Nuclear Physics B116 (1976) 301–316

RESCATTERING EFFECTS IN THE DECAY OF THE A_1

Gösta GUSTAFSON and Carsten PETERSON
Department of Theoretical Physics, University of Lund, Lund, Sweden ★

Received 25 June 1976
(Revised 16 September 1976)

Rescattering effects in the 3π system in the A_1 region are studied with a method, which fulfils unitarity and has the proper analytic structure. Fairly small effects are obtained which confirm the result found in amplitude analyses, neglecting rescattering corrections, that the phases of the $J^P = 1^+$ amplitudes vary only very little over the A_1 region. Similar results hold for the Q enhancement.

1. Introduction

The existence of a $J^{PC} = 1^{++}$ meson is predicted by the quark model with $SU(6) \otimes O(3)$ symmetry, which has been very successful in classifying the hadrons. Also current algebra predicts chiral symmetric spectral sum rules [1]. If these are saturated only by the ρ-meson and a $J^{PC} = 1^{++}$ meson denoted by A_1, it is found that $m_{A_1} \approx 1100$ MeV. The enhancement at 1100 MeV observed in the diffractively produced 3π-system in $\pi p \to \pi\pi\pi p$ reactions [2] has for a long time been considered as a candidate for the A_1. If the A_1 is a pure Breit-Wigner resonance, then the phase of the production amplitude should increase by 90° as the mass is varied through the resonance region. Existing partial-wave analyses show essentially no phase variation of the dominant $J^P = 1^+$ amplitude when measured relative to the other partial waves [2]. Thus the nature of the A_1 is much questioned.

It has been suggested that the A_1 structure is a pure kinematical effect caused by double Regge exchange, one of the trajectories being the pion [3]. This so-called reggeized Deck model provides a peak about 200 MeV above the $\rho\pi$ threshold with a width ~400 MeV. Even if this peak strongly resembles the A_1 enhancement, it is evident that the model does not fit the data on the high-mass side of the A_1 [3].

Hybrid models have been suggested [4] where one part, corresponding to production of a real A_1 resonance, is added to a large diffractive Deck background in such a way that no phase variation is observed.

★ Supported by the Swedish Atomic Research Council.

A more clear evidence would be if A_1 could be observed in charge-exchange reactions like $\pi^+ p \to \pi^+\pi^-\pi^0\Delta^{++}$ and $\pi^- p \to \pi^+\pi^-\pi^0 n$ [5] in which, however, no A_1 signal is observed.

Also recent investigations of the spectral sum rules mentioned above [6], show that these cannot be saturated by the ρ-meson and the 1^+ wave, extracted from the diffractively produced 3π system. In that case the 1^+ enhancement is not the chiral partner of ρ and should therefore not be identified with the A_1.

The situation is somewhat similar for the $K\pi\pi$ system produced in $K^\mp p \to K^\mp\pi^+\pi^- p$ where one observes a sharp rise near threshold and a rapid drop at $M(K\pi\pi) \simeq$ 1300 MeV in the 1^+ wave [7]. Also in this Q phenomenon the situation concerning resonance characteristic phase variation is unclear. However, the nature of this enhancement is expected to be more complicated since the quark model predicts two $J^P = 1^+$ resonances in this region, belonging to the same SU(3) multiplets as A_1 and B respectively.

It has been suggested that final state interactions could make the phase-shift analyses in the diffractive case unreliable and in this paper we want to estimate the effect of rescattering between the produced pions.

The partial-wave analyses [2] are based on the isobar model. In this model it is assumed that the production amplitude for the three-particle state is dominated by terms, where two particles form a resonance, which together with the third particle can be treated as a quasi-two-body system.

With resonances in the pairs (23) and (13) of particles, the partial-wave analyses use the following ansatz for the production amplitude

$$F(s, s_1, s_2, s_3) = C_1(s)T_1(s_1) + C_2(s)T_2(s_2) \tag{1}$$

with $s = (k_1 + k_2 + k_3)^2$, $s_i = (k_j + k_l)^2$.

We here assume that all particles move in relative S-waves. In other cases we have to include threshold factors and dependence upon the angles. $T_1(s_1)$ is the scattering amplitude for the (2,3) pair and $C_1(s)$ is the amplitude for production of particle 1 and a resonance in the (2,3) pair. An important assumption is that the amplitudes C_i depend upon s only.

However, this ansatz violates unitarity because unitarity implies the presence of diagrams like those in fig. 1, which have a more complicated dependence on the subenergies s_i and the total energy s.

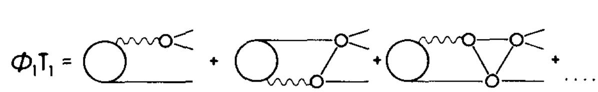

Fig. 1. Rescattering diagrams included in the first term in eq. (2), where only rescatterings with the last scattering in the pair (2,3) are taken into account.

It is possible to write the amplitude in the form [8]

$$F(s,s_1,s_2) = \phi_1(s,s_1)T_1(s_1) + \phi_2(s,s_2)T_2(s_2)\,, \tag{2}$$

where the amplitudes ϕ_i are more complicated functions which depend on both s and s_i. The first term in eq. (2) corresponds to all diagrams where the last scattering occurs in the pair (2,3). Unitarity implies the following discontinuity relation for ϕ_1:

$$\phi_1(s,s_{1+}) - \phi_1(s,s_{1-}) = i\rho_1(s_1)\int_{-1}^{1} dx_1\phi_2(s,s_2)T_2(s_2)\,, \tag{3}$$

where $\rho_1(s_1)$ is the two-particle phase space:

$$\rho_1(s_1) = \frac{\sqrt{\lambda(s,m_2^2,m_3^2)}}{16\pi s_1}\,, \tag{4a}$$

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc\,, \tag{4b}$$

x_1 is the cosine of the angle between particle 1 and 2 in the (2,3) c.m.s.

