

System Theory

FYS244

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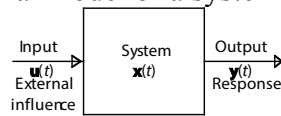
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1 On systems

System theory is about studying systems and make mathematical models for them. Some examples of systems are

- The solar system
- A bicycle
- A population
- A heart
- Chemical reactions
- Weather systems

A general model of a system may be illustrated in the figure below.



The system variables are denoted by $\mathbf{x}(t)$, the input signal by $\mathbf{u}(t)$, and the output signal by $\mathbf{y}(t)$. They have not necessarily the same dimension. $\mathbf{x}(t)$ describes the internal state of the system.

1.1 Some kinds of systems

The time t can be *continuous*. Often it is practical to let it be *discrete*, for instance in computer simulations or when it is the time between generations in a populations.

In a *static* system we have

$$\mathbf{x}(t) = \mathbf{f}(\mathbf{u}(t))$$

Example 1.1. A simple system that follows Newton's laws $F = ma$, with $x = a$ and $u = F$ is an example of a system where the system variables only depends on the input signal i.e. it is static.

In a dynamic system the state of the system depends on what has happened earlier, that is the system has a memory. The change of the system thus depends also on its actual state. Many systems can be described by differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Some systems are described by differential-difference equations, like the development of a population with a specific fertility age T

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-T), \mathbf{u}(t), t)$$

We can further classify systems in *linear* and *non-linear systems*. For linear systems we have powerful mathematical methods. This is in general not the case for non-linear systems. By *linearizing* such systems around suitable state points we can often extract interesting information.

Some system models are stochastic, that is we can only specify signals and system states with certain probabilities. If $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ does not depends explicitly on time t we

have an *autonomous* system. The systems we will treat will almost always be such systems.

We will often be interested in whether a system is stable or not. Often this is the only thing we are interested in.

The description $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ is despite its form (first order) a quite general description.

Example 1.2. Consider Newton's law

$$\frac{d^2 z}{dt^2} = \frac{F(z, t)}{m} = f(z, t)$$

If we introduce the system variables $x_1 = z$ and $x_2 = \dot{z}$ we get the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, t) \end{cases}$$

that is a description of the type above.

The choice of system variables is not unique. In some cases we study systems without output signals and let the system variables themselves be output signals. One way to get such systems is to enlarge the system to contain also suitable parts of the environment.

1.2 Linear autonomous systems

Study the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix}$$

As the system is linear, every linear combination of solutions is also a solution. \mathbf{A} is called the system matrix.

Put $\mathbf{x}(t) = T(t)\mathbf{z}$, where T is a scalar function of time and \mathbf{z} a vector that is independent of time. Then

$$\mathbf{A}\mathbf{z} = \frac{\dot{T}}{T}\mathbf{z}$$

As $\mathbf{A}\mathbf{z}$ is independent on time $\dot{T}/T = \lambda$ is independent on time i.e.

$$\begin{cases} \dot{T} = \lambda T \\ \mathbf{A}\mathbf{z} = \lambda\mathbf{z} \end{cases} \Rightarrow T(t) = ce^{\lambda t}$$

The second equation is an eigenvalue equation where λ is the eigenvalue. We rewrite the equation as a homogeneous equation

$$(\mathbf{A} - \lambda\mathbf{1})\mathbf{z}$$

where $\mathbf{1}$ denotes the unit matrix. This equation has non-trivial solutions when

$$\det(\mathbf{A} - \lambda \cdot \mathbf{1}) = 0$$

This is an n degree equation in λ , the *secular equation*. We have a set of n roots $\{\lambda_i\}$ that is called the *spectrum* of the matrix \mathbf{A} .

Assume that all the roots are different. We can then insert the roots, one at a time, in the homogeneous equation and determine the corresponding eigenvectors \mathbf{z}_i . The general solution is then

$$\mathbf{x}(t) = \sum_i c_i e^{\lambda_i t} \mathbf{z}_i$$

where the constants c_i are determined by the boundary conditions

$$\mathbf{x}(0) = \sum_i c_i \mathbf{z}_i$$

If two eigenvalues are equal, say $\lambda_1 = \lambda_2 = \lambda$, the contribution to the solution will be

$$(c_1 \mathbf{z}_1 + t \cdot c_2 \mathbf{z}_2) e^{\lambda t}$$

In general if m roots are equal we get a polynomial of degree $m-1$ in t

$$\sum_{k=0}^{m-1} c_k t^k e^{\lambda t} \mathbf{z}_k$$

Physical interpretation of the solution

In most cases the matrix \mathbf{A} is real. The eigenvalues then are real or complex conjugate pairs.

- Eigenvalues with $\text{Re } \lambda > 0$ give exponentially growing terms in the solution, i.e. we have instability.
- $\text{Re } \lambda < 0$ give exponentially decreasing solutions. If all the eigenvalues have negative real part the system is stable.
- $\text{Im } \lambda \neq 0$ gives oscillatory solutions.

Another method of solution

Formally the equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with boundary condition $\mathbf{x}(0) = \mathbf{a}$ is solved by

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{a}$$

What is the meaning of $e^{\mathbf{A}t}$? We *define* this quantity as the series expansion

$$e^{\mathbf{A}t} = \mathbf{1} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$$

It can be shown that the series converges and for many matrices \mathbf{A} the number of terms in the series is finite. It is also easy to see the expression above is a solution (Exercise)

Using the binomial theorem you can show that (Exercise) $e^{\mathbf{A}s} \cdot e^{\mathbf{A}t} = e^{\mathbf{A}(s+t)}$

If we put $s = -t$ we have $e^0 = \mathbf{1} = e^{\mathbf{A}t} \cdot e^{-\mathbf{A}t}$ that is $e^{\mathbf{A}t}$ is non-singular with inverse $e^{-\mathbf{A}t}$.

We can now solve the inhomogeneous equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Put $\mathbf{x}(t) = e^{At} \mathbf{c}(t) \Rightarrow \dot{\mathbf{c}} = e^{-At} \cdot \mathbf{B} \mathbf{u}$

The last equation is solved by

$$\mathbf{c}(t) = \mathbf{c}(0) + \int_0^t e^{-As} \cdot \mathbf{B} \mathbf{u}(s) ds$$

giving

$$\mathbf{x}(t) = e^{At} \mathbf{a} + \int_0^t e^{-A(t-s)} \cdot \mathbf{B} \mathbf{u}(s) ds$$

1.3 Linearization

Example 1.3 (Pendulum) Consider a mathematical pendulum with mass m and length L . Let the pivot point be freely movable along a horizontal line in a vertical plane. The pendulum oscillates in a vertical plane. Let the system variables be the angle of the pendulum with the vertical and the time derivative of this angle, $x_1 = \varphi$, $x_2 = \dot{\varphi}$. Let the acceleration of the pivot point (i.e. the external applied force) be the input signal, $u = \ddot{z}$. Let the output signal be the angle of the pendulum. The system equations for this system can be derived by for instance Lagrange formalism and are (Exercise)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 - \ddot{z} \cos x_1 \end{cases}$$

If we study the system without external forces we have

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}, \text{ with system matrix } \begin{pmatrix} 0 & 1 \\ -\sin x_1 & 0 \end{pmatrix}$$

The *equilibrium points* or *fix-points* are given by

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0 \Rightarrow x_2 = 0, \quad \sin x_1 = 0$$

We have two fix-points

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \quad \begin{cases} x_1 = \pi \\ x_2 = 0 \end{cases}$$

Taylor expansion around the first point and only keeping first order terms gives

$$\begin{cases} \dot{\varepsilon}_1 = \varepsilon_2 \\ \dot{\varepsilon}_2 = -\varepsilon_1 \end{cases} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The matrix has eigenvalues $\lambda = \pm i$ i.e. oscillatory solutions as expected. It is easy to show that the second fix-point gives unstable solutions.

In general we have a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with fix-points given by $\mathbf{f}(\mathbf{x}^*) = 0$.

Linearization with $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\varepsilon}$ gives

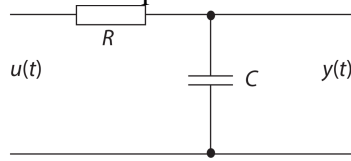
$$\dot{\varepsilon}_i = \sum_j \varepsilon_j \frac{\partial f_i(\mathbf{x}^*)}{\partial x_j} \text{ i.e. } A_{ij} = \frac{\partial f_i(\mathbf{x}^*)}{\partial x_j}$$

1.4 Models for input and output signals – the black box model

The number of system variables can be very large. In many cases we are only interested in a few of them. When you drive a car you need to know very little of what happens in the car system. We will therefore study methods where we concentrate on relations between input and output signals.

When can we use such a description? The development of the system depends not only on the input signals but also on the state at the beginning. If we assume that we have a stable system (all the eigenvalues of the system matrix are less than zero) the system will, if we wait long enough, have been damped into a well-defined state.

Example 1.4. A low-pass filter
Consider the low-pass filter below.



It is easy to show (Exercise) that the relation between the input and output signals is

$$\dot{y} = -y / RC + u / RC$$

Assuming that the output signal is limited when $t \rightarrow -\infty$ the solution is

$$y(t) = \frac{1}{RC} \int_{-\infty}^t e^{-(t-s)/RC} u(s) ds$$

We see that the output signal at a certain time depends on the input signal for all earlier times!

