1. **Saving energy** (8p)

A ball of radius $b$, consisting of a dielectric substance with $\epsilon_r = 2$, has a small spherical hole of radius $a < b$ at its center. In the middle of the hole a point charge $Q$ is placed.

a) [1p] In each of the three domains ($r < a$, $a < r < b$, $r > b$), find the electric displacement field, $D(r)$.

b) [2p] Show that the electric field $E(r)$ in three domains is given by, respectively

$$E(r) = \frac{Q}{4\pi \epsilon_0 r^2}, \quad \frac{Q}{4\pi \epsilon_r \epsilon_0 r^2}, \quad \frac{Q}{4\pi \epsilon_0 r^2}.$$

c) [1p] Calculate the total surface charge at $r = b$ and $r = a$, respectively, from the discontinuity in $E_r$. ($\pm Q/2$.)

d) [2p] Use the $E$-field from (b) to calculate the electrostatic potential $V$ in each domain. (Demand that $V$ be continuous, and that $V \to 0$ for $r \to \infty$.)

e) [1p] Calculate the difference $\Delta W$ in the total electrostatic energy (the energy stored in the fields) between the cases with and without the dielectric.

f) [1p] Comment on the sign of $\Delta W$. Which of the cases is the energetically more favourable?

2. **Potentially simple** (6p) When a varying current $I_1$ is sent through a closed loop $C_1$, it will induce an emf $E_2$ in another, separate loop $C_2$, according to the formula $E_2 = -\partial_t \Phi_{21}$, where $\Phi_{21}$ is the magnetic flux through the second loop from the field generated by the current in the first loop. If the current variation is slow, one can calculate the flux as if static, and $\Phi_{21}$ will be proportional to $I_1$.

a) [2p] The magnetic flux $\Phi$ through a loop $C$ can be conveniently expressed in terms of the vector potential $A$, using Stokes’ theorem. Show that $\Phi = \oint_C A \cdot dl$.

b) [3p] In Lorenz gauge, the vector potential $A_1$ from a slowly varying line current $I_1$ in $C_1$ can be expressed as

$$A_1(r_2) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{dl_1}{|r_2 - r_1|}.$$

Use this to show that the flux $\Phi_{21}$ above is proportional to $I_1$, i.e.

$$\Phi_{21} = M_{21} I_1,$$

where $M_{21}$ is a mutual inductance, that only depends on the geometries and relative positions of the loops. Give the resulting expression for $M_{21}$ as a double loop integral.

c) [1p] Show that the mutual inductance is symmetric, i.e. $M_{21} = M_{12}$. 
3. **Magnetic ball** (8p)
Consider a solid sphere with radius \( b \) and uniform magnetization \( \mathbf{M} = M \hat{z} \) for \( r < b \). To find out the magnetic field one can introduce a so called *magnetic scalar potential* \( \Psi \) (in analogy with the electrostatic case) as follows.

a) [2p] Use Maxwell’s equations to show that, in this particular case, the \( \mathbf{H} \)-field can be written as

\[
\mathbf{H} = -\nabla \Psi,
\]
for some scalar field \( \Psi \), and that

\[
\nabla^2 \Psi = \nabla \cdot \mathbf{M}.
\]
Argue carefully from first principles.

b) [2p] Show that the discontinuity in the \( \mathbf{H} \)-field at \( r = b \) can be written as

\[
\hat{r} \cdot (\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}) = -\hat{r} \cdot (\mathbf{M}_{\text{out}} - \mathbf{M}_{\text{in}}) = M \cos \theta.
\]

c) [2p] Use separation of variables together with (a) and (b) to show that

\[
\Psi_{r>b} = \frac{M b^3}{3} \frac{1}{r^2} \cos \theta, \quad \Psi_{r<b} = \frac{M b}{3} \cos \theta.
\]
(Hint: Also use that \( \Psi \) is finite and continuous, and that it vanishes as \( r \to \infty \).)

d) [2p] Calculate the \( \mathbf{H} \)-field and the magnetic field both inside the sphere and outside of it.

4. **The vanishing of the force** (8p)
Consider a plane wave travelling in vacuum along the \( x \)-axis with wave vector \( \mathbf{k} = k \hat{x} \). The corresponding three-vector potential \( \tilde{\mathbf{A}} \) and scalar potential \( \tilde{V} \) can, in complex notation, be chosen as

\[
\tilde{\mathbf{A}} = (\tilde{a}_x \hat{x} + \tilde{a}_y \hat{y} + \tilde{a}_z \hat{z}) e^{i(kx - \omega t)}, \quad \tilde{V} = c \tilde{a}_x e^{i(kx - \omega t)},
\]
respectively, where \( \tilde{a}_x, \tilde{a}_y \) and \( \tilde{a}_z \) are (possibly complex) constants, and \( c \) is the speed of light.

a) [1p] Show that the potentials given above fulfil the Lorenz gauge condition.

b) [2p] Show that the electric and magnetic fields are given by

\[
\tilde{\mathbf{E}} = i \omega (\tilde{a}_y \hat{y} + \tilde{a}_z \hat{z}) e^{i(kx - \omega t)}, \quad \tilde{\mathbf{B}} = i k (-\tilde{a}_x \hat{x} + \tilde{a}_y \hat{y}) e^{i(kx - \omega t)},
\]
respectively, and that \( \tilde{\mathbf{E}} \cdot \tilde{\mathbf{B}} = \mathbf{B} \cdot \mathbf{k} = \mathbf{k} \cdot \tilde{\mathbf{E}} = 0.\)

c) [2p] Show that both \( \tilde{\mathbf{E}} \) and \( \tilde{\mathbf{B}} \) are solutions to the homogeneous wave-equation.

d) [1p] Calculate the time-averaged Poynting vector.

e) [2p] Calculate the force (by the electric and magnetic fields) on a point charge \( q \) located at the origin at time \( t = 0 \) moving along the \( x \)-axis with speed \( v \). For what speed does the force vanish (irrespectively of the phases of \( \tilde{a}_y \) and \( \tilde{a}_z \))?

——— Good Luck! ————