

Mie Theory

We consider scattering of an electromagnetic wave against a homogeneous sphere with radius a .

Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

We will deal with waves having the time dependence described by the factor $e^{-i\omega t}$, then Maxwell's equation take the form

$$\nabla \times \mathbf{H} = (\sigma - i\omega\epsilon) \mathbf{E} = -\frac{i\omega n^2 c^2}{\mu} \mathbf{E}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = i\omega\mu \mathbf{H}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\text{with the refractive index } n = \sqrt{\left(\epsilon + \frac{i\sigma}{\omega}\right)\mu} \text{ and } c = \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$

The equations imply the wave equations

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \nabla^2 \mathbf{B} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\text{or } \nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \quad \nabla^2 \mathbf{B} + k^2 \mathbf{B} = 0 \text{ with } k = \frac{n\omega}{c} = \frac{2\pi}{\lambda}$$

The scalar wave equation is

$$\nabla^2 \psi - \frac{n^2}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \Rightarrow \nabla^2 \psi + k^2 \psi = 0$$

If ψ satisfies the scalar wave equation then the vectors \mathbf{L} , \mathbf{M} and \mathbf{N} , defined by

$$\mathbf{L} = \nabla \psi \quad \mathbf{M} = \nabla \times (\mathbf{r} \psi) \quad \mathbf{N} = \frac{1}{k} \nabla \times \mathbf{M}$$

satisfy the vector wave equation and with $\mathbf{M} = \frac{1}{k} \nabla \times \mathbf{N}$

The three vectors are mutually orthogonal and

$$\nabla \times \mathbf{L} = 0 \quad \nabla \cdot \mathbf{M} = 0 \quad \nabla \cdot \mathbf{N} = 0 \quad \nabla \cdot \mathbf{L} = \nabla^2 \psi = -k^2 \psi$$

Let r, θ, ϕ be polar coordinates. Then solutions of the scalar wave functions are

$$\psi_{e_{lm}^o}(r) = z_l(kr) P_l^m(\cos\theta) \frac{\cos m\phi}{\sin m\phi}$$

The radial part of the wave equation satisfies

$$\frac{\partial^2(r\psi)}{\partial r^2} + \frac{l(l+1)}{r^2} r\psi + k^2 r\psi = 0$$

In polar coordinates we have

$$\begin{aligned}\nabla &= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \\ \nabla \times \mathbf{a} &= \mathbf{e}_r \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin a_\phi) - \frac{\partial a_2}{\partial \phi} \right] \\ &\quad \mathbf{e}_\theta \left[\frac{1}{r \sin \theta} \frac{\partial a_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (ra_\phi) \right] + \mathbf{e}_\phi \frac{1}{r} \left[\frac{\partial}{\partial r} (ra_\theta) - \frac{\partial a_r}{\partial \theta} \right]\end{aligned}$$

We then get

$$\begin{aligned}L_r &= \frac{\partial \psi}{\partial r} & L_\theta &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} & L_\phi &= \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \\ M_r &= 0 & M_\theta &= \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \phi} & M_\phi &= -\frac{\partial \psi}{\partial \theta} \\ kN_r &= \frac{\partial^2(r\psi)}{\partial r^2} + k^2 r\psi & kN_\theta &= \frac{1}{r} \frac{\partial^2(r\psi)}{\partial r \partial \theta} & kN_\phi &= \frac{1}{r \sin \theta} \frac{\partial^2(r\psi)}{\partial r \partial \phi}\end{aligned}$$

Using the scalar wave equation we get

$$kN_r = \frac{l(l+1)}{r} \psi$$

This gives the fundamental vector solutions

$$\begin{aligned}\mathbf{L}_{lm}^{e,o}(r) &= \frac{d}{dr} z_l(kr) P_l^m(\cos\theta) \frac{\cos m\phi}{\sin m\phi} \mathbf{e}_r \\ &\quad \frac{1}{r} z_l(kr) \frac{d}{d\theta} P_l^m(\cos\theta) \frac{\cos m\phi}{\sin m\phi} \mathbf{e}_\theta \\ &\quad \mp \frac{m}{r \sin \theta} z_l(kr) P_l^m(\cos\theta) \frac{\sin m\phi}{\cos m\phi} \mathbf{e}_\phi \\ \mathbf{M}_{lm}^{e,o}(r) &= \mp \frac{m}{\sin \theta} z_l(kr) P_l^m(\cos\theta) \frac{\sin m\phi}{\cos m\phi} \mathbf{e}_\theta \\ &\quad + z_l(kr) \frac{d}{d\theta} P_l^m(\cos\theta) \frac{\cos m\phi}{\sin m\phi} \mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\mathbf{N}_{lm}^{e,o}(r) &= \frac{l(l+1)}{kr} z_l(kr) P_l^m(\cos\theta) \frac{\cos m\phi}{\sin m\phi} \mathbf{e}_r \\ &\quad \frac{1}{kr} \frac{d}{dr} [rz_l(kr)] \frac{d}{d\theta} P_l^m(\cos\theta) \frac{\cos m\phi}{\sin m\phi} \mathbf{e}_\theta \\ &\quad \mp \frac{m}{kr \sin\theta} \frac{d}{dr} [rz_l(kr)] P_l^m(\cos\theta) \frac{\sin m\phi}{\cos m\phi} \mathbf{e}_\phi\end{aligned}$$