Since $\rho_1(s_{1+}) - \rho_1(s_{1-}) = 2\,\rho_1(s_{1+})$ it is possible to fulfil eq. (3) with a K-matrix ansatz of the form

$$\phi_1(s,s_1) = C_1'(s) + \tfrac{1}{2}\,i\rho_1(s_1)\int_{-1}^{1} dx_1\,\phi_2(s,s_2)T_2(s_2)\,, \tag{5}$$

where the $C_i'(s)$ are interpreted as kinds of "bare" production amplitudes, which are modified by subsequent final state interactions. This ansatz has been used by Ascoli and Wyld [9] in a new analysis of the 3π data. This analysis gave very large rescattering corrections. However, the fit to the data turned out to be worse than for the original simpler analysis, and no phase variation of the 1^+ amplitudes was obtained.

It has been pointed out by Aitchison and Golding [10] that this K-matrix formalism gives spuriously large rescattering effects, because although the amplitudes fulfil unitarity exactly, they do not have the correct analyticity properties. This is due to the fact that the contribution from the integral in eq. (4) has a singularity close to the physical region, whereas the amplitudes corresponding to the rescattering diagrams in fig. 1 have the singularity on another Riemann sheet [11].

In this paper we discuss another ansatz which also contains functions $C_i(s)$ which are interpreted as "bare" production amplitudes followed by rescattering in the final state. However, this ansatz both fulfils the unitarity conditions and has the correct analyticity properties.

The outline of this paper is as follows: In sect. 2 our method of treating rescattering corrections is presented and sect. 3 contains our calculations and results. Finally, a short summary and discussion is included in sect. 4.

2. Unitary treatment of rescattering corrections

In order to illustrate our method we first consider the S-matrix for three-body scattering in the case with a resonance only in pair (23). For simplicity we here neglect the angular momentum and study only pure S-waves. In the isobar model we now make the following ansatz for the S-matrix (see fig. 2). (Our normalization is $\langle k|k'\rangle = (2\pi)^3\, 2E\, \delta(k-k')$

$$S = S_\mathrm{d} + iF\,, \tag{6}$$

where

$$S_\mathrm{d} = 1\!\!1 + i(2\pi)^7\, 2E_1\, \delta^{(3)}(k_1^\mathrm{i} - k_1^\mathrm{f})\delta^{(4)}(p^\mathrm{i} - p^\mathrm{f})T_1(s_1)\,, \tag{7}$$

$$F = \delta^{(4)}(p^\mathrm{i} - p^\mathrm{f})\frac{2\pi^2}{\sqrt{M\gamma}}\, T_1(s_1^\mathrm{i})\, A(s)\frac{2\pi^2}{\sqrt{M\gamma}}\, T_1(s_1^\mathrm{f})\,. \tag{8}$$

$T_1(s_1)$ is the two-particle scattering amplitude in the pair (23) for which we assume the Breit-Wigner form

$$T_1(s_1) = \frac{M\gamma}{M^2 - s_1 - iM\gamma\rho_1(s_1)}\,, \tag{9}$$

where M and $\Gamma \equiv \gamma\rho_1(s_1)$ are the mass and width of the resonance. The superscripts i and f denote the initial and the final state respectively.

$A(s)$, which is assumed to have no subenergy dependence, is interpreted as the resonance particle scattering amplitude. The normalization factor $(2\pi^2/\sqrt{M\gamma})^2$ is introduced in order to make $A(s)$ behave as an ordinary two-particle scattering amplitude in the limit when the width of the resonance goes to zero. The normalization factor is divided into two parts in order to more easily see the generalization to the case when the angular momentum l of the resonance is different from 0. In such cases γ must have the threshold behaviour $\gamma \sim q_1^{2l}$, where q_1 is the momentum in the c.m.s of particles 2 and 3. Thus the factor $(2\pi^2/\sqrt{M\gamma})T_1$ in eq. (7) will get the proper threshold behaviour $\sim (q_1^\mathrm{f})^l$.

T_1 fulfills the two-particle unitarity relation

$$\mathrm{Im}\, T_1 = \rho_1 |T_1|^2\,, \tag{10}$$

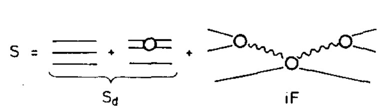

Fig. 2. Diagrammatic representation of the isobar approximation in the case of a resonance in only one pair of particles.

which implies the following relations

$$S_d \cdot S_d^+ = 1\!\!1 , \tag{11a}$$

$$T_1 \cdot S_d^+ = T_1^* , \tag{11b}$$

$$S_d \cdot T_1^* = T_1 . \tag{11c}$$

3-particle unitarity for S gives

$$1 = SS^+ = S_d S_d^+ + \delta^{(4)}(p^i - p^f) \frac{2\pi^2}{\sqrt{M\gamma}} T_1^*(s_1^i) [i(A(s) - A^*(s)) + 2A(s) I(s) A^*(s)]$$

$$\times \frac{2\pi^2}{\sqrt{M\gamma}} T_1(s_1^f) , \tag{12}$$

which implies

$$\mathrm{Im}\, A(s) = \mathrm{I}(s)\, |A(s)|^2 , \tag{13}$$

where

$$I(s) = \frac{1}{(2\pi)^9} \frac{1}{2} \int \frac{d^3k_1}{2E_1} \frac{d^3k_2}{2E_2} \frac{d^3k_3}{2E_3} \frac{4\pi^4}{M\gamma} |T_1(s_1)|^2 \delta^{(4)}(p^i - p^f)$$

$$= \int ds_1 \rho(s, s_1) \frac{1}{\pi M\gamma} \rho_1(s_1)\, |T_1(s_1)|^2 . \tag{14}$$

$\rho(s, s_1)$ is the phase space for two particles with masses $\sqrt{s_1}$ and m_1, and if their c.m.s. energy is $\sqrt{s}$

$$\rho(s, s_1) = \frac{\sqrt{\lambda(s, s_1, m_1)}}{16\pi s} . \tag{15}$$

The range of integration in eq. (14) is determined by those values of s_1 which are kinematically allowed.

