Calculating the electrostatic potential:

We have seen that the scalar potential can be used to calculate the electric field from a charge distribution.

\[ V = \frac{1}{4\pi \varepsilon_0} \int \frac{q(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d^3\vec{r}' \]

Problems:
1. Integral may be (too) difficult
2. \( q \) may not be known in detail (conductor)

Three general methods:
1. Solving Poisson's eqn \( \nabla^2 V = -\frac{1}{\varepsilon_0} q \)
2. Method of images
3. Multipole expansion

Start by looking at Laplace eqn \( \nabla^2 V = 0 \)
- True outside localized charge distort.
- To solve it, we will need to know the boundary conditions.
Simple 1-d example: (capacitor) y two "infinite" parallel conducting plates with potential difference $V_0$, a distance $d$ apart.

Put one plate at $V=0$ (ground)

$$\frac{d^2 V}{dx^2} = 0 \Rightarrow V = ax + b$$

Define $x=0$ at $V=0$ plate boundary cond $\Rightarrow$

$$b=0, \quad a = \frac{V_0}{d}$$

$$\therefore V = \frac{V_0}{d} x \Rightarrow \vec{E} = -\frac{V_0}{d} \hat{x}$$

Now assume that we instead know the surface charge $\sigma$ on one of the plates

$$\begin{align*}
\vec{E}_x &= 0 \\
\vec{E}_y &= \sigma
\end{align*}$$

Put

$$\vec{E}_m = 0 \Rightarrow \vec{E}(x=0) = \frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \frac{dV}{dx}(x=0) = -\frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow a = -\frac{\sigma}{\varepsilon_0}$$

$$\begin{align*}
V(x=d) &= 0 \\
\therefore V &= \frac{\sigma}{\varepsilon_0} (d-x)
\end{align*}$$
Characteristics of solutions to Laplace eqn

\[ \nabla^2 V = 0 \]

in some volume with a boundary

1) the potential at one point \( \vec{r} \) is given by the average on a spherical surface surrounding it

\[ V(\vec{r}) = \frac{1}{4\pi R^2} \int V \, dS \]

"proof: - see book. Essentially it shows that the average pot. on a spherical surface from a point charge outside is the same as the pot. at the center of the sphere"

\( \Rightarrow 2) V \) has no local minima or maxima except on the boundary

\( \Rightarrow 3) V \) is smooth and continuous inside volume

\( \Rightarrow 4) V \) is uniquely determined by boundary conditions (\( V \) or \( \frac{\partial}{\partial n} V \), \( n \) normal to surface) if \( g \) is known in region of interest

proof: Suppose two different sol'n \( V_1, V_2 \)

consider \( V_3 = V_1 - V_2 \) on boundary

know:

\[ \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 \]

\[ V_3 = 0 \text{ on boundary} \]

(1-3) \( \Rightarrow \) \( V_3 \equiv 0 \) \( \therefore \) \( V_1 = V_2 \)

called **Uniqueness theorem**
Method of images:

Powerful tool to solve problems with conductors based on uniqueness theorem - leave for math. meth. of physics
Solving $\nabla^2 V = 0$ by separation of variables is useful when we know the potential or surface charge on the boundary of a region

Answer: $V(x, y, z) = \Sigma(x) \ Y(y) \ Z(z)$

If we can find a form that satisfies the boundary conditions we are done.

Important simplification

$$\frac{\partial^2}{\partial x^2} V = \left[ \frac{d^2}{dx^2} \Sigma(x) \right] \ Y(y) \ Z(z)$$

↑

cost. wrt $x$

total derivative

$$\Rightarrow \ \nabla^2 V = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V =$$

$$= \left( \frac{d^2}{dx^2} \Sigma \right) Y Z + \Sigma \left( \frac{d^2}{dy^2} Y \right) Z + \Sigma Y \left( \frac{d^2}{dz^2} Z \right)$$

dividing by $V = \Sigma Y Z$ gives

$$\frac{1}{\Sigma} \frac{d^2}{dx^2} \Sigma + \frac{1}{Y} \frac{d^2}{dy^2} Y + \frac{1}{Z} \frac{d^2}{dz^2} Z = 0$$

Sum of ordinary differential eqn's!

