Examination, Computational Physics (FYTN03), October 30 2009

Allowed calculational aids: "TEFYMA" and pocket calculator.
The examination consists of eight problems (8×5 points).

1. Solve the initial value problem $y'(x) = 2xy(x)$, $y(0) = 1$. What is the analytic solution? Use the Euler method with step size $h = 0.1$. Compare the results from the Euler method with the exact result at $x = 3h$. What is the relative error?

2. Calculate
$$\int_1^2 \frac{dx}{x^2}$$
with the trapezoidal rule for the step sizes 0.5 and 0.25. What are the relative errors (compare to the analytic solution)? Improve the values by Richardson extrapolation.

3. Consider the probability distribution for the random variable X:
$$p(x) = \begin{cases} x^{-1/2}e^{-x/2}/\sqrt{2\pi} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
What is the characteristic function for X: $\phi(k) = \langle e^{ikX} \rangle$? What are the first two cumulants?

4. Describe how a random number with the probability distribution $p(x) = 1/(2\sqrt{1-x})$ if $0 \leq x \leq 1$ and $p(x) = 0$ otherwise can be obtained from uniformly distributed random numbers between 0 and 1.

5. Suppose the explicit Euler method is applied to the initial value problem
$$\frac{d\mathbf{y}(x)}{dx} = \mathbf{A}\mathbf{y}(x) \qquad \mathbf{y}(0) = \mathbf{y}_0 \qquad \mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$
where $\mathbf{y}(x)$ is a two-component vector. How small must the step size h be for the method to be stable?

6. Consider a system of N Ising spins $s_i \in \{\pm 1\}$ $(i = 1, \ldots, N)$. A classic optimization problem is the 2-SAT problem for which each configuration $s = (s_1, \ldots, s_N)$ can be assigned an Energy
$$E = \sum_{m=1}^{M} (1 - c_{m1}s_{m1})(1 - c_{m2}s_{m2})$$
where s_{m1}, s_{m2} are selected from the N spins and c_{m1}, c_{m2} are constants with values -1 or 1. Assume an Boltzmann distribution
$$P(s, T) \propto \exp(-E/T)$$
where T is an artificial temperature. Suggest a Metropolis update that satisfies detailed balance and show that this update leads to a stationary $P(s, T)$.

7. Describe a numerical method for finding out whether a state with Energy equal to zero exist in the previous problem. Argue for your choice of method.

8. Describe the implicit Euler method for the system

$$i\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}t} = \mathbf{H}\bar{\psi}\,,$$

where $\bar{\psi}$ is a complex vector with components $\psi_i(t)$ $(i = 1, \ldots, N)$ and $\mathbf{H}$ is a constant Hermitian matrix (i.e., $H_{ij}^* = H_{ji}$, where * means complex conjugation). Discuss its relation to stability and normalization

$$|\bar{\psi}|^2 = \sum_{i=1}^{N} |\psi_i|^2$$