Eq. (13) resembles very much the 2-particle unitarity relation in eq. (10). In particular when the width of the resonance goes to zero we have

$$\frac{1}{\pi M\gamma} \rho_1(s_1)\, |T_1(s_1)|^2 \to \delta(s_1 - M^2) , \tag{16}$$

yielding

$$\mathrm{Im}\, A(s) = \rho(s, M^2)\, |A(s)|^2 \tag{17}$$

in exact analogy with eq. (10).

We also notice that the continuation of $A(s)$ into the lower-half s-plane has a cut starting at $(M_R + m_1)^2$. Here M_R^2 is the position of the pole of $T_1(s_1)$ in the complex

s_1 plane. M_R^2 is the solution to the equation

$$M^2 - M_R^2 - iM\rho_1(M_R^2) = 0\,, \tag{18}$$

which is of 3rd degree. If the resonance is not too broad M_R can be well approximated by the expression

$$M_R^2 \approx M^2 - iM\gamma\rho_1(M^2)\,. \tag{19}$$

Similarly the pole residue of $T_1(s_1)$, which we call r, is well approximated by $-M\gamma$,

$$r \approx -M\gamma\,. \tag{20}$$

It is possible to show [12] that the discontinuity of A over the cut is given by

$$\frac{1}{2i}(A^U - A^L) = A^U(I^U(s) - I^L(s))A^L\,, \tag{21}$$

where the indices U and L indicate the continuations to the upper and lower sides of the cut respectively. I^U and I^L differ because in eq. (14) the integration contours in the s_1 plane lie on different sides of the pole in T_1. Thus $I^U - I^L$ is given by the residue at this pole, and using eq. (10) we find:

$$I^U(s) - I^L(s) = \rho(s, M_R^2)\,\frac{-r}{M\gamma} \approx \rho(s, M_R^2)\,. \tag{22}$$

Thus

$$\frac{1}{2i}(A^U - A^L) \approx \rho(s, M_R^2)A^U A^L\,, \tag{23}$$

which again is analogous to eq. (10).

It is possible to renormalize A slightly in order to make eq. (23) an exact equality. This is done in ref. [12] for the simpler static, non-relativistic case, where eq. (18) is of 2nd degree.

In the case of resonances in two channels the division of the S-matrix into connected and disconnected parts is shown in fig. 3. However, a simple isobar ansatz for the disconnected part, S_d, is not possible because unitarity implies a more complicated relation for S_d. Unitarity means

$$1 = SS^+ = S_d S_d^+ + S_d S_c^+ + S_c S_d^+ + S_c S_c^+\,. \tag{24}$$

In this case $S_d S_d^+$ is not equal to unity but contains contributions from the exchange of particle (3) which give rise to singularities in the physical region. Such terms must

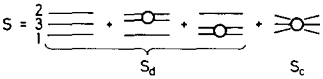

Fig. 3. Division of the S-matrix into connected and disconnected parts.

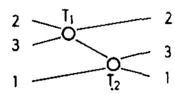

Fig. 4. Exchange diagram contained in $S_d S_d^+$ and S_c.

hence be present in S_c (see fig. 4). Since these rescattering terms are determined by the two-particle scatterings and forced upon us by unitarity, we want to treat them separately and write the S-matrix in the form (see fig. 5)

$$S = S_F + iF . \tag{25}$$

S_F contains all repeated scatterings between two particles. (In ref. [9] only this part is studied.) Thus

$$S_F = 1\!\!1 + i(2\pi)^7 \delta^{(4)}(p^i - p^f)\{2E_1 \delta^{(3)}(k_1^i - k_1^f) T_1(s_1) + 2E_2 \delta^{(3)}(k_2^i - k_2^f) T_2(s_2)\}$$

$$- \tfrac{1}{2} i(2\pi)^4 \delta^{(4)}(p^i - p^f) \left\{ T_1(s_1^i) \frac{1}{-(k_2^i + k_3^i - k_2^f)^2 + m_3^2 - i\epsilon} T_2(s_2^f) \right.$$

$$\left. + T_2(s_2^i) \frac{1}{-(k_1^i + k_3^i - k_2^f)^2 + m_3^2 - i\epsilon} T_1(s_1^f) \right\} + \ldots\ldots \tag{26}$$

It is easy to show that S_F fulfils unitarity exactly

$$S_F S_F^+ = 1 . \tag{27}$$

The remainder term F contains the pure three-particle interaction. This is assumed to be dominated by scattering between one particle and a resonance of the other two. We make the following ansatz

$$F = \sum_{i,j}^{2} \psi_i A_{ij} \psi_j , \tag{28}$$

which contains 4 terms of the form shown in fig. 6. The functions $A_{ij}(s)$ are assumed to depend only on s and not on the subenergies and can be interpreted as resonance-particle scattering amplitudes. ψ_i corresponds to the production of a resonance in the

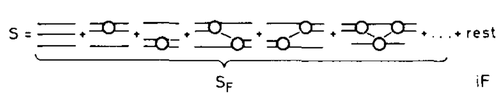

Fig. 5. Division of the S-matrix into one part S_F, which contains repeated 2-particle scattering and a remainder.