We now study linear (or linearized) systems. We assume that the system is autonomous, that the system matrix is explicitly independent on time. We also assume that $x(t \rightarrow -\infty) = 0$. The general solution to the equations

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

is then

$$\mathbf{x} = e^{\mathbf{A}t} \mathbf{a} + \int_{-\infty}^t e^{\mathbf{A}(t-s)} \mathbf{B}\mathbf{u} ds \quad \mathbf{y} = \mathbf{D}\mathbf{u} + \mathbf{C} \int_{-\infty}^t e^{\mathbf{A}(t-s)} \mathbf{B}\mathbf{u} ds$$

The solution for \mathbf{y} is independent of the system variables. We realise this if we put $\mathbf{z} = \mathbf{T}\mathbf{x}$ where \mathbf{T} is a diagonal matrix. We get

$$\dot{\mathbf{z}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{T}^{-1}\mathbf{z} + \mathbf{D}\mathbf{u} \Rightarrow \mathbf{y} = \mathbf{D}\mathbf{u} + \mathbf{C} \int_{-\infty}^t \mathbf{T}^{-1} e^{\mathbf{T}\mathbf{A}\mathbf{T}^{-1}(t-s)} \mathbf{T}\mathbf{B}\mathbf{u} ds$$

It is easy to show that $e^{\mathbf{T}\mathbf{A}\mathbf{T}^{-1}(t-s)} = \mathbf{T} e^{\mathbf{A}(t-s)} \mathbf{T}^{-1}$ (Exercise) giving

$$\mathbf{y} = \mathbf{D}\mathbf{u} + \mathbf{C} \int_{-\infty}^t e^{\mathbf{A}(t-s)} \mathbf{B}\mathbf{u} ds, \text{ independent of } \mathbf{T}.$$

Assume for a moment that $\mathbf{D} = 0$, that the input signal does not influence the output signal. The output signal is then a weighted average of the input signal in the interval $(-\infty, t)$. In out case the weight function (in general a matrix) is

$$\mathbf{h}(\tau) = \mathbf{C}e^{-\mathbf{A}\tau}\mathbf{B} \quad \tau \geq 0$$

τ is the time from the arrival of the input signal that will be weighted. If $\mathbf{D} \neq 0$ we can define the weight function

$$\mathbf{h}(\tau) = \mathbf{D}\delta(\tau) + \mathbf{C}e^{-\mathbf{A}\tau}\mathbf{B} \quad \tau \geq 0 \Rightarrow$$

$$\mathbf{y}(t) = \int_{-\infty}^t \mathbf{h}(t-s)\mathbf{u}(s)ds = \int_0^{\infty} \mathbf{h}(s')\mathbf{u}(t-s')ds'$$

We can give the weight function a physical interpretation that also implies a practical method of measuring it for an arbitrary linear system. Let one of the input signals u_i , be a δ pulse at time $\xi < t$ and study one of the output signals y_j

$$y_j(t) = \int_{-\infty}^t h_{ji}(t-s)\delta(s-\xi)ds = h_{ji}(t-\xi)$$

The weight function is the output signal that results from an input δ pulse! The weight function is often called the impulse response. By using different input signals and study all the responses we can measure the weight function.

So far we have only got the weight function for positive arguments of time. A natural generalisation is to put its value for negative arguments to zero (or the system would be able to produce an output signal *before* the input signal), we say that the weight function is *causal*. By this trick we can also extend the time integration over all times.

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(t-s)\mathbf{u}(s)ds$$

We will now show that for a linear autonomous system there is a direct relation between the input and output signals that can be formulated using *Laplace transforms*. These transforms are also very handy to solve differential equations.

2 The Laplace transform

The Laplace transform $\mathcal{L}(f) = F(s)$ of a function $f(t)$ is defined by

$$\mathcal{L}(f) \equiv F(s) \equiv \int_0^{\infty} dt e^{-st} f(t), \quad \operatorname{Re} s \geq s_0$$

where s_0 is a suitably chosen such that the integral converges. We will denote the Laplace transforms by capital letters. The integral above defines the Laplace transform in a complex half-plane. If needed also other parts of the complex plane can be defined by analytical continuation. By the Laplace transform we can convert differential equations unto algebraical equations. It can be shown (Exercise) that

$$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0)$$

We Laplace transform the system equations

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \rightarrow \begin{cases} s\mathbf{X} - \mathbf{x}(0) = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} \\ \mathbf{Y} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{U} \end{cases}$$

where \mathbf{X} , \mathbf{Y} , and \mathbf{U} are Laplace transforms. We can immediately solve the equations

$$\begin{aligned} \mathbf{X} &= [\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} \cdot \mathbf{x}(0) + [\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} \cdot \mathbf{B}\mathbf{U} \\ \mathbf{Y} &= \mathbf{C}([\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} \cdot \mathbf{x}(0) + [\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} \cdot \mathbf{B}\mathbf{U}) + \mathbf{D}\mathbf{U} \end{aligned}$$

If $\mathbf{x}(0) = 0$ we have $\mathbf{Y} = (\mathbf{C}[\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} \cdot \mathbf{B} + \mathbf{D})\mathbf{U} \equiv \mathbf{G}\mathbf{U}$.

$\mathbf{G}(s)$ is called the *transfer function*. Very often we can assume that $\mathbf{x}(0) = 0$ if the system is stable and we wait long enough.

Example 2.1 Pendulum continued.

We return to the pendulum that we studied earlier.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \mathbf{D} = 0$$

We solve the equation system using the inverse

$$[\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}^{-1} = \frac{1}{1+s^2} \begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}$$

It is easily seen that this is the inverse by testing. We now get

$$\mathbf{G}(s) = \mathbf{C}[\mathbf{s}\mathbf{1} - \mathbf{A}]^{-1} \cdot \mathbf{B} + \mathbf{D} = \frac{1}{1+s^2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{1+s^2}$$

$$\mathbf{Y} = \mathbf{G}\mathbf{U} \Rightarrow (1+s^2)\mathbf{Y} = \mathbf{U}$$

Make an inverse Laplace transformation to get

$$\frac{d^2 y}{dt^2} + y = u$$

Note that the transfer function has *poles* for $s = \pm i$, the eigenvalues that we found earlier for the system matrix.

Example 2.2 Heat conduction

We study the heat conduction in a rod. The temperature in the rod is described at location x , $0 \leq x \leq L$ and time t by the function $T(x, t)$. Let the rod have density ρ , specific heat capacity c , and heat conduction coefficient k . Let $u(t) = T(0, t)$ be the input signal and $y(t) = T(L, t)$ be the output signal. The equation for heat conduction is

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{a^2} \frac{\partial T}{\partial t} = 0 \quad a^2 = k / c\rho$$

Introduce the Laplace transform

$$\Theta(s, x) = \mathcal{L}(T) = \int_0^\infty dt e^{-st} T(x, t)$$

Assume that $T(x, 0) = 0$ and the Laplace transformed equation is

$$\frac{\partial^2 \Theta}{\partial x^2} - \frac{s}{a^2} \Theta = 0 \Rightarrow \Theta(s, x) = Ae^{x\sqrt{s}/a} + Be^{-x\sqrt{s}/a}$$

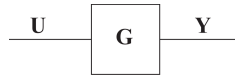
In order that Θ be finite when x is large we put $A = 0$. We have

$$\Theta(s, 0) = U(s) \Rightarrow \Theta(s, x) = U(s) e^{-x\sqrt{s}/a}$$

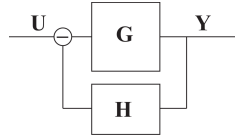
$$Y(s, L) = U(s) e^{-L\sqrt{s}/a} \Rightarrow G(s) = e^{-L\sqrt{s}/a}$$

Example 2.3. Feedback

Consider the system illustrated in the following graph with transfer function G and thus $\mathbf{Y} = \mathbf{G}\mathbf{U}$



We now modify this system by adding a feedback system with transfer function \mathbf{H} . The output signal is taken through this system and the subtracted from the input signal (negative feed-back)



Starting at the output and passing through \mathbf{H} and then back to the output we have

$$\mathbf{G}(\mathbf{U} - \mathbf{H}\mathbf{Y}) = \mathbf{Y} \quad \text{or} \quad \mathbf{Y} = \frac{\mathbf{G}}{1 + \mathbf{G}\mathbf{H}} \mathbf{U}$$

The feed-backed system has transfer function $\frac{\mathbf{G}}{1 + \mathbf{G}\mathbf{H}}$.

Suppose now that the original system is unstable, for instance that $\mathbf{G} = \frac{1}{1-s}$ that has a real pole for $s = 1 > 0$. Suppose that $\mathbf{H} = 2s$. The transfer function of the system with feed-back is then

$$\mathbf{G}' = \frac{\frac{1}{1-s}}{1 + \frac{1}{1-s} \cdot 2s} = \frac{1}{1-s+2s} = \frac{1}{1+s}$$

we have moved the pole and it is now at $s = -1 < 0$ meaning that the system is now stable. Feedback is a very common method to improve the stability of a system. An example from the real world is when to want to grab a cup of coffee from a table. The

feedback is the information you get from your eye and via the brain to your muscles. If the feedback loop is disturbed as for people with Parkinson's it will be very hard to perform the task.

