1. [6p]
Consider a pendulum consisting of a massless rod of length $L$ and a mass $m$ attached at the end of the rod.

**a) [2p]** Write down the kinetic and potential energy of the pendulum when the rod forms an angle $\phi$ w.r.t. the vertical axis.

**b) [2p]** Write down the Hamiltonian expressed in terms of $\phi$ and the corresponding generalized momentum $p_{\phi}$.

**c) [2p]** Write down and solve Hamilton’s equations in the small angle limit, when the pendulum is hanging almost straight down.

2. [8p]
At the Large Hadron Collider (LHC) at CERN, protons will be accelerated in opposite directions, and collide head on with a total center of momentum energy of 13 TeV (i.e. $E = 6.5$ TeV per proton). The mass $m$ of the proton is approximately $1\text{GeV}/c^2$.

**a) [2p]** Choose suitable coordinates and write down the four-momenta of the protons in the LHC frame in terms of $E$ and $m$.

**b) [1p]** Calculate the invariant mass using the four-momenta.

**c) [3p]** Make sensible approximations and determine the energy of one proton in the other proton’s rest system.

**d) [2p]** A bumblebee has a mass of about 0.2g. How fast should the bumblebee fly to have the same kinetic energy as the moving proton in (c)? Use $1\text{GeV}/c^2 \approx 1.8 \times 10^{-27} \text{kg}$. (For full points your answer may deviate with up to 30% from the correct value.)

3. [6p]
An intergalactic voyager wants to visit the Andromeda galaxy, 2.5 million light years away. She is prepared to spend 50 years of her life on the trip.

**a) [2p]** Neglecting the time for acceleration, how fast does she have to travel?

**b) [1p]** Let us temporarily ignore relativistic effects, how much time does it take to reach this speed if she is accelerating with $g \approx 10 \text{m}/\text{s}^2$?

**c) [3p]** How long time does it take to reach this speed if we consider special relativistic effects, and she is accelerating with $g$ in her own frame? *Hint:* $\text{arctanh}(1 - 2 \times 10^{-10}) \approx 11.5$. 

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4. [6p] A so-called Wilberforce pendulum consists of a mass \( m \) hanging in a spiral spring that is free to oscillate in vertical and in torsional mode. The Lagrangian can be written

\[
L = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} k z^2 - \frac{1}{2} \delta \theta^2 - \frac{1}{2} \epsilon z \theta
\]  

where \( z \) is the deviation from the vertical equilibrium, \( \theta \) describes the angle of rotation from equilibrium, \( I \) is the moment of inertia for rotation around the center of the spring, while \( k, \delta \) and \( \epsilon \) are constants.

a) [3p] Write down Lagrange’s equations.

b) [3p] Find the frequencies of the normal modes.

5. [4p] In so-called \( \varphi^4 \) field theory the Lagrangian density can be written

\[
\mathcal{L}(\varphi, \partial^\mu \varphi) = \frac{1}{2} [\partial^\mu \varphi \partial_\mu \varphi - m^2 \varphi^2] - \frac{1}{4} \lambda \varphi^4,
\]

where \( \varphi \) is a scalar field, \( m \) is the mass of the field and \( \lambda \) is a constant. The action is given by

\[
\int \mathcal{L}(\varphi, \partial^\mu \varphi) dx^0 dx^1 dx^2 dx^3.
\]

Consider \( \varphi(x^\mu) \) as a generalized coordinate and derive the Lagrange equation.

GOOD LUCK!