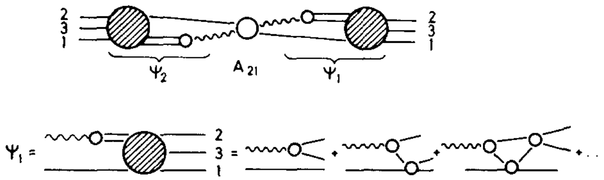

Fig. 6. (a) Diagrammatic form of terms in F. (b) Expansion of ψ_1 according to eq. (29).

pair (j,k), followed by all possible final state interactions. ψ_1 is e.g. given by

$$\psi_1(s,s_i^{\mathrm{f}}) = \frac{(2\pi)^2}{i}\int \mathrm{d}R_3^{\mathrm{i}}\,\frac{1}{\rho_1(s_1^{\mathrm{i}})}\,\frac{1}{\sqrt{M\gamma}}\left[\begin{array}{l}\text{those parts in } S_{\mathrm{F}}(s,s_i^{\mathrm{i}},s_i^{\mathrm{f}}) \text{ which have}\\ \text{the first scattering in the pair (2,3)}\end{array}\right], \tag{29}$$

where $\mathrm{d}R_3^{\mathrm{i}}$ is the 3-body phase-space element

$$\mathrm{d}R_3^{\mathrm{i}} = \frac{1}{(2\pi)^9}\,\frac{\mathrm{d}^3k_1^{\mathrm{i}}}{2E_1}\,\frac{\mathrm{d}^3k_2^{\mathrm{i}}}{2E_2}\,\frac{\mathrm{d}^3k_3^{\mathrm{i}}}{2E_3}\,. \tag{30}$$

In fig. 6 ψ_1 is expressed in a series expansion

$$\psi_1 = \sum_{k=1}^{\infty} \psi_1^{(k)}\,, \tag{31}$$

where the first terms are given by

$$\psi_1^{(1)} = \frac{2\pi^2}{\sqrt{M\gamma}}\,T_1(s_1^{\mathrm{f}})\,, \tag{32a}$$

$$\psi_1^{(2)} = \tfrac{1}{2}(2\pi)^4\int \mathrm{d}R_3^{\mathrm{i}}\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\,\frac{1}{\rho_1(s_1)}\,\frac{2\pi^2}{\sqrt{M\gamma}}\,T_1(s_1^{\mathrm{i}})\,\frac{1}{-(k_2^{\mathrm{i}}+k_3^{\mathrm{i}}-k_2^{\mathrm{f}})^2+m_3^2-i\epsilon}\,T_2(s_2^{\mathrm{f}}) \tag{32b}$$

It is possible to show that

$$\psi_k S_{\mathrm{F}}^{+} = \psi_k^{*}\,, \tag{33a}$$

$$S_{\mathrm{F}}\,\psi_k^{*} = \psi_k\,, \tag{33b}$$

which gives the unitarity relation

$$1 = SS^{+} = \sum_{\substack{k,j\\ l,m}} \psi_k\,[i(A_{kj}-A_{jk}^{*}) + 2A_{kl}I_{lm}A_{jm}^{*}]\,\psi_j^{*} = 0\,, \tag{34}$$

where

$$I_{lm}(s) = \tfrac{1}{2}\int \mathrm{d}R_3\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\,\psi_l(s,s_l)\,\psi_m^{*}(s,s_m)\,. \tag{35}$$

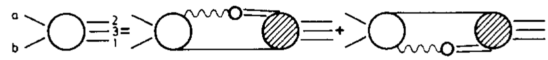

Fig. 7. Diagrammatic form of the ansatz in eq. (37).

Thus our ansatz fulfils unitarity if the amplitudes A_{kj} satisfy the equation

$$-i(A_{kj} - A_{jk}^*) = 2 \sum_{l,m} A_{kl} I_{lm} A_{jm}^* \ . \tag{36}$$

Here all the terms depend upon s only and these relations resemble very much the ordinary unitarity relations for coupled 2-particle channels. An essential difference is, however, that we also have non-diagonal terms in the phase-space factors I_{lm}.

Similar results could be obtained with an ansatz where, in the expression for S_F eq. (26), we allowed a dependence of T_i on the mass of an intermediate particle when it is taken off-shell. However, we assume that the rescattering terms are dominated by the singularities which are needed from unitarity, and we therefore neglect this dependence. In a way, our ansatz is the simplest possible which fulfils uniatrity and analyticity requirements.

We now turn to the production reaction a + b → 1 + 2 + 3 and make the following ansatz for the amplitude (cf. fig. 7)

$$T = \mathcal{C}_1(s)\, \psi_1(s,s_1) + \mathcal{C}_2(s)\, \psi_2(s,s_2)\ . \tag{37}$$

Let $B(s)$ denote the amplitude for the elastic reaction a + b → a + b. Then we can show that this ansatz fulfils unitarity if the amplitudes A, B and $\mathcal{C}$ fulfil the relations

$$-\frac{i}{2}(B - B^*) = B\rho' B^* + \sum_{kl} \mathcal{C}_k I_{kl} \mathcal{C}_l^* \ , \tag{38a}$$

$$-\frac{i}{2}(\mathcal{C}_k - \mathcal{C}_k^*) = B\rho' \mathcal{C}_k^* + \sum_{ij} \mathcal{C}_j I_{ji} A_{ki}^* \ , \tag{38b}$$

$$-\frac{i}{2}(A_{kl} - A_{lk}^*) = \mathcal{C}_k \rho' \mathcal{C}_l^* + \sum_{ij} A_{kj} I_{ji} A_{li}^* \ , \tag{38c}$$

where ρ' is the phase-space factor for system ab.

From these equations we see that a resonance which couples to the systems ab and 123 should show up as a pole in B, $\mathcal{C}_i$ and A_{ij}. We also note that if the rescattering effects were so strong that the series expansion diverges and S_F has a pole, then it follows from eq. (38) that $\mathcal{C}_i$ and A_{ij} have zeroes at this point. (We here assume that $\mathcal{C}_i$ is regular in the upper half plane and thus $\mathcal{C}_i^*$ is regular in the lower half plane.) This also implies that B is regular. Thus the elastic amplitude for scattering of particles a and b is regular although the system ab couples to 123. Hence such a pole in S_F would not behave as an ordinary resonance pole. This point is further discussed in ref. [12].