We have from Maxwell's equations

$$\mathbf{E} = \frac{i\omega\mu}{k^2} \nabla \times \mathbf{H} \quad \mathbf{H} = \frac{1}{i\omega\mu} \nabla \times \mathbf{E}$$

Introduce the conventional scalar and vector potentials Φ and \mathbf{A} such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \quad \mathbf{H} = \nabla \times \mathbf{A}$$

Develop \mathbf{A} in the fundamental vectors

$$\mathbf{A} = \frac{i}{\omega} \sum_{l,m} (a_{ml} \mathbf{M}_{ml} + b_{ml} \mathbf{N}_{ml} + c_{ml} \mathbf{L}_{ml})$$

This gives

$$\mathbf{H} = -\frac{k}{i\omega\mu} \sum_{l,m} (a_{ml} \mathbf{N}_{ml} + b_{ml} \mathbf{M}_{ml})$$

$$\mathbf{E} = -\sum_{l,m} (a_{ml} \mathbf{M}_{ml} + b_{ml} \mathbf{N}_{ml})$$

The incident plane wave is $\mathbf{E}^{(i)} = \mathbf{e}_x e^{ik_2 z} \quad \mathbf{H}^{(i)} = \mathbf{e}_y e^{ik_2 z}$

In polar coordinates we have

$$\begin{aligned}\mathbf{e}_x &= \mathbf{e}_r \sin\theta \cos\phi + \mathbf{e}_\theta \cos\theta \cos\phi - \mathbf{e}_\phi \sin\phi \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta \sin\phi + \mathbf{e}_\theta \cos\theta \sin\phi + \mathbf{e}_\phi \cos\phi \\ \mathbf{e}_z &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta\end{aligned}$$

When we develop the incident electromagnetic wave we see that only the components with $m = 1$ will contribute. Choosing the combinations that give the correct component ϕ dependence we have

$$\mathbf{e}_x e^{ikz} = \sum_{l=1}^{\infty} (a_{1l} \mathbf{M}_{1l}^o + b_{1l} \mathbf{N}_{1l}^e)$$

Using orthogonality relations we get

$$a_{1l} = \frac{2l+1}{l(l+1)} i^l \quad b_{1l} = -\frac{2l+1}{l(l+1)} i^{l+1}$$

and thus

$$\mathbf{e}_x e^{ikz} = \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} (\mathbf{M}_{1l}^o - i \mathbf{N}_{1l}^e)$$

In the same way

$$\mathbf{e}_y e^{ikz} = - \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} (\mathbf{M}_{1l}^e + i\mathbf{N}_{1l}^o)$$

To have finite fields as $r \rightarrow \infty$ we have to take $z_l(k_2 r) = j_l(k_2 r)$

The outside scattered wave is

$$\begin{aligned}\mathbf{E}^{(r)} &\sim \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} (a_l^{(r)} \mathbf{M}_{1l}^o - i b_l^{(r)} \mathbf{N}_{1l}^e) \\ \mathbf{H}^{(r)} &\sim \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} (b_l^{(r)} \mathbf{M}_{1l}^e + i a_l^{(r)} \mathbf{N}_{1l}^o)\end{aligned}$$

now with $z_l(k_2 r) = h_l^{(1)}(k_2 r)$.

For the inside scattered wave we have

$$\begin{aligned}\mathbf{E}^{(t)} &\sim \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} (a_l^{(t)} \mathbf{M}_{1l}^o - i b_l^{(t)} \mathbf{N}_{1l}^e) \\ \mathbf{H}^{(t)} &\sim \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} (b_l^{(t)} \mathbf{M}_{1l}^e + i a_l^{(t)} \mathbf{N}_{1l}^o)\end{aligned}$$

now with $z_l(k_1 r) = j_l(k_1 r)$.

