y} + (\nabla^2 a_z) \hat{z} \)

\[ \nabla \times (\vec{\nabla} f) = 0 \]

\[ \nabla \left( \nabla \cdot \vec{a} \right) \text{ not interesting} \neq \nabla^2 \vec{a} \]

\[ \nabla \cdot (\vec{\nabla} \times \vec{a}) = 0 \]

\[ \nabla \times (\vec{\nabla} \times \vec{a}) = \nabla \left( \nabla \cdot \vec{a} \right) - \nabla^2 \vec{a} \]

**Note:** \( \nabla^2 \) sometimes written \( \nabla^2 \) but strictly speaking this is an abuse of notation
Integral calculus

1-dim calc. In general the integral
\[ \int_{x_i}^{x_2} g(x) \, dx \]
depends on the value of \( g \) everywhere in \( x \in [x_i, x_2] \).
But if \( g(x) = \frac{df}{dx} \) then the integral relates
derivative of \( f \) with its value on the boundaries.
\[ \int_{x_i}^{x_2} \frac{df}{dx} \, dx = f(x_2) - f(x_i) \] depends on \( x_i \) and \( x_2 \)!

generalises to several dimensions

We are interested in

line, surface, and volume integrals:

\[ \int_{\vec{r}_2}^{\vec{r}_1} \vec{a} \cdot d\vec{r} \], \[ \int_{\Gamma} \vec{a} \cdot d\vec{s} \], \[ \int f \, dV \]
closed integrals indicated by \( \oint \)

Line integrals:

\[ \int_{\vec{r}_i}^{\vec{r}_2} \vec{a} \cdot d\vec{l} \]
depends in general on path taken but if \( \vec{a} = \vec{v} f \)
then, \( (\vec{v} \cdot f) \cdot d\vec{l} = df \) and
\[ \oint \vec{v} f \cdot d\vec{l} = f(\vec{r}_2) - f(\vec{r}_1) \] \[ \oint \vec{v} f \cdot d\vec{l} = 0 \]
**Surface Integrals**

\[ \int_{\Delta} \hat{a} \cdot d\Delta \], \hspace{1cm} d\Delta = \hat{n} \, d\Delta \hspace{1cm} \text{outward surface element}

depends in general on shape of surface but if \( \hat{a} = \nabla \times \vec{b} \) then

\[ \int (\nabla \times \vec{b}) \cdot d\Delta = \oint_{P=\Delta} \vec{b} \cdot d\vec{l} \hspace{1cm} \text{Stokes' theorem} \]

relation of \( \hat{n} \) and \( d\vec{l} \) by right-hand rule

**Volume Integral**

\[ \int_{\Omega} f \, d\Omega \]

\( d\Omega = dx \, dy \, dz \)

depends in general on values of \( f \) inside volume but if \( f = \nabla \cdot \vec{a} \) then

\[ \int (\nabla \cdot \vec{a}) \, d\Omega = \oint_{\Omega=\Delta} \vec{a} \cdot d\Delta \hspace{1cm} \text{Gauss' theorem} \]