If particles 1 and 2 are identical, all A_{ij} must be equal. With the notation $A = A_{ij}$, $\psi = (\psi_1 + \psi_2)$, $\mathcal{C} = \mathcal{C}_i$. We have [12]

$$F = \psi A \psi \,, \tag{39}$$

$$T_{\mathrm{ab}\to 123} = \mathcal{C}\,\psi \,, \tag{40}$$

and with $I = \Sigma I_{kl}$ we find again eq. (38), only with the change that A and C are not matrices but simple amplitudes.

The generalization to the case when there are more than one resonance in one pair of particles (e.g. ϵ and ρ in the A_1) is straightforward.

3. Calculations and results

The test of the resonance status of the diffractively produced A_1 is based on the phase variation. As discussed after eq. (38) a resonance should show up as a pole in the amplitudes $\mathcal{C}_i$ which therefore should have a phase increase of 90° over the width of the resonance. $\mathcal{C}_i$ are determined by a fit of the data to the expression in eq. (37). Let us study the expansions of ψ_i in eq. (31). In ref. [12] it is shown that in the non-relativistic case these expansions converge like power series for not too large values of the width compared to the kinetic energy of the resonance. In the present case of $\epsilon\pi$ and $\rho\pi$ in a 1^+ wave we also find, as will be seen below, that $\psi_1^{(2)}$ is much smaller than $\psi_1^{(1)}$. Thus we conjecture that the expansions will converge also in the present relativistic case and that higher order terms can be neglected. From eq. (30) we see that $\psi_1^{(2)}$ contains T_2 or $\psi_2^{(1)}$ as a factor. Thus, if we write

$$\psi_1^{(2)} = \lambda_1(s, s_2)\,\psi_2^{(1)} \tag{41}$$

and retain only the first rescattering corrections $\psi_j^{(2)}$ we have

$$T = \mathcal{C}_1\psi_1 + \mathcal{C}_2\psi_2 \approx \mathcal{C}_1(\psi_1^{(1)} + \lambda_1\psi_2^{(1)}) + \mathcal{C}_2(\psi_2^{(1)} + \lambda_2\psi_1^{(1)})$$

$$= (\mathcal{C}_1 + \lambda_2\mathcal{C}_2)\psi_1^{(1)} + (\mathcal{C}_2 + \lambda_1\mathcal{C}_1)\psi_2^{(1)} \,. \tag{42}$$

In the case of identical particles and resonances we have $\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2$ and eq. (42) simplifies to

$$T \approx \mathcal{C}(1 + \lambda)(\psi_1^{(1)} + \psi_2^{(1)}) \,. \tag{43}$$

The form in eq. (42) or (43) is convenient because, in the case of A_1, it will turn out that λ_i are either negligibly small or only very weakly dependent on the sub-energies. Hence the functions C_i defined by

$$C_1 = \mathcal{C}_1 + \lambda_2\,\mathcal{C}_2 \,,$$

$$C_2 = \mathcal{C}_2 + \lambda_1\,\mathcal{C}_1 \tag{44}$$

are, to a very good approximation, functions of s only. Eq. (42) now reads

$$T \approx C_1 \psi_1^{(1)} + C_2 \psi_2^{(1)} \sim C_1 T_1 + C_2 T_2 \ , \tag{45}$$

which is the form used in the non-unitarized isobar analyses. Thus we expect that these non-unitarized analyses should give a good fit to the data, which is also actually the case.

These analyses determine C_i and form eq. (44) we can determine the physically more interesting functions $\mathcal{C}_i$.

With an obvious notation we find for the case of $\rho\pi$ and $\epsilon\pi$ in 1^+ states (we only study $\rho\pi$ S-wave and $\epsilon\pi$ P-wave and neglect the small contributions for $\rho\pi$ D-wave and $f\pi$ P- and F-wave):

$$T \sim [\mathcal{C}_\rho(1 + \lambda_{\rho\rho}) + \mathcal{C}_\epsilon \lambda_{\epsilon\rho}]\ (\psi_{\rho 1}^{(1)} + \psi_{\rho 2}^{(1)})$$

$$+ [\mathcal{C}_\epsilon(1 + \lambda_{\epsilon\epsilon}) + \mathcal{C}_\rho \lambda_{\rho\epsilon}]\ (\psi_{\epsilon 1}^{(1)} + \psi_{\epsilon 2}^{(1)}) \ . \tag{46}$$

Explicit expressions for the factors λ are given in the appendix together with a discussion of threshold factors and angular dependence in the case of non-zero angular momenta. We study the case $\pi^-\pi^+\pi^-$, which can be obtained from $\rho^0\pi^-$ and $\epsilon\pi^-$.

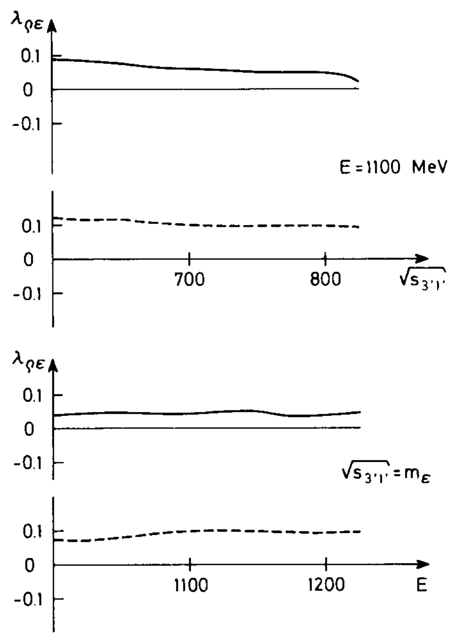

Fig. 8. (a) $\lambda_{\rho\epsilon}$ as a function of $\sqrt{s_{3'1'}}$ at fixed total energy E = 1100 MeV. (b) $\lambda_{\rho\epsilon}$ as a function of E at fixed two-particle subenergy $\sqrt{s_{3'1'}} = M_\epsilon$. Solid and dashed lines represent real and imaginary parts respectively.