2.1 An important theorem

We will now show that the transfer function is the Laplace transform of the weight function. Generally we have

$$y(t) = \int_{-\infty}^t h(t-\tau)u(\tau)d\tau$$

Let the input signal be zero for $t < 0$ and let h be causal.

$$y(t) = \int_0^{\infty} h(t-\tau)u(\tau)d\tau$$

Take the Laplace transform

$$\begin{aligned} Y(s) &= \int_0^{\infty} dt e^{-st} \int_0^{\infty} h(t-\tau)u(\tau)d\tau = \int_0^{\infty} dt \int_0^{\infty} e^{-s(t-\tau)} e^{-s\tau} h(t-\tau)u(\tau)d\tau = \\ &= \int_0^{\infty} d\tau e^{-s\tau} \left[\int_0^{\infty} e^{-s(t-\tau)} h(t-\tau)dt \right] u(\tau) \stackrel{t-\tau \rightarrow t'}{=} \int_0^{\infty} d\tau e^{-s\tau} \left[\int_{-\tau}^{\infty} e^{-st'} h(t')dt' \right] u(\tau) \stackrel{h \text{ causal}}{=} \\ &= \int_0^{\infty} d\tau e^{-s\tau} \left[\int_0^{\infty} e^{-st'} h(t')dt' \right] u(\tau) = H(s) \int_0^{\infty} d\tau e^{-s\tau} u(\tau) = H(s)U(s) \end{aligned}$$

2.2 Partial fraction expansion

The transfer function is often of the type $G(s) = N(s)/D(s)$ where N and D are polynomials with $\text{grad } N < \text{grad } D$. The fundamental theorem of algebra then says

$$G(s) = K \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$$

where $\{z_i\}$ are the zeros of N and $\{p_j\}$ the zeros of D , i.e. the pole of G . G is the uniquely determined by its poles and zeros and the value of the constant K . Assume that G has no double poles or double zeros. We then have

$$G(s) = \sum_j \frac{R_j}{s - p_j} \text{ with the residue } R_j \stackrel{s \rightarrow s_j}{\rightarrow} G(p_j)(s - p_j) \text{ because}$$

$$G(s)(s - p_k) = K \frac{\prod_i (s - z_i)}{\prod_{j \neq k} (s - p_j)} = \frac{R_1(s - p_k)}{s - p_1} + \dots R_k + \dots \stackrel{s \rightarrow s_j}{\rightarrow} R_k$$

A term in the expansion of type $\frac{1}{s - p}$ corresponds in the inverse Laplace transform to

a term e^{pt} . If the pole p is located in the right complex half-plane this will give rise to an exponentially growing signal. We formulate a general theorem:

A stable system has a transfer function that has poles only in the left complex half-plane.

2.3 Harmonic input signal

Assume that $u(t) = u_0 e^{i\omega t} \Rightarrow U(s) = \frac{u_0}{s - i\omega}$

Assume that we have a transfer function $G = G(s)$ and that the system is stable, i.e. lacks poles in the right half-plane. We have

$$Y(s) = G(s)U(s)$$

Do a partial fraction expansion of $Y(s)$ we have a pole at $s = i\omega$

Using the results of the previous section we can write

$$Y(s) = \frac{u_0 G(i\omega)}{s - i\omega} + \text{other terms}$$

The other terms will go to zero as time goes to infinity as the system is stable. So we neglect these terms and do an inverse Laplace transform

$$y(t) = u_0 G(i\omega) e^{i\omega t}$$

In general $G(i\omega)$ is complex and we write $y(t) = u_0 |G(i\omega)| e^{i(\omega t + \delta)}$.

Thus $|G(i\omega)|$ is the amplitude amplification and δ the phase lag.

Example 2.3 Low-pass filter continued.

We return to our previous example 1.4. We have

$$\dot{y} = -y / RC + u / RC$$

$$sY = -Y / RC + U / RC \Rightarrow Y = \frac{1}{1 + sRC} U \Rightarrow G(s) = \frac{1}{1 + sRC}$$

We now compute

$$G(i\omega) = \frac{1}{1 + i\omega RC} \Rightarrow |G(i\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad \tan \delta = -\omega RC$$

We see that the filter behaves correctly: as $\omega \rightarrow \infty$, $|G(i\omega)| \rightarrow 0$ and $\delta \rightarrow -\pi / 2$.

When analysing circuits like this, we often make a log-log plot of $|G(i\omega)|$ and $\delta(\omega)$.

This is called a *Bode plot*.

Example 2.4. Heat conduction continued

We now use our new tool for the heat conduction problem, Example 2.2. We apply the result where the rod is a piece of ground, the rod stretching downwards in the ground. The transfer function is $G(s) = e^{-L\sqrt{s}/a}$. Thus

$$G(i\omega) = e^{-L\sqrt{i\omega}/a} = e^{-L\sqrt{\omega/2}/a} \cdot e^{\pm iL\sqrt{\omega/2}/a}$$

where we have used that $\sqrt{i} = (1 \pm i) / \sqrt{2}$ (Exercise: Show this!)

Then

$$|G(i\omega)| = e^{-L\sqrt{\omega/2}/a}$$

We see that oscillations in temperature on the surface will be damped as we go down in the ground. A typical damping depth will be $L_D = a\sqrt{2/\omega}$, i.e. high frequencies will be more damped, the ground acts as a low-pass filter. The phase will be given by

$$\delta = L\sqrt{\omega/2a^2}$$

At a certain depth the oscillations will be in anti-phase, $\delta = \pi$, with the ground oscillations. We want to build a cellar at this depth. Inserting reasonable numbers for k , c , and ρ we get $L_\pi = 5$ m.

We want to build a cellar at this depth. Inserting reasonable numbers for k , c , and ρ we get $L_\pi = 5$ m.


2.3 Stability and trajectories

Look at the followings system

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2) & \frac{dx_2}{dt} &= f_2(x_1, x_2) \\ \frac{dx_1}{dx_2} &= \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = g(x_1, x_2)\end{aligned}$$

By studying $g(x_1, x_2)$ we can get a picture of system trajectories in the phase plane.

Equilibrium points are singular points ("0/0") for g . In the neighbourhood of the equilibrium points the system matrix will determine the kind of equilibrium we have. Below we enumerate the different cases.

λ_1, λ_2 real > 0  instable node

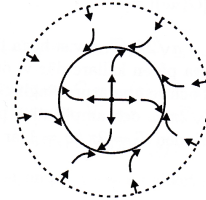
λ_1, λ_2 real < 0  stable node

$\lambda_1 \cdot \lambda_2$ real < 0  saddle point

$\lambda_i \in \mathbb{C}$, $\text{Re } \lambda_i < 0$  stable spiral

$\lambda_i \in \mathbb{C}$, $\text{Re } \lambda_i > 0$  instable spiral

At the instable fix-points there is often a limit cycle when higher order terms are included. The picture illustrates a typical limit cycle. The full drawn circle is the limit cycle itself, points starting inside the circle will go away from the instable fix-point and approach the limit cycle. Starting from points outside the limit cycle, within a certain area, will approach the limit cycle from the outside.

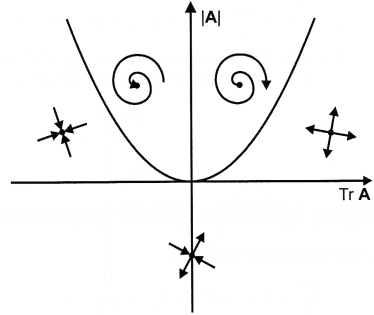


The eigenvalues in two dimensions are determined by

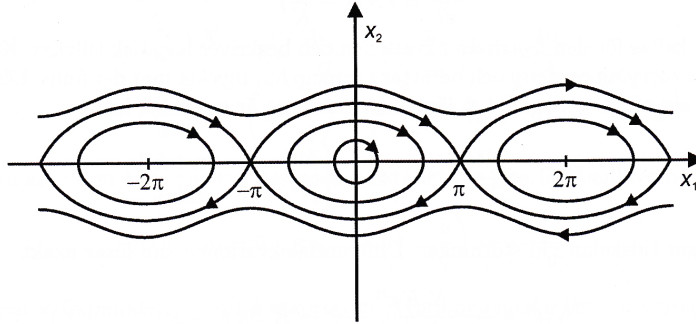
$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \lambda \operatorname{Tr} \mathbf{A} + |\mathbf{A}| = 0 \Leftrightarrow \lambda = \frac{\operatorname{Tr} \mathbf{A}}{2} \pm \sqrt{\frac{\operatorname{Tr}^2 \mathbf{A}}{4} - |\mathbf{A}|}$$

The last expression can be associated to the properties of the equilibrium points, see the figure below. From the properties of the system matrix, we can determine the kinds of eigenvalues and establish if they give stable or non-stable fix-points. According to the table above we have stability when $\operatorname{Tr} \mathbf{A} < 0$ and $|\mathbf{A}| > 0$. The curve in the figure corresponds to $|\mathbf{A}| = \operatorname{Tr}^2 \mathbf{A} / 4$.



We can also draw trajectories that show how the system moves in phase-space with time. As an example we take the mathematical pendulum in Example 1.3. with system variables $(x_1, x_2) = (\varphi, \dot{\varphi})$. There are stable fix-points at $(n \cdot 2\pi, 0)$ and saddle points at $(\pi + n \cdot 2\pi, 0)$, $n = 1, 2 \dots$. In the figure below we show a sketch of some trajectories with different boundary conditions.



3 Continuous one-dimensional population models

An example of a simple population model is

$$\frac{dN}{dt} = fN - dN = rN, \quad r = f - d$$

The change in the number of individuals is proportional to the number of individuals with a proportionality constant that is determined by the difference between the fraction of born, f , and dead, d . This leads to either an exponentially decreasing or exponentially increasing population depending on the sign of r , in the long run both developments are unrealistic. The solution to the differential equation, $N(t) = N_0 e^{rt}$, is called Malthus' law and was stated in 1798. A more reasonable model was suggested by Verhulst in 1836:

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right)$$

The equation is called the *logistic* equation and describes a logistic growth. The constant K is called *carrying capacity* and in principle tells of how much food there is. The solution is stable as $t \rightarrow \infty$. The two equilibrium points are easily found:

$$rN^* \left(1 - \frac{N^*}{K} \right) = 0 \Leftrightarrow \begin{cases} N^* = 0 \\ N^* = K \end{cases}$$

$\tau = 1/r$ gives the time scale for disturbances. We can solve the differential equation exactly

$$N(t) = \frac{N_0 K e^{rt}}{K + N_0 (1 - e^{rt})} \rightarrow K \text{ as } t \rightarrow \infty$$

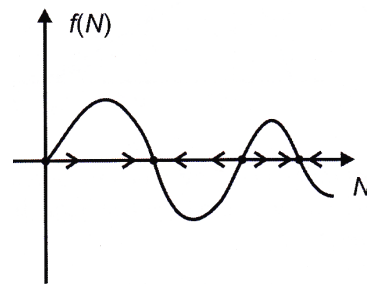
Whether the fix-points are stable or not is determined by looking at $N(t) = N^* + n(t)$, $n \ll 1$. Inserting this in the differential equation we get

$$\frac{dn}{dt} \approx f(N^* + n)$$

$$\text{where } f(N) = rN \left(1 - \frac{N}{K} \right) = r(N^* + n) \left(1 - \frac{N^* + n}{K} \right) \approx rn \left(1 - \frac{N^*}{K} \right) - rn \frac{N^*}{K}.$$

We have used that $f(N^*) = 0$ and skipped second order terms. If $N^* = 0$ we have obviously exponentially growing solutions for $r > 0$, i.e. the fix-point is unstable. For $N^* = K$, the solution is stable, something already indicated by the exact solution.