Only possible if

$$\frac{1}{\Sigma} \frac{d^2}{dx^2} \Sigma = C_1, \ \frac{1}{Y} \frac{d^2}{dy^2} Y = C_2, \ \frac{1}{Z} \frac{d^2}{dz^2} Z = C_3$$

↑

cost.

With $C_1 + C_2 + C_3 = 0$
Example in 2-d

Metal square gutter

\[ V(0,0) = V(a,0) = V(0,a) = V(a,a) = 0 \]

boundary conditions:

- \[ V(x=0) = V(x=a) = 0 \quad \text{for} \quad y > 0 \]
- \[ V(y=0) = V_0(x) \]
- \[ V \to 0 \quad \text{as} \quad y \to \infty \]

Ansatz: \[ V = \overline{X}(x) \overline{Y}(y) \]

Separation of variables:

- \[ \frac{d^2}{dx^2} \overline{X}(x) = C_1 \overline{X}(x) \]
- \[ \frac{d^2}{dy^2} \overline{Y}(y) = C_2 \overline{Y}(y) \]
- \[ C_1 + C_2 = 0 \]

What are \( C_1 \) and \( C_2 \)?

- \( C_1 > 0 \): \[ \overline{X}(x) = A e^{\sqrt{C_1}x} + B e^{-\sqrt{C_1}x} \]
  cannot get \( \overline{X}(0) = \overline{X}(a) = 0 \)
- \( C_1 = 0 \): \[ \overline{X}(x) = A + Bx \]
  cannot get \( \overline{X}(0) = \overline{X}(a) = 0 \)
\( C_1 < 0 \) : \( \mathbf{V}(x) = A \sin kx + B \cos kx \), \( k^2 = -C \), \( C_2 = -C_1 = k^2 \Rightarrow \mathbf{V}(y) = C e^{ky} + D e^{-ky} \)

1. \( V \to 0 \) as \( y \to \infty \) \( \Rightarrow \ C = 0 \)
2. \( V(x = 0) = 0 \) \( \Rightarrow \ B = 0 \)

\[ V = \frac{A D e^{-ky}}{C} \sin kx \]

3. \( V(x = a) = 0 \) \( \Rightarrow \ ka = n \pi \), \( n = 1, 2, \ldots \)

Label solutions with \( n \)

\[ V_n(x, y) = C_n e^{\frac{n \pi y}{a}} \sin \frac{n \pi x}{a} \]

4. Coefficients \( C_n \) determined such that

\[ \sum_{n=1}^{\infty} C_n \sin \frac{n \pi x}{a} = V_0(x) \]

Use orthogonality of \( \sin \frac{n \pi x}{a} \)

\[ \int_0^a \sin \frac{n \pi x}{a} \cdot \sin \frac{m \pi x}{a} \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{a}{2} & \text{if } n = m \end{cases} \]

\[ C_n = \frac{2}{a} \int_0^a V_0(x) \sin \frac{n \pi x}{a} \, dx \]
What about other geometries?

In spherical coordinates $(r, \theta, \phi)$

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) +$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Assume $V$.

The general solution given by spherical harmonics is a limit to the case when $V$ has no $\phi$-dependence.

Ansatz: $V(r, \theta) = R(r) \Theta(\theta)$

$$= \frac{1}{r^2} \left( \frac{d}{dr} r^2 \frac{dR(r)}{dr} \right) \Theta(\theta) + \frac{1}{r^2} R(r) \frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \Theta(\theta) \right) = 0$$

Divide with $R(r) \Theta(\theta)$ and multiply with $r^2$

$$= \frac{1}{R(r)} \left( r^2 \frac{d}{dr} \frac{dR(r)}{dr} \right) + \frac{1}{\Theta(\theta)} \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \Theta(\theta) \right) = 0$$

$$= \ell (\ell + 1) \quad \Theta(\theta) = P_\ell(\cos \theta), \quad \text{Legendre polynomials}$$

$$P_0(x) = 1, \quad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1), \ldots$$

$$\int P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell+1} \delta_{\ell \ell'} \quad \text{orthogonal on } -1 \leq \cos \theta \leq 1$$
Examples: Start with a uniform electric field $\mathbf{E} = E_0 \hat{z}$. Add an uncharged metal sphere (radius $R$) in the field. Calculate the resulting potential outside the sphere.