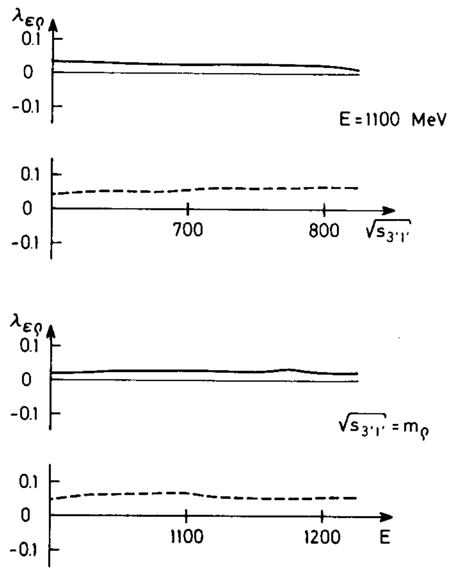

Fig. 9. $\lambda_{\epsilon\rho}$ with the same notation as in fig. 8.

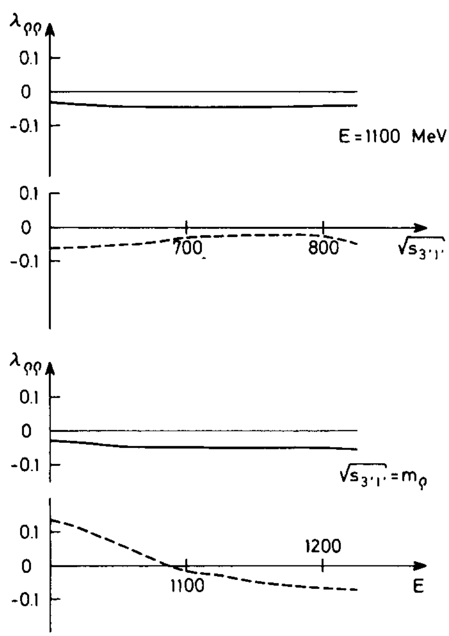

Fig. 10. $\lambda_{\rho\rho}$ with the same notation as in fig. 8.

The results for $\lambda_{\rho\epsilon}$, $\lambda_{\epsilon\rho}$ and $\lambda_{\rho\rho}$ are shown in figs. 8,9 amd 10. $\lambda_{\epsilon\epsilon}$ is approximately zero. To get the value for $\lambda_{\rho\epsilon}$ we have made the further assumption that the $(\rho\pi)^-$ system is produced in a state with $I = 1$. Such an assumption is necessary because $\epsilon\pi^-$ can be obtained from rescattering from both $\rho^0\pi^-$ and $\rho^-\pi^0$. From the figures we see that the λ factors are either small or very weakly dependent on the sub-energies, and thus $C_\rho = \mathcal{C}_\rho(1 + \lambda_{\rho\rho}) + \mathcal{C}_\epsilon \lambda_{\epsilon\rho}$ and $C_\epsilon = \mathcal{C}_\epsilon(1 + \lambda_{\epsilon\epsilon}) + C_\rho \lambda_{\rho\epsilon}$ are approximately functions of s only. The amplitudes $\mathcal{C}_\rho$ and $\mathcal{C}_\epsilon$ which determine the production of $\rho\pi$ and $\epsilon\pi$ systems before rescattering are obtained from

$$\mathcal{C}_\rho = \frac{(1 + \lambda_{\epsilon\epsilon})\, C_\rho - \lambda_{\epsilon\rho} C_\epsilon}{(1 + \lambda_{\epsilon\epsilon})(1 + \lambda_{\rho\rho}) - \lambda_{\rho\epsilon}\lambda_{\epsilon\rho}} \approx (1 - \lambda_{\rho\rho}) C_\rho - \lambda_{\epsilon\rho} C_\epsilon \,, \tag{47a}$$

$$\mathcal{C}_\epsilon = \frac{(1 + \lambda_{\rho\rho})\, C_\epsilon - \lambda_{\rho\epsilon} C_\rho}{(1 + \lambda_{\epsilon\epsilon})(1 + \lambda_{\rho\rho}) - \lambda_{\rho\epsilon}\lambda_{\epsilon\rho}} \approx (1 - \lambda_{\epsilon\epsilon}) C_\epsilon - \lambda_{\rho\epsilon} C_\rho \,. \tag{47b}$$

In the final expressions we have neglected terms of the order λ^2.

From ref. [2] we see that C_ρ and C_ϵ for the 1^+ wave have very little phase variations relative to structureless background waves, and that $C_\epsilon \sim 0.4\, e^{i \cdot 85^\circ}\, C_\rho$. From fig. 8 we see that the most important λ-factor is $\lambda_{\rho\epsilon}$ which is $\sim 0.04 + 0.10i$. Hence we conclude that also $\mathcal{C}_\rho$ and $\mathcal{C}_\epsilon$ show a smooth behaviour with $\mathcal{C}_\rho$ not much different from C_ρ but with $\mathcal{C}_\epsilon$ significantly smaller in magnitude than C_ϵ

$$\mathcal{C}_\epsilon \approx C_\epsilon(1 - (0.04 + 0.10i)/0.4\, e^{i\,85^\circ}) \approx C_\epsilon(0.74 + 0.08i) \approx C_\rho 0.3i \,. \tag{48}$$

These results rely on our conjecture that the series expansions for S_F and ψ_i converge. However, if we suppose that the series diverge and S_F and ψ_i have poles, then, as discussed after eq. (38), the unitarity relations in eq. (38) imply that $\mathcal{C}_i$ are zero at the pole position and the production amplitude $T \sim \mathcal{C}\psi$ is finite.