In general a nice method of finding the stability of the fix points is to plot the function $f(N)$. The fix-points will be the zeros of this function. Where $f(N) > 0$, N will increase, where $f(N) < 0$, N will decrease.



Fix-points where the arrows points will be stable otherwise unstable.

Example 3.1 Spruce budworm, Ludwig 1978.

We study a model for the development of a spruce budworm population. They feed on spruce and are in turn eaten by birds and are modelled by

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B} \right) - p(N), \quad p(N) = \frac{BN^2}{A^2 + N^2}$$

N is the number of worms and the new term $p(N)$ represents the birds. We simplify the problem by making it dimensionless:

$$u = N / A, r = Ar_B / B, q = K_B / A, \tau = Bt / A$$

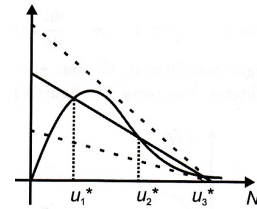
to get

$$\frac{du}{d\tau} = ru \left(1 - \frac{u}{q} \right) - \frac{u^2}{1+u^2} = f(u; r, q)$$

Not that we have reduced the number of parameters from four to two. The fix-pointa are determined by

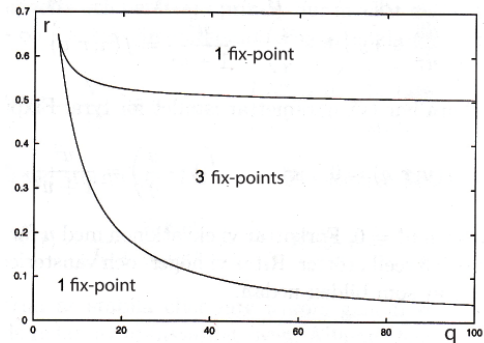
$$f(u; r, q) = 0 \Leftrightarrow ru \left(1 - \frac{u}{q} \right) = \frac{u^2}{1+u^2}$$

Trivially we have the (instable) fix-point $u^* = 0$. If we divide the equation by u we get a third degree polynomial with 1 or 3 real roots. If we draw the left and right members of the equation in the same graph we get the figure.



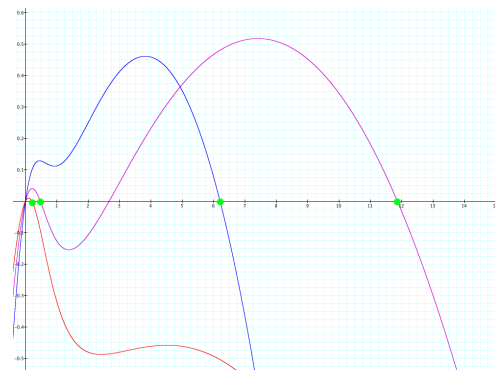
The number of solutions depends on r and q . By looking at the condition for double roots, we can establish (Exercise) that the region with three solutions is bordered by the parameterised curves

$$r(s) = \frac{2s^3}{(1+s^2)^2} \quad q(s) = \frac{2s^3}{s^2-1}$$



A plot of the curves is shown to the right in the region $1.01 < s < 30$, $s = 1$ cannot be included because of the singularity there. In the next figure there is a plot of $f(u)$ for some different values of r and q and you can see why there is sometimes only one solution. The curve with three solutions (except the $u = 0$ solution) has parameters

$(r, q) = (0.4, 15)$, violet curve. The curves with only one solution (except $u = 0$) have parameters $(0.2, 10)$, red curve, and $(0.7, 8)$, blue curve. Stable points are marked by a green dot.



This simple model exhibits hysteresis. Changing r in a certain direction will force the system to jump from one stable fix-point to another when the first one disappears. When r is changed in the other direction the jump will occur for another value of r and some regions can only be reached by entering from one direction, this is what is

called hysteresis. In the last figure we have also changed the value of q in order to get a reasonable scale for the curves.

How should we fight the worm? Looking at the parameters a suggestion is

- decrease r : r_B small, A small or B large *or*
- decrease q : K_B small or A large

Often it is possible to reduce the number of parameters by rescaling like we did here. This considerably makes the analysis of the system simpler.

3.1 Time-lag models

Generally we can describe a model with time-lag by

$$\frac{dN(t)}{dt} = f(N(t), N(t-\tau), t) \quad \tau > 0$$

Example 3.2 Verhulst equation with time-lag

$$\frac{dN(t)}{dt} = rN(t) \left(1 - \frac{N(t-\Delta)}{K} \right)$$

Put $u = N/K$ $\tau = rt$ $\theta = r\Delta$ to get

$$\frac{du(\tau)}{d\tau} = u(\tau)(1 - u(\tau - \theta))$$

We have fix-points at $u^* = 0$ and $u^* = 1$. We study the second fix-point.

Put $u = 1 + \varepsilon$, $\varepsilon \ll 1$

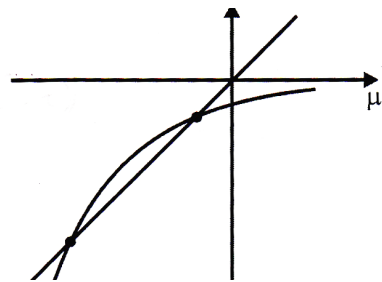
$$\frac{d\varepsilon(\tau)}{d\tau} = (1 + \varepsilon(\tau))(1 - \varepsilon(\tau - \theta) - 1) \approx -\varepsilon(\tau - \theta)$$

Try the ansatz $\varepsilon(\tau) = ce^{\lambda\tau}$ that implies $\lambda = -e^{-\lambda\theta}$. Put $\lambda = \mu + i\omega$ to get

$$\mu = -e^{\mu\theta} \cos \omega\theta$$

$$\omega = e^{\mu\theta} \sin \omega\theta$$

Real solutions are given by $\omega = 0$, $\mu = -e^{\mu\theta}$. We show a graphical solution where we plot the two sides of the last equation.



When $\theta = 0$ we have two solutions $\mu = -1$ (stable)

and $\mu = -\infty$. As θ becomes larger these solutions approach and when $\theta = 1/e$ the coincide at $\mu = -e$. There are also an infinite set of damped oscillatory solutions.

If $1/e < \theta < \pi/2$ there is an infinite set of damped oscillation. For larger θ the solutions have $\mu > 0$, i.e. they are instable oscillatory solutions. However, in the exact differential equation the non-linear terms will give rise to a limit cycle and the amplitude of the oscillations will be finite.

As this example shows even a very simple time-lag model has a quite complicated behaviour. In general they have been to be handled with computer simulations.

3.2 Fishing and depletion fishing

Example 3.3 Fishing with constant relative intensity

A simple model can here be formulated

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right) - EN$$

where E measures the intensity of the fishing. Fix-points are $N_1^* = 0$ (instable) and $N_2^* = K(1 - E/r)$ that is positive and stable if $E < r$. The result of the fishing is

$$Y = EN = EK(1 - E/r)$$

at the fix-point. The optimal profit will be when $E = r/2$ i.e. $Y_{\max} = rK/r$ that implies $N_2^* = K/2$.

We now study the system around the fix-point for two values of E , $E = 0$ and $E \neq 0$.

$E = 0$:

$$N = K + n, \quad n \ll N \Rightarrow \frac{dn}{dt} = -rn \Rightarrow n \propto e^{-rt}$$

The typical damping time $T(0) = 1/r$.

$E \neq 0$:

$$N = K(1 + E/r) + n \Rightarrow \frac{dn}{dt} = -(r - E)n \Rightarrow T(E) = 1/(r - E)$$

At the optimal result we then have $T(E = r/2) = 2/r$ that is disturbances dampen out twice as slow.

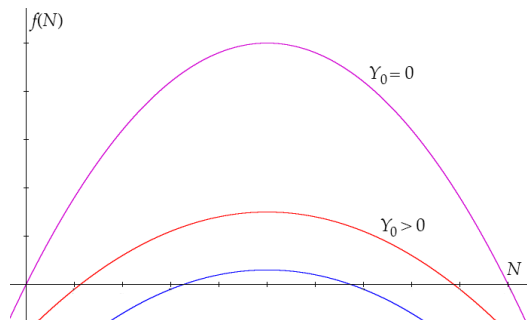
Example 3.4 Fishing quotient

Now assume that we fish with a constant result quotient $Y = Y_0$. We then have

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right) - Y_0 = f(N)$$

We have two fix-points different from zero with the same stability properties, the smaller one is instable, the larger one stable. We plot $f(N)$ for some values of Y_0

Assume that Y_0 is set close to the optimum profit, the lowest curve. A small fluctuation can then take us from the right stable fix-point, to just below the left instable fix-point. Then $N \rightarrow 0$ in a finite time. A system with fishing quota can lead very quickly to extermination if set close to the optimum profit.