![Sphere with uniform electric field](image)

The charges in the sphere will move until the field inside vanishes - the sphere becomes polarized.

General solution (spherical symmetry)

$$V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos \theta)$$

Proper boundary conditions:

- We know $V(r < R, \theta) = V_0 = \text{const}$ (equid potential)
- For simplicity, take $V_0 = 0$
- Also know $V(r \to \infty, \theta) = -E_0 \hat{z} = -E_0 r \cos \theta$

As $r \to \infty$:

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = -E_0 r \cos \theta$$

$A_l = -E_0$, $A_e = 0$ for $l \neq 1$
\[ z, r = R : A_1 R \cos \theta + \sum_{l=1} B_l \frac{1}{r^{l+1}} P_l(\cos \theta) = 0 \]

\[ B_1 = -A_1 R^3, \quad B_l = 0 \quad l \neq 1 \]

\[ \therefore V(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta \]

\text{external field} \quad \text{induced charge in sphere}

\text{electric field}

\[ \vec{E} = E_r \hat{r} + E_\theta \hat{\theta} \]

\[ E_r = -\frac{\partial V}{\partial r} \bigg|_{r=R} = E_0 \left( 1 + \frac{2R^3}{r^3} \right) \cos \theta \]

\[ E_\theta = -\frac{1}{r} \frac{\partial V}{\partial \theta} \bigg|_{r=R} = -E_0 \left( 1 - \frac{R^3}{r^3} \right) \sin \theta \]

\text{induced surface charge}

\[ \sigma(\theta) = E_0 \left( E_r^\text{out} - E_r^\text{in} \right) \bigg|_{r=R} =
\]

\[ = E_0 E_0 \left( 1 + 2 \right) \cos \theta = 0 = 3 E_0 E_0 \cos \theta \]

also note that

\[ (E_\theta^\text{out} - E_\theta^\text{in}) \bigg|_{r=R} = 0 - 0 = 0 \]
Multipole expansion

method for calculating

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{G(r')}{|r - r'|} \, d^3r'$$

approximately. Far away from a localized charge distribution it will look like a point charge

\[\frac{1}{|r - r'|} \text{ as a power expansion in } \frac{1}{r} \]

\[(r^2 - r')^2 = r^2 + r'^2 - 2r \cdot r' = r^2 + r'^2 - 2rr'\cos\alpha \]

\[\Rightarrow |r - r'| = r \left( 1 + \frac{r'}{r^2} - 2\frac{r'}{r}\cos\alpha \right)^{1/2} \approx r \left( 1 + \frac{r'}{r} + \ldots \right) \]

\[\delta << 1 \]

\[\Rightarrow \frac{1}{|r^2 - r'|} = \frac{1}{r} \left( 1 - \frac{\delta}{2} + \frac{3}{8}\delta^2 - \frac{5}{16}\delta^3 + \ldots \right) \]

\[= \frac{1}{r} \left[ 1 - \frac{1}{2} \frac{r'}{r} \left( \frac{r}{r} - 2\cos\alpha \right) + \frac{3}{8} \left( \frac{r'}{r} \right)^2 \left( \frac{r}{r} - 2\cos\alpha \right)^2 + \ldots \right] \]

\[= \frac{1}{r} \left[ 1 - \frac{r'}{r} \cos\alpha + \left( \frac{r'}{r} \right)^2 \frac{3\cos^2\alpha - 1}{2} + \ldots \right] \frac{P_1(\cos\alpha)}{P_2(\cos\alpha)} \]