In this situation we would expect large deviations of T from the simple form in eq. (44) and it remains a puzzle why the non-unitarity isobar analyses give a good fit to the experimental data.

In the case of the Q phenomenon we have a very similar situation with final state interaction where pion exchange couples the states $K^*\pi$ and $\kappa\pi$ with $K\rho$ and $K\epsilon$. We have only calculated the simpler case when all three particles move in S-waves. In this case we get results similar to, although somewhat smaller than, the corresponding case with three pions.

Thus we conclude that it should be sensible to use the non-unitarized isobar model to fit the data, and that the differences between the $\mathcal{C}$ and C amplitudes should not be important.

4. Conclusions

We conclude that the rescattering corrections in our treatment are fairly small, about 10–20%. This is much smaller than the results obtained in the K-matrix formalism of Ascoli and Wyld [9], which gives a spuriously large effect as shown by Aitchison and Golding [10]. In the K-matrix formalism the triangle diagram gives a singularity close to the physical region although this singularity should be on another Riemann sheet. This spurious feature is avoided in our calculation.

This result justifies the neglection of rescattering effects in the nonunitarized isobar analyses and explains why these analyses give good fits to the data. We also deduce a relation between the amplitudes obtained in these analyses and the 'bare' amplitudes, which describe the production of resonance-particle states before rescattering.

We confirm the results that the production amplitudes in the A_1 case do not show a phase increase of 90° as expected for a normal resonance. However we cannot conclude from this that the A_1 does not exist because our result also justifies the neglection of rescattering effects in calculations like those of Morgan [4], which study the effect of a resonance producing amplitude on top of a diffractive background.

A more crucial test of the existence of A_1 should be in non-diffractive reactions. Also here rescattering effects should be small according to our analysis.

Similar conclusions also apply in the Q region.

Appendix

The functions λ in eq. (41) are defined as the ratios between the graphs in fig. 11 and in the cases considered in this investigation they are given by

$$\lambda_{\rho\epsilon} = \frac{\psi_{1\rho\epsilon}^{(2)}}{\psi_{2\epsilon}^{(1)}} , \qquad \lambda_{\rho\rho} = \frac{\psi_{1\rho\rho}^{(2)}}{\psi_{2\rho}^{(1)}} , \qquad \lambda_{\epsilon\rho} = \frac{\psi_{1\epsilon\rho}^{(2)}}{\psi_{2\rho}^{(1)}} , \qquad \lambda_{\epsilon\epsilon} = \frac{\psi_{1\epsilon\epsilon}^{(2)}}{\psi_{2\epsilon}^{(1)}} , \tag{A.1}$$

with an obvious notation.

Let us first study $\lambda_{\epsilon\epsilon}$. For the production of an $\epsilon\pi$ P-wave system with $J_z = 0$ we have (eq. (32a) and fig. 11a):

$$\psi_{2\epsilon}^{(1)} = \sqrt{3}\sqrt{\tfrac{2}{3}}\frac{2\pi^2}{\sqrt{M_\epsilon \gamma_\epsilon}} \cos\theta_{1'z}\, T_{2\epsilon}(s_2) , \tag{A.2}$$

where $\theta_{1'z}$ is the angle between $\boldsymbol{k}_{1'}$ and $\boldsymbol{e}_z$ in the c.m.s. The factor $\sqrt{3}$ originates

Fig. 11. Graphs corresponding to the first and second terms in the axpansions of ψ_1 and ψ_2 respectively. Notations are the same as in ref. [9].

from $\sqrt{2L+1}$ and $\sqrt{\frac{2}{3}}$ is an isospin factor for $\epsilon \to \pi^+\pi^-$ and $T_{2\epsilon}(s_2)$ is given by eq. (9). $\psi^{(2)}_{1\epsilon\epsilon}$ is given by the expression (cf. eq. (32b) and fig. 11b)

$$\psi^{(2)}_{1\epsilon\epsilon} = \tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,\frac{(2\pi)^4}{2}\int \mathrm{d}R^{\mathrm{i}}_3\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\frac{2\pi^2}{\sqrt{M_\epsilon\gamma_\epsilon}}\,\frac{1}{\rho_1(s_1)}\;T_{1\epsilon}(s^{\mathrm{i}}_1)\,P(k_{3''})\,T_{2\epsilon}(s^{\mathrm{f}}_2)\,, \tag{A.3}$$

where $P(k_{3''}) = (-k^2_{3''} + m^2 - i\epsilon)^{-1}$ is the propagator for the exchanged pion. Again the factor $\frac{2}{3}\sqrt{\frac{2}{3}}$ is due to isospin Clebsch-Gordan coefficients.

For the production of a $\rho\pi$ S-wave with $J_z = 0$ we have

$$\psi^{(1)}_{2\rho} = \sqrt{3}\,\frac{2\pi^2}{\sqrt{M_\rho\beta_\rho}}\,\frac{\sum_\lambda(-\epsilon(\lambda)k_{2'})}{M^2_\rho - s_1 - iM_\rho\beta_\rho q^2_1}\,, \tag{A.4}$$

where the constant β_ρ is defined so that $\gamma_\rho = \beta_\rho q^2_1$ and the ρ-meson width is given by

$$\Gamma_\rho(s_1) = \beta_\rho\,\rho_1(s_1)q^2_1\,. \tag{A.5}$$

q_1 is the c.m.s. momentum of the particles 2 and 3, and $\sqrt{3}$ is a factor $\sqrt{2S+1}$. We notice that $\psi^{(1)}_{2\rho}$ has the threshold behaviour $\psi^{(1)}_{2\rho} \sim q_1$ for small q_1, in accordance with the discussion after eq. (9). $\psi^{(2)}_{1\rho\epsilon}$ is given by

$$\psi^{(2)}_{1\rho\epsilon} = 2\sqrt{3}\,\frac{2}{3}\,\frac{(2\pi)^4}{2}\int \mathrm{d}R^{\mathrm{i}}_3\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\,\frac{1}{\rho_1(s_1)}\;\psi^{(1)}_{1\rho}(s,s^{\mathrm{i}}_1)\,P(k_{3''})\,T_{2\epsilon}(s^{\mathrm{f}}_2)\,. \tag{A.6}$$

The factor 2 originates from an assumption that the original $\rho\pi$ system is produced in a state with $I = 1$. In this case both the processes $\rho^0\pi^- \to \epsilon\pi^-$ and $\rho^-\pi^0 \to \epsilon\pi^-$ contribute with equal strength. Thus, this factor has to be changed if it is found that states with $I = 2$ contribute significantly.