3.3 Some other one-dimensional models. Bifurcations and hysteresis

Example 3.5 The overdamped harmonic oscillator

The overdamped harmonic oscillator is described by the following equation

$$m\ddot{q} + \gamma\dot{q} = F(q)$$

where q is a position variable. Assume that m is small and γ large.

$$\dot{q} \approx \frac{F(q)}{\gamma} = f(q)$$

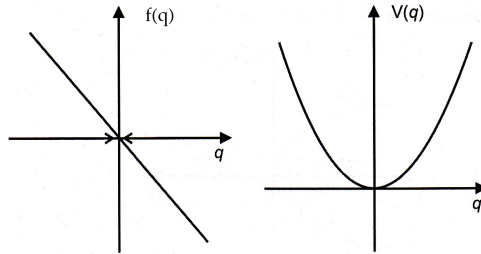
An ordinary harmonic overdamped oscillator would be described by

$$\dot{q} = -kq$$

This system has a stable fix-point at $q = 0$. We can also determine the potential

$$f(q) = -\frac{dV}{dq} \Rightarrow V(q) = \frac{1}{2}kq^2$$

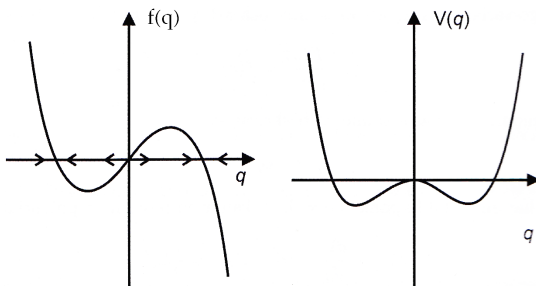
From the graph of the potential we see that the particle will try to reach the bottom of the potential well.



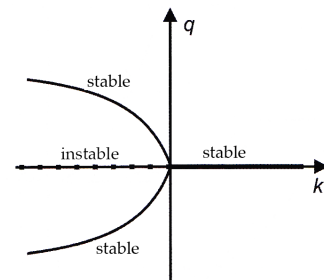
Now study an anharmonic oscillator $f(q) = -kq - k_1q^3$ $\dot{q} = -kq - k_1q^3$

The potential is $V(q) = \frac{1}{2}kq^2 + \frac{1}{4}k_1q^4$

The system has three possible fix-points, $q = 0$ and $q_{\pm} = \pm\sqrt{-k/k_1}$. If $k > 0$ only the first fix-point exists and is stable. If $k < 0$, the fix-point $q = 0$ becomes unstable and instead the fix-points $q_{\pm} = \pm\sqrt{-k/k_1}$ will be stable as $q = q_{\pm} + \varepsilon \Rightarrow \dot{\varepsilon} = -2|k|\varepsilon$.



If we plot the fix-points as a function of k we have the graph, i.e. a so called *bifurcation*. Notice that although the potential is symmetric in q , the system will end up in one of the valleys. The state of the system is not symmetric. This is called *spontaneous symmetry breaking* and you will find it for instance in particle physics in the Higgs mechanism giving mass to particles.



Example 3.6 The tree

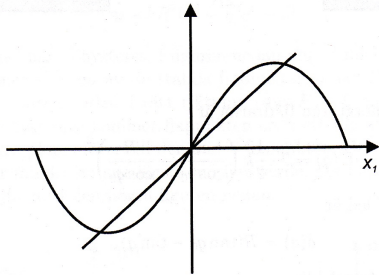
Very simplified a tree can be described as a trunk without mass with all mass gathered in at the top. The roots, trying to keep the tree vertical can be assumed to have a torque that is proportional to the angle of the trunk with the vertical. Newton's second law gives

$$I\ddot{\theta} = -k\theta + mgL \sin \theta$$

Define the system variables $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then

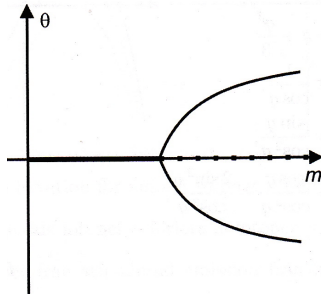
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{I}x_1 + \frac{mgL}{I}\sin x_1 \end{cases}$$

Fix-points are given by $x_2 = 0$ and $-kx_1 + mgL \sin x_1 = 0$. The second equation can be solved by drawing a graph



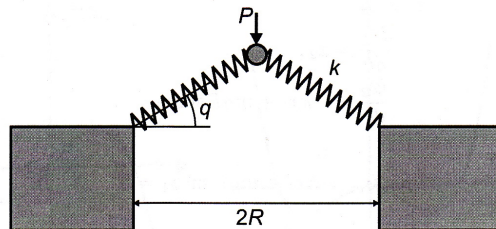
If $k / mgL > 1$ or $m < k / gL$ we have one stable fix-point: $x_1 = 0$. If $m > k / gL$ this fix-point will be unstable and there will be additional two stable fix-points at $x_1 = \pm X$.

We have a bifurcation:



Example 3.7 The bridge

Consider the bridge consisting of two straight springs with a load P applied to the middle.



The potential energy of the springs is $U(q) = 2 \cdot \frac{1}{2} k \left(\frac{R}{\cos q_0} - \frac{R}{\cos q} \right)^2$, q_0 is the angle when $P = 0$.

The deflection downwards is $\delta(q) = R(\tan q_0 - \tan q)$

The total energy of the system is $V(q) = U(q) - P\delta(q)$

In order to study this system we need some Taylor expansions:

$$\tan q \approx q + q^3 / 3$$

$$f(q) = 1 / \cos q \quad f(0) = 1$$

$$f'(q) = \sin q / \cos^2 q \quad f'(0) = 0$$

$$f''(q) = \cos q / \cos^3 q + 2 \sin^2 q / \cos^3 q \quad f''(0) = 1$$

$$f^{(3)}(0) = 0$$

$$f^{(4)}(0) = A$$

Thus

$$V(q) \approx kR^2 \left(1 + \frac{q^2}{2} + \frac{Aq^4}{24} - 1 - \frac{q_0^2}{2} - \frac{Aq_0^4}{24} \right)^2 - P \left(q + \frac{q^3}{3} - q_0 - \frac{q_0^3}{3} \right)$$

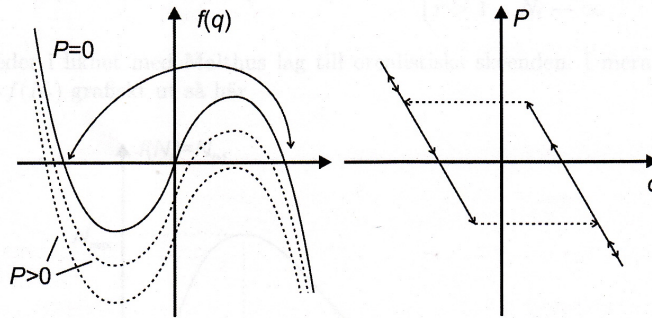
Saving terms up to order q^4 we have

$$V(q) \approx \frac{kR^2}{4} (q^4 + q_0^4 - 2q^2 q_0^2)^2 - P \left(q + \frac{q^3}{3} - q_0 - \frac{q_0^3}{3} \right)$$

We compute

$$f(q) = -\frac{dV}{dq} = -kR^2 (q^3 - q_0^2 q) - P(1 + q^2)$$

Also in this example we have hysteresis. In the graph below we illustrate how it comes about. Assume that the system is in the the stable fix-points to the right when $P = 0$. Increase P that lowers the curve and moves the right stable fix-point left and the instable middle fix-point to the right. Finally the stable fix-point to the right will cease to exist and the system jumps to the other stable fix-point to the left.



Example 3.8 Laser

An equation for the number of photons in a laser is

$$\dot{n} = \text{produced photons} - \text{lost photons} = A - F$$

The produced photons come from stimulated emission of excited atoms, assume the number of excited atoms is N . Then

$$A = GNn, \quad G = \text{constant}$$

The photons leak out at the two ends of the laser (required in order that we see a laser beam)

$$F = 2\kappa n$$

The number of excited atoms decrease when they are de-excited and send out photons. Assume that we with pumping the laser produce N_0 excited atoms. Also assume that the real number of excited atoms is smaller

$$N = N_0 - \alpha n$$

with more photons in the laser we will have more de-excited atoms. This gives

$$\dot{n} = GNn - 2\kappa n = GN_0n - 2\kappa n - G\alpha n^2 = -kn - k_1n^2$$

$$k = 2\kappa - GN_0 \quad k_1 = G\alpha$$

If N_0 is small (weak pumping), $k > 0$. Of the two fix-points $n = 0$ and $n = -k / k_1$ only the first one is stable and physical. If $N_0 > 2\kappa / G$ then $k < 0$ and we have two physical fix-points, $n = 0$ is instable and $n = |k| / k_1$ is stable. There is a kind of phase transition.