The continuity conditions on the surface of the sphere:

$$\begin{aligned}\mathbf{e}_r \times (\mathbf{E}^{(i)} + \mathbf{E}^{(r)}) &= \mathbf{e}_r \times \mathbf{E}^{(t)} \\ \mathbf{e}_r \times (\mathbf{H}^{(i)} + \mathbf{H}^{(r)}) &= \mathbf{e}_r \times \mathbf{H}^{(t)}\end{aligned}$$

imply

$$j_l(x) + a_l^{(r)} h_l^{(1)}(x) = a_l^{(t)} j_l(y) \quad (\mathbf{e}_\theta, \mathbf{E})$$

$$\mu_1 [x j_l(x)]' + \mu_1 a_l^{(r)} [x h_l^{(1)}(x)]' = \mu_2 a_l^{(t)} [y j_l(y)]' \quad (\mathbf{e}_\theta, \mathbf{H})$$

$$\mu_1 j_l(x) + \mu_1 b_l^{(r)} h_l^{(1)}(x) = \mu_2 b_l^{(t)} n j_l(y) \quad (\mathbf{e}_\phi, \mathbf{H})$$

$$n [x j_l(x)]' + n b_l^{(r)} [x h_l^{(1)}(x)]' = b_l^{(t)} [y j_l(y)]' \quad (\mathbf{e}_\phi, \mathbf{E})$$

where $x = k_2 a$ and $y = k_1 a = n k_2 a$

With little less generality we will now assume $\mu_1 = \mu_2$

Using the Riccati-Bessel functions we can solve the system above

$$\begin{aligned}a_l^{(r)} &= - \frac{\psi_l(y) \psi'_l(x) - n \psi'_l(y) \psi_l(x)}{\psi_l(y) \zeta'_l(x) - n \psi'_l(y) \zeta_l(x)} \\ b_l^{(r)} &= - \frac{\psi_l(x) \psi'_l(y) - n \psi'_l(x) \psi_l(y)}{\psi'_l(y) \zeta_l(x) - n \psi_l(y) \zeta'_l(x)}\end{aligned}$$

Using the far field approximation for the scattered wave

$$h_l^{(1)}(x) \approx (-i)^{l+1} \frac{e^{ix}}{x} \text{ as } x \rightarrow \infty$$

gives

$$E_{\theta}^{(r)} = H_{\phi}^{(r)} \simeq \frac{e^{ik_2 r}}{k_2 r} \cos \phi S_2(\theta)$$

$$E_{\phi}^{(r)} = -H_{\theta}^{(r)} \simeq \frac{e^{ik_2 r}}{k_2 r} \sin \phi S_1(\theta)$$

where

$$S_1(\theta) = \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} (a_l^{(r)} \tau_l(\cos \theta) + b_l^{(r)} \pi_l(\cos \theta))$$

$$S_2(\theta) = \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} (a_l^{(r)} \pi_l(\cos \theta) + b_l^{(r)} \tau_l(\cos \theta))$$

Riccati-Bessel functions

$$\begin{aligned}\psi_n(x) &= x j_n(x) \quad \chi_n(x) = x n_n(x) \\ \zeta_n(x) &= x h_n^{(2)}(x) = x(j_n(x) + n_n(x))\end{aligned}$$

$$\begin{aligned}\psi_0(x) &= \sin x \quad \chi_0(x) = -\cos x \\ \psi_1(x) &= \frac{\sin x}{x} - \cos x \quad \chi_1(x) = -\frac{\cos x}{x} - \sin x \\ \zeta_n(x) &= \psi_n(x) + i\chi_n(x) \\ f_{n+1}(x) &= (2n+1) \frac{f_n(x)}{x} - f_{n-1} \\ f'_n(x) &= f_{n-1}(x) - (n+1) \frac{f_n(x)}{x}\end{aligned}$$

Associated Legendre polynomials

$$\begin{aligned}x &= \cos\theta \\ P_0^1(x) &= 0 \quad P_1^1(x) = \sqrt{1-x^2} = \sin\theta \\ nP_{n+1}^1(x) &= (2n+1)xP_n^1(x) - (n+1)P_{n-1}^1(x) \\ \pi_n(x) &= \frac{P_n^1(x)}{\sqrt{1-x^2}} \quad \tau_n(x) = \frac{1}{\sqrt{1-x^2}}(nxP_n^1 - (n+1)P_{n-1}^1(x)) = \frac{dP_l^1(\cos\theta)}{d\theta} \\ \pi_n(\pm 1) &= (\pm 1)^n \frac{n(n+1)}{2} \\ \tau_n(\pm 1) &= (\pm 1)^{n+1} \frac{n(n+1)}{2}\end{aligned}$$