For the process $\rho\pi(S) \to \rho\pi$, corresponding to fig. 11b, we get the expressions

$$\sqrt{3}\,\frac{(2\pi)^4}{2}\int \mathrm{d}R^{\mathrm{i}}_3\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\,\frac{1}{\rho_1(s_1)}\;\psi^{(1)}_{1\rho}(s,s^{\mathrm{i}}_1)\,P(k_{3''})\,\frac{I}{M^2_\rho - s_2 - iM_\rho\beta_\rho\rho_2(s_2)q^2_2} \tag{A.7}$$

where I is given by

$$I = -3M_\rho\beta_\rho\left[k_1k_{1'} - \frac{k_1(k_{3''}+k_1)(k_{2'}+k_{1'})k_{1'}}{s_{2'3'}}\right]. \tag{A.8}$$

The expression in eq. (A.8) corresponds to both S- and D-waves. In order to find the S-wave contribution we make the decomposition

$$\sum_\lambda D^1_{0\lambda}\epsilon^{in}_\lambda k_{2'}\cdot I = A\sum_\lambda D^1_{0\lambda}\epsilon^{\mathrm{out}}_\lambda k_{1'} + B\Big(\sum_{\lambda=\pm1} D^1_{0\lambda}\epsilon^{\mathrm{out}}_\lambda k_{1'} - 2D^1_{00}\epsilon^{\mathrm{out}}_0 k_{1'}\Big) \tag{A.9}$$

and identify the quatities A and B. Here the first term represents an S-wave and the second a D-wave. Thus the rescattering $\rho\pi(S) \to \rho\pi(S)$ corresponds to the following expression

$$\psi^{(2)}_{1\rho\rho} = \sqrt{3}\,\frac{(2\pi)^4}{2}\int \mathrm{d}R^{\mathrm{i}}_3\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\,\frac{1}{\rho_1(s_1)}\,\psi^{(1)}_{1\rho}(s,s^{\mathrm{i}}_1)\,P(k_{3''})\,\frac{A}{M^2_\rho - s_2 - iM_\rho\,\beta_\rho\,\rho_2(s_2)q^2_2}$$

$$\times \sum_\lambda D^1_{0\lambda}\epsilon^{\mathrm{out}}_\lambda k_{1'}\,. \qquad \text{(A.10)}$$

Finally for $\epsilon\pi(P) \to \rho\pi(S)$ we have an expression analogous to eq. (A.7) and again we make a decomposition similar to the one in eq. (A.9). We identify the factor $A_{\epsilon\rho}$ which corresponds to an S-wave $\rho\pi$ state and we thus obtain

$$\psi^{(2)}_{1\epsilon\rho} = \sqrt{3}\,\sqrt{\tfrac{2}{3}}\,\frac{(2\pi)^4}{2}\int \mathrm{d}R^{\mathrm{i}}_3\,\delta^{(4)}(p^{\mathrm{i}}-p^{\mathrm{f}})\,\frac{2\pi^2}{\sqrt{M_\epsilon\gamma_\epsilon}}\,\frac{1}{\rho_1(s_1)}\,T_{1\epsilon}(s_1)\,P(k_{3''})$$

$$\times \frac{A_{\epsilon\rho}}{M^2_\rho - s_2 - iM_\rho\,\beta_\rho\,\rho_2(s_2)q^2_2}\sum_\lambda D^1_{0\lambda}\epsilon^{\mathrm{out}}_\lambda k_{1'}\,. \qquad \text{(A.11)}$$

The resonance parameters used in the calculations are M_ρ = 770 MeV, Γ_ρ = 140 MeV, M_ϵ = 700 MeV and Γ_ϵ = 400 MeV.

References

[1] S. Weinberg, Phys. Rev. Letters 18 (1967) 507.
[2] Yu.M. Antipov et al., Nucl. Phys. B63 (1973) 153;
G. Ascoli et al., Phys. Rev. D7 (1973) 669.
[3] G. Ascoli et al., Phys. Rev. D8 (1973) 3894;
G. Ascoli et al., Phys. Rev. D9 (1974) 1963.
[4] M.G. Bowler and M.A.V. Game, Oxford University preprint 38/74 (1974);
D. Morgan, Rutherford Lab. preprint RL-75-100.
[5] M.J. Emms et al., Charge-exchange production of A_1 in 4 GeV/c π^+d interactions. F1-14 contr. to the Palermo conference;
N.R. Stanton, Proc. Summer Symposium at ANL 1976, p. 19.
[6] M.S. Chen, G.L. Kane and J. Krisch, Phys. Rev. D13 (1976) 1499.
[7] G. Otter et al., Nucl. Phys. B93 (1975) 365;
G. Otter et al., Nucl. Phys. B106 (1976) 77;
G.W. Brandenburg et al., Phys. Rev. Letters 36 (1976) 703,706.
[8] C. Lovelace, Phys. Rev. B135 (1964) 1225.
[9] G. Ascoli and H.W. Wyld, Phys. Rev. D12 (1975) 43.
[10] I.J.R. Aitchison and R. Golding, Phys. Letters 59B (1975) 288.
[11] C. Schmid, Phys. Rev. 154 (1967) 1363.
[12] G. Gustafson, Nucl. Phys. B63 (1973) 325.