4 Discrete one-dimensional population models

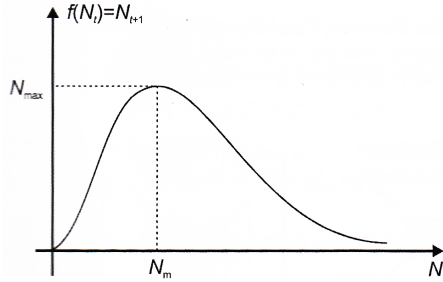
All one-dimensional discrete population models can be gathered in a difference equation of the type

$$N_{t+1} = f(N_t) = N_t F(N_t)$$

A simple example is

$$N_{t+1} = rN_t \Rightarrow N_t = r^t N_0 \Rightarrow \begin{cases} r > 1 & N_t \rightarrow \infty \\ r < 1 & N_t \rightarrow 0 \end{cases}$$

but this like Malthus' law leads to unrealistic descriptions. In more realistic models $f(N_t)$ may graphically look like this



A simple model is the discrete logistic model

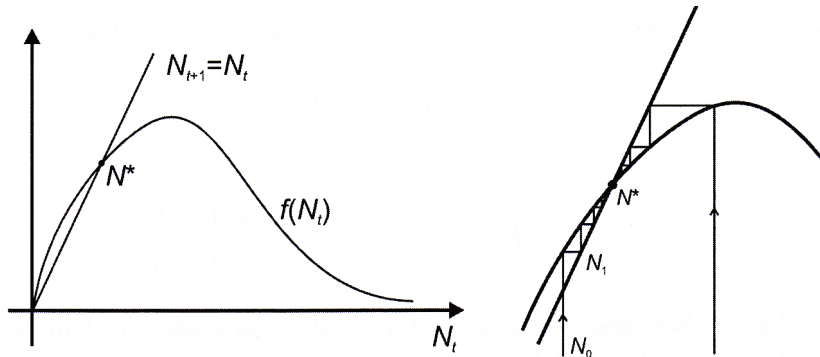
$$N_{t+1} = rN_t \left(1 - \frac{N_t}{K}\right), \quad r > 0, \quad K > 0$$

Observe that if $N_t > K$, $N_{t+1} < 0$ that is unphysical, this is because the equation is quadratic and lacks the asymptotic shape illustrated in the figure above. A more realistic alternative is

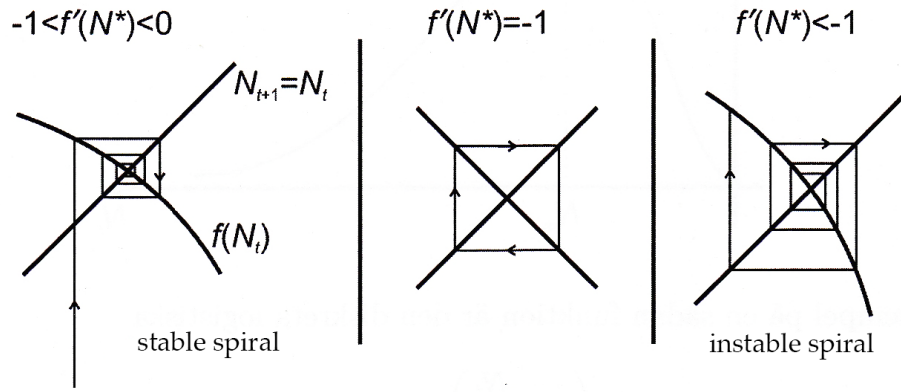
$$N_{t+1} = N_t e^{r(1-N_t/K)}$$

4.1 Graphical solution

As for the continuous models $f(N)$ can be used to find fix-points but not in the same way as earlier; instead we look at the intersections between the curves $N_{t+1} = f(N_t)$ and $N_{t+1} = N_t$, the last being the condition for a fix-point $N^* = f(N^*)$.



By stepping from a point in the neighbourhood of the fix-point we can determine if it is stable or not, in the figure above the fix-point is stable as we approach the fix-point. In general the fix-point N^* is stable if $0 < |f'(N^*)| < 1$. Some alternatives are shown in the figure below.



In general we have at a fix-point $N_{t+1} = f(N_t)$ or $N^* = f(N^*)$. We linearize around the fix-point:

$$N^* + \varepsilon_{t+1} = f(N^* + \varepsilon_t) \approx f(N^*) + f'(N^*)\varepsilon_t \Rightarrow$$

$$\varepsilon_{t+1} = f'(N^*)\varepsilon_t = \lambda\varepsilon_t$$

$\lambda = f'(N^*)$ is called the *eigenvalue*, and we have

$$0 < \lambda < 1, \quad \varepsilon_t \rightarrow 0, \quad \text{stable}$$

$$-1 < \lambda < 0, \quad \varepsilon_t \rightarrow 0, \quad \text{stable, alternating}$$

$$|\lambda| > 1, \quad \text{unstable}$$

Example 4.1 Discrete logistic model

We describe the system by

$$u_{t+1} = ru_t(1 - u_t), \quad r > 0$$

Assume that $0 < u_0 < 1$, such that $u_t > 0$. The fix-points are

$$u_1^* = 0 \quad \lambda_1 = f'(0) = r$$

$$u_2^* = (r-1)/r \quad \lambda_2 = f'(u_2^*) = 2-r$$

If we first consider the case $0 < r < 1$, we have that u_1^* is stable and that u_2^* is unstable. When $r = 1$ we have a bifurcation: u_1^* becomes unstable and u_2^* stable.

Now study $u_{t+2} = ru_{t+1}(1 - u_{t+1}) = r[ru_t(1 - u_t)][1 - ru_t(1 - u_t)]$. Are there fix-points such that $u^* = u_{t+2} = u_t$? We try

$$u^* = r[ru^*(1 - u^*)][1 - ru^*(1 - u^*)]$$

We see directly that $u^* = 0$ still is a fix-point. Put $K = r(1 - u^*)$ in the equation where we have divider out u^*

$$1 = rK[1 - K(1 - K/r)] = rK - rK^2 + K^3$$

We see that $K = 1$ satisfies the equation

$$K^3 - rK^2 + rK - 1 = (K-1)(K^2 + K(1-r) + 1) = 0$$

Inserting the expression for u^* we solve the second order equation

$$u^* = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r} \text{ real and } > 0 \text{ if } 3 < r < 4$$

If $r > 3$ we get two new stable fix-points A and B . It can be shown that $u_{A+1}^* = u_B^*$ and $u_{B+1}^* = u_A^*$. We have a two-cycle. This continues such that for $r = r_4$, we get a new bifurcation and $u_{t+4} = u_t$, for $r = r_8$ we have $u_{t+8} = u_t$ and so on. The difference between the r 's becomes less and less. For a certain $r = r_c$, we have instability for all cycles 2^n . When this occurs we have a three-cycle $u^* = u_{t+3} = u_t$. Numerically $r = r_c = 3.828...$ For $r > r_c$ the solutions are aperiodic or *chaotic*.

Feigenbaum 1978 showed that generally for maps

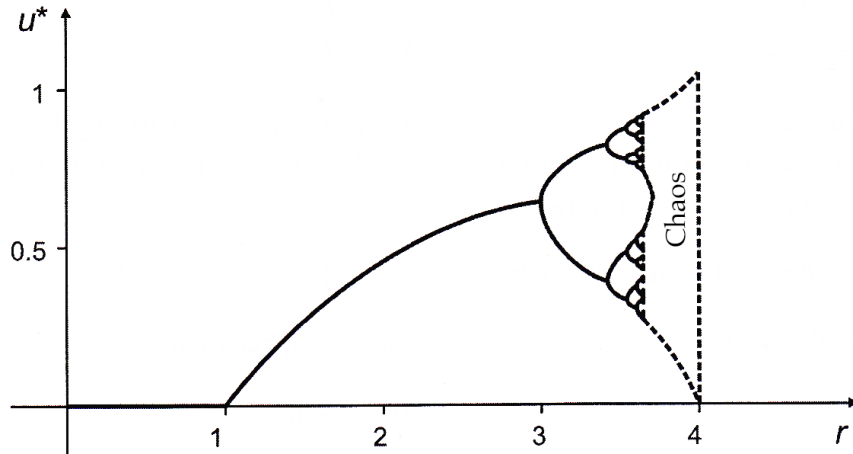
$$u_{t+1} = f(u_t)$$

we have

$$\lim_{n \rightarrow \infty} \frac{r_{2(n+1)} - r_{2n}}{r_{2(n+2)} - r_{2(n+1)}} = \delta \approx 1.66920...$$

where δ is a *universal* constant.

In the graph below we sketch the bifurcation behaviour for our example. As seen it has a fractal structure.



We can define chaos using the correlation function

$$C(n_t, n_{t'}) = \langle (n_t - \langle n_t \rangle)(n_{t'} - \langle n_{t'} \rangle) \rangle$$

If $C(n_t, n_{t'}) = 0$ when $|t - t'|$ is large we say that we have a chaotic system.

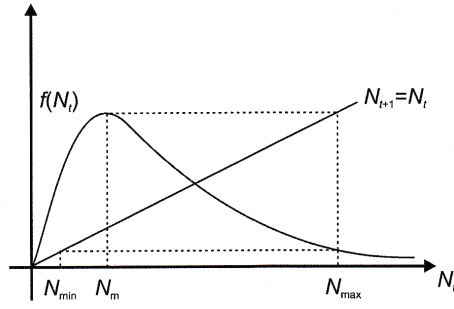
4.2 A digression on ecology

Example 4.2

We study a model for population dynamics

$$N_{t+1} = N_t e^{r(1-N_t/K)} = f(N_t)$$

The fix-points are $N^* = 0$ (unstable if $r > 0$) and $N^* = K$ (stable if $0 < r < 2$). We want to investigate the intervals within the population can vary. Graphically we have



We have that N_m is determined by $\frac{df}{dN_t} = 0$ or

$$e^{r(1-N_t/K)}(1 - rN_t/K) = 0 \Rightarrow N_m = K/r$$

We then have

$$N_{\max} = f(N_m) = Ke^{r-1}/r$$

From the graph we see that

$$N_{\min} = f(N_{\max}) = Ke^{2r-1-e^{r-1}}/r$$

If $N_{\min} < 1$ the population disappears, as it cannot reproduce, i.e. for survival we require

$$Ke^{2r-1-e^{r-1}}/r > 1$$

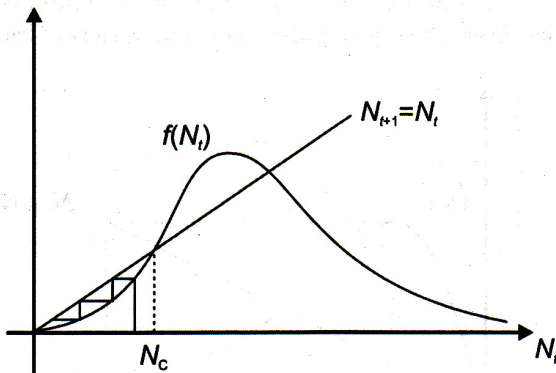
If we now study for instance $r = 3.5$, we have that K must be larger than about 1600, And $r = 5$ implies that $K > 10^{20}$. If $r = 3.5$ we note that in the next to last population we have

$$N = N_{\max} = Ke^{r-1}/r \approx 3500$$

we have a population collapse.