In fact

\[\frac{1}{|r^2 - r'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r'}{r} \right)^n P_n(\cos\alpha) \]
\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n (\cos \alpha) \rho(\vec{r}') \, d^3 \vec{r}' = \]

\[ = \frac{1}{4\pi\varepsilon_0} \left[ \frac{1}{r} \int \rho(\vec{r}') \, d^3 \vec{r}' \right. + \frac{1}{r^2} \left. \int r' \cos \alpha \rho(\vec{r}') \, d^3 \vec{r}' + \ldots \right] \]

\( \Theta \) - monopole \quad \vec{r} \cdot \int \frac{\rho(\vec{r}') \, d^3 \vec{r}'}{r^2} \quad \) quadropole

\( \vec{p} \) - dipole

\[ \frac{\alpha}{4\pi\varepsilon_0 r} + \frac{\vec{p} \cdot \vec{r}}{4\pi\varepsilon_0 r^2} + \frac{\sum \mathcal{Q}_i \cdot \hat{r} \cdot \hat{r}}{4\pi\varepsilon_0 r^3} + \ldots \]

\( V_{\text{mon}} \quad V_{\text{dip}} \quad V_{\text{quad}} \)
To be more concrete:
- consider a physical dipole with
  \[ \mathbf{P}_{\text{dipole}} = q \mathbf{g}(\mathbf{r}' - \mathbf{r}_+) - q \mathbf{g}(\mathbf{r}' - \mathbf{r}_-) \]

  \[ Q = \int \mathbf{P}_{\text{dipole}}(\mathbf{r}') d^3r' = q - (-q) = 0 \]

  \[ \mathbf{p} = \int \mathbf{r}' \mathbf{P}_{\text{dipole}}(\mathbf{r}') d^3r' = q \left( \frac{\mathbf{r}_+ - \mathbf{r}_-}{d} \right) \]

  \[ Q_{ij} \approx 0 \]

- and a physical quadrupole

  \[ +q \quad -q \]

  \[ \text{or} \quad \uparrow \quad \downarrow \]

  \[ -q \quad +q \]

  \[ \Rightarrow Q = 0, \quad \mathbf{p} = 0 \]

**Choice of origin?**
- \( Q \) indep of origin (follows from def)
- if \( Q = 0 \) then \( \mathbf{p} \) is indep of origin
- etc
\[ \vec{F}_s = \int \vec{r}_s \cdot \vec{f}(\vec{r}') \, d^3r' = [\vec{r}' = \vec{r}_s + \vec{a}] = \int \vec{r}' \cdot \vec{f}(\vec{r}') \, d^3r' - \vec{a} \int \vec{f}(\vec{r}') \, d^3r' \]

**Electric Field**

\[ \vec{E} = -\vec{\nabla} V = -\vec{\nabla} V_{\text{mon}} - \vec{\nabla} V_{\text{dipole}} - \vec{\nabla} V_{\text{quad}} \ldots \]

we already know \( \vec{E}_{\text{mon}} = -\vec{\nabla} V_{\text{mon}} \)

what about \( \vec{E}_{\text{dipole}} \)?

\[ \vec{E}_{\text{dipole}} = -\vec{\nabla} \frac{\vec{\hat{r}} \cdot \vec{p}}{4\pi \varepsilon_0 r^2} \]

\[ = -\vec{\nabla} \frac{p \cos \theta}{4\pi \varepsilon_0 r^2} \]

\[ = \frac{2p \cos \theta}{4\pi \varepsilon_0 r^3} \hat{\hat{r}} + \frac{p \sin \theta}{4\pi \varepsilon_0 r^3} \hat{\hat{\theta}} + 0 \hat{\hat{\phi}} \]

[prob 3.36] \[ \frac{3(\vec{E} \cdot \vec{\hat{r}}) \vec{\hat{r}} - \vec{\hat{p}}}{4\pi \varepsilon_0 r^3} \]

coordinate free form

and \( \vec{E}_{\text{quad}} \)?

\[ \vec{E}_{\text{quad}} = \frac{1}{r^4} \]