Example 4.3 Allee effect

Assume that $f(N_t)$ looks as in the following graph.



If N_c gets below N_c the, $N_t \rightarrow 0$ as $t \rightarrow \infty$. Thus we have a threshold when the population disappears (Allee effect).

Example 4.4 Another model that has been used in population dynamics is

$$N_{t+1} = f(N_t) = \frac{rN_t}{(1 + aN_t)^b} \quad b > 0$$

5 Interacting populations

5.1 Lotka-Volterra

The Lotka-Volterra equations is an early (1926) example of a continuous model for interacting populations. The model treats fishing in the Adriatic Sea and tries to explain periodic variations in the catch using an interaction between predator and prey fishes. The equations are

$$\begin{aligned}\frac{dN}{dt} &= N(a - bP) \\ \frac{dP}{dt} &= P(cN - d)\end{aligned}$$

where N is the number of predators and P the number of preys. All parameters a, b, c , and d are larger than zero. As usual we start by finding the fix-points

$$(N^*, P^*) = (0, 0) \text{ and } (N^*, P^*) = (d/c, a/b)$$

Introduce dimensionless units

$$\tau = at \quad u = cN/d \quad v = bP/a \quad \alpha = d/a$$

and get

$$\begin{aligned}\frac{du}{d\tau} &= u(1 - v) = f(u, v) \\ \frac{dv}{d\tau} &= \alpha v(u - 1) = g(u, v)\end{aligned}$$

reducing the number of parameters to one. The fix-points are

$$(u^*, v^*) = (0, 0) \text{ and } (u^*, v^*) = (1, 1)$$

The stability in the fix-points is given by the matrix (using the notation $\frac{\partial f}{\partial u} = f_u$)

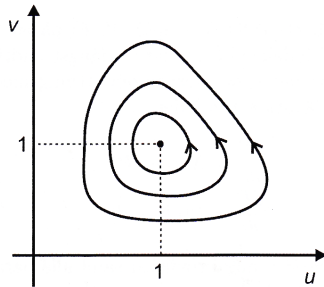
$$\mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} 1 - u & -u \\ \alpha v & \alpha(u - 1) \end{pmatrix}$$

Insert the fix-points

$$(0, 0): \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \quad |\mathbf{A}| = -\alpha \quad \text{Tr } \mathbf{A} = 1 - \alpha \Rightarrow \text{instable}$$

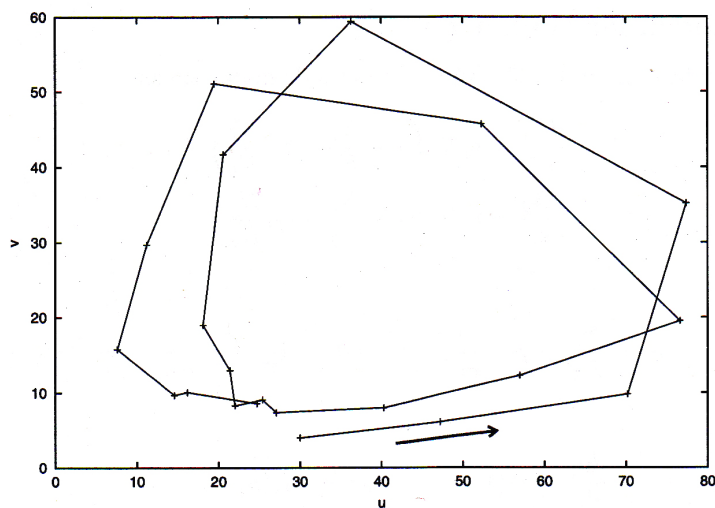
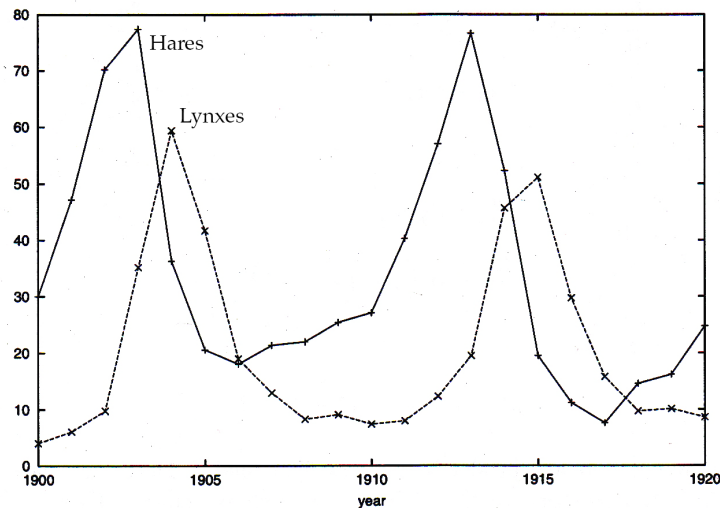
$$(1, 1): \quad \mathbf{A} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} \quad |\mathbf{A}| = \alpha \quad \text{Tr } \mathbf{A} = 0 \Rightarrow \text{stable oscillatory}$$

The trajectories around the second fix-point are sketched below



One of the problems with this model is that random variations can have the system jump between the trajectories. Sooner or later u or v will be zero and the system collapses. This can be remedied by adding a so called Verhulst term for the prey proportional to $-eN^2$ resulting in stable spirals.

The model has been tested on hares and lynxes in Canada. Below is a plot of the number of hare and lynx pelts collected (in thousands) between 1900 and 1920¹ that shows the expected oscillatory behaviour. In the bottom plot are the trajectories. We can see the oscillations around the fix-point and the random jumps between the trajectories.



If we add a term representing fishing with intensity f in the Lotka-Volterra model we get the equations

$$\begin{aligned}\frac{du}{d\tau} &= u(1-v) - \beta u & \beta &= f\alpha / c \\ \frac{dv}{d\tau} &= \alpha v(u-1) - \gamma v & \gamma &= f / b\end{aligned}$$

¹ Data from <http://www-rohan.sdsu.edu/~jmahaffy/courses/f00/math122/labs/labj/q3v1.htm>

The second fix-point will move to $(1 + \gamma / \alpha, 1 - \beta)$

The consequences of the fishing is that the number of prey fishes *increase* due to the decrease of the number of predators. This can only happen up to a certain limit. When f becomes so large that $\beta = 1$, $v = 0$ and the system collapses. If f is increased further $u \propto e^{(1-\beta)t}$ and goes exponentially to zero. As before we can have random fluctuations around the fix-point that may cause v to be zero earlier.

5.2 A model for competition

We will now formulate a model for two competing populations,

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_2} \right) \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_1} \right)\end{aligned}$$

Introduce

$$u_1 = N_1 / K_1, u_2 = N_2 / K_2, \rho = r_2 / r_1, \tau = r_1 t, a_{12} = b_{12} K_2 / K_1, a_{21} = b_{21} K_1 / K_2$$

to get

$$\begin{aligned}\frac{du_1}{d\tau} &= u_1 (1 - u_1 - a_{12} u_2) = f_1(u_1, u_2) \\ \frac{du_2}{d\tau} &= \rho u_2 (1 - u_2 - a_{21} u_1) = f_2(u_1, u_2)\end{aligned}$$

The fix-points are

$$\begin{aligned}(u_1^*, u_2^*) &= (0, 0), \quad (u_1^*, u_2^*) = (1, 0), \quad (u_1^*, u_2^*) = (0, 1) \\ (u_1^*, u_2^*) &= \left(\frac{1 - a_{12}}{1 - a_{12} a_{21}}, \frac{1 - a_{21}}{1 - a_{12} a_{21}} \right) \quad a_{12} a_{21} \neq 1\end{aligned}$$

The system matrix is

$$\mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} 1 - 2u_1 - a_{12}u_2 & -a_{12}u_1 \\ -\rho a_{21}u_2 & \rho(1 - 2u_2 - a_{21}u_1) \end{pmatrix}$$

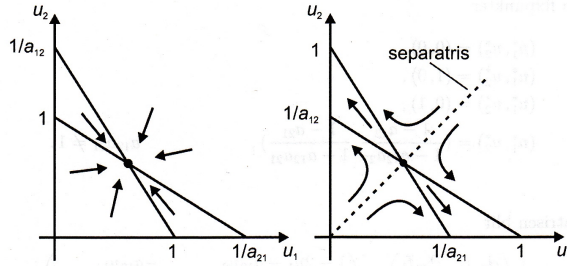
We insert the fix-points and determine the eigenvalues of the matrix

$$\begin{aligned}(0, 0): \quad & \begin{vmatrix} -1 - \lambda & 0 \\ 0 & \rho - \lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = \rho \quad \text{instable} \\ (1, 0): \quad & \lambda_1 = -1, \lambda_2 = \rho(1 - a_{21}) \quad \begin{cases} a_{21} > 1 & \text{stable} \\ a_{21} < 1 & \text{instable} \end{cases} \\ (0, 1): \quad & \lambda_1 = 1 - a_{12}, \lambda_2 = -\rho \quad \begin{cases} a_{12} > 1 & \text{stable} \\ a_{12} < 1 & \text{instable} \end{cases}\end{aligned}$$

The last fix-point is more complicated. We first note that it is determined by the intersections of the lines

$$1 - u_1 - a_{12}u_2 = 0 \quad \text{and} \quad 1 - u_2 - a_{21}u_1 = 0$$

The first line intersects the fix-point (1, 0) the second one the fix-point (0, 1). We know the stability properties of these points depending on the size of a_{12} and a_{21} . By extrapolating the trajectories in the neighbourhood of these points we can get a picture of how the last fix-point behaves. In the figure below we show this.



In the first case there is a possible way for the two populations to co-exist. In the second case one of them will disappear.

Example 5.1 Symbiosis.

We can remake the model for competition to a model with symbiosis by simply changing the signs of the competition terms

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1} + b_{12} \frac{N_2}{K_2} \right) \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2} + b_{21} \frac{N_1}{K_1} \right)\end{aligned}$$

The new fix-points are

(0,0): instable node

(1,0): saddle point

(0,1): saddle point

$$\left(\frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}} \right) \text{ stable if } a_{12}a_{21} < 1$$

5.3 Discrete models

The analytical treatment of discrete model easily gets complicated. We study here a model that is manageable but will turn out to be too simple to give reasonable results.

Example 5.2 A discrete prey-predator model is

$$\begin{aligned}N_{t+1} &= r N_t e^{-a P_t} \\ P_{t+1} &= N_t (1 - e^{-a P_t}) \quad a > 0\end{aligned}$$

The fix-points are

$$(N^*, P^*) = (0, 0)$$

and

$$\begin{aligned}1 &= r e^{-a P^*} \quad P^* = N^* (1 - e^{-a P^*}) \Rightarrow \\ P^* &= \frac{1}{a} \ln r \quad N^* = \frac{r}{a(r-1)}\end{aligned}$$

We investigate the stability by linearizing around the fix-points

$$N_t = N^* + n_t \quad |n_t / N^*| \ll 1$$

$$P_t = P^* + p_t \quad |p_t / P^*| \ll 1$$

For the first fix-point we get

$$n_{t+1} = rn_t, \quad p_{t+1} = 0 \Rightarrow (n_t, p_t) \xrightarrow{t \rightarrow \infty} \begin{cases} (0, 0); & r < 1 \\ (\infty, 0); & r > 1 \end{cases}$$

i.e. it is stable if $r < 1$. For the other fix-point we have

$$n_{t+1} = n_t - aN^*p_t, \quad p_{t+1} = n_t(1 - 1/r) + aN^*p_t/r$$

where we have used that $1 = re^{-aP^*}$. We iterate the first equation once

$$\begin{aligned} n_{t+2} &= n_{t+1} - aN^*p_{t+1} = n_{t+1} - aN^*(n_t(1 - 1/r) + aN^*p_t/r) = \\ &= n_{t+1} - aN^*(n_t(1 - 1/r) + (n_t - n_{t+1})/r) = \\ &= (1 + aN^*/r)n_{t+1} + aN^*n_t \end{aligned}$$

Assume solutions of type $n_t = Ax^t$. The last equation becomes

$$x^2 - (1 + aN^*/r)x + aN^* = 0$$

or

$$x^2 - \left(1 + \frac{1}{r-1} \ln r\right)x + \frac{r}{r-1} \ln r = 0$$

The second order equation has solutions

$$x_{1,2} = \frac{1}{2} \left[1 + \frac{\ln r}{r-1} \pm \sqrt{\left(\frac{r + \ln r}{r-1}\right)^2 - \frac{4r \ln r}{r-1}} \right]$$

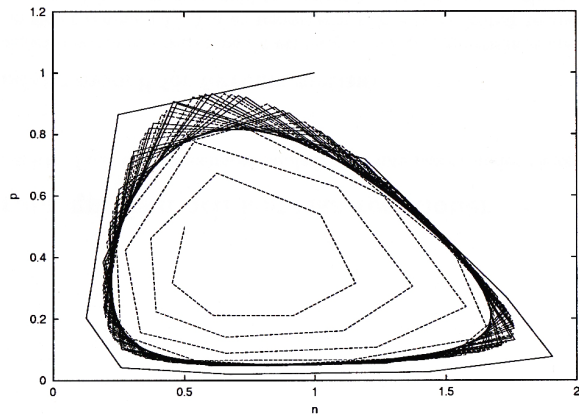
The general solution is $n_t = A_1x_1^t + A_2x_2^t$ and in the same way $p_t = B_1y_1^t + B_2y_2^t$.

The *stability* in this fix-point depends on the modulus $|x_{1,2}|$ and $|y_{1,2}|$. If all four are less than 1, the fix-point is stable. It can be shown that $|x_1| > 1$ if $r > 1$, i.e. that the positive fix-point is instable with growing oscillations. The model is too simple a possible variant is

$$N_{t+1} = N_t e^{r(1-N_t/K - aP_t)}$$

$$P_{t+1} = N_t (1 - e^{-aP_t}) \quad a > 0$$

This can be handled in the same way as before but you get complicated equations that have to be solved numerically. In the figure below you see the result of 100 iterations with starting condition $(n_0, p_0) = (1.0, 1.0)$ (full curve) and $(0.5, 0.5)$ (hatched), the parameter values are $r = 1.2$, $a = 2$, $K = 2$, it is evident that there is a limit cycle.



Example 5.3 Another model has been used for competition between two commercial variants of the same product

$$N_{t+1} = \frac{rN_t}{1 + \alpha N_t + AP_t}$$

$$P_{t+1} = \frac{rP_t}{1 + \alpha P_t}$$

where A stands for *advantage* and denotes the advantage of one of the products. In New Scientist #1859, 6. February 1993, there are examples of different competing products that historically have followed this model.

More generally we can study a linearization of the kind

$$\begin{pmatrix} n_{t+1} \\ p_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} n_t \\ p_t \end{pmatrix}$$

In example 5.2 above we have

$$\mathbf{A} = \begin{pmatrix} 1 & -N^*a \\ 1 - 1/r & N^*a/r \end{pmatrix}$$

We again assume a solution of type

$$\begin{aligned} \begin{pmatrix} n_t \\ p_t \end{pmatrix} &= \mathbf{B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^t \Rightarrow \mathbf{A} \begin{pmatrix} n_t \\ p_t \end{pmatrix} = \mathbf{AB} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^t \Rightarrow \begin{pmatrix} n_{t+1} \\ p_{t+1} \end{pmatrix} = \mathbf{AB} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^t \\ &\Rightarrow \mathbf{B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^{t+1} + 1 = \mathbf{AB} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^t \Rightarrow \mathbf{B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x = \mathbf{AB} \end{aligned}$$

i.e. $\mathbf{B} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x$ is an eigenvalue of \mathbf{A} .

6 Magnetism and chemistry

6.1 Landau's model for ferromagnetism

In a ferromagnet, the magnetism is due to that a number of atomic magnetic moments spontaneously direct themselves in the same direction. At a temperature larger than a critical temperature T_c (the *Curie temperature*) the magnetism disappears – the elementary magnets will be randomly oriented. If the temperature T is lowered below T_c the spontaneous magnetism will reappear. Further ferromagnets display different phenomena around the Curie temperature that are typical of a *phase transition*. So for instance is the heat capacity a discontinuous function of temperature.

Assume that the ferromagnet contains elementary magnets with magnetic moment m that can point in two different directions, up and down. The magnetization q then is

$$q = (N_{\uparrow} - N_{\downarrow})m$$

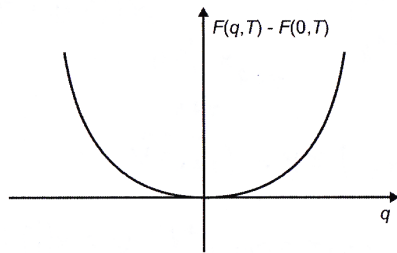
In general a thermodynamical system is in equilibrium when the free energy $F = F(q, T)$ has a minimum. Close to the equilibrium point $q = 0$ we can write

$$F(q, T) = F(0, T) + qF'(0, T) + \frac{q^2}{2}F''(0, T) +$$

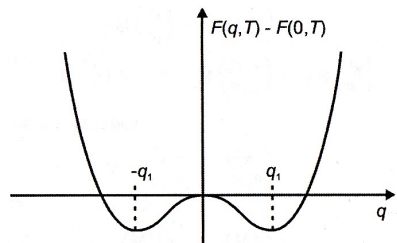
In the absence of an external magnetic field $F(q, T) = F(-q, T)$ and we have only even power terms in the expansion

$$F(q, T) = F(0, T) + \alpha \frac{q^2}{2} + \beta \frac{q^4}{4}$$

Landau's model assumes that $\alpha = a(T - T_c)$, $a > 0$ and $\beta > 0$. A plot of $F(q, T)$ when $T > T_c$ looks like this, the point $q = 0$ is stable.



If $T < T_c$ the factor in front of q^2 is negative and the plot will look like this



The point $q = 0$ is now unstable and we have got two stable points $\pm q_1$. There is a phase transition (bifurcation) into a system with non-zero magnetism. Also not there is a spontaneous symmetry breaking.

For $T < T_c$ we have $\frac{dF}{dq} = qa(T - T_c) + \beta q^3 \Rightarrow q_1 = \sqrt{\frac{a}{\beta}} \sqrt{T - T_c}$

From statistical mechanics we have that the entropy S is given by $S = -\frac{\partial F}{\partial T}$ and the

heat capacity C by $C = T \frac{\partial S}{\partial T}$.

For $T > T_c$ we have ($q = 0$) $S = -\frac{\partial F(0, T)}{\partial T} = S_0$

And for $T < T_c$

$$F(q_1, T) = F(0, T) + \frac{aq_1^2}{2}(T - T_c) + \frac{\beta q_1^4}{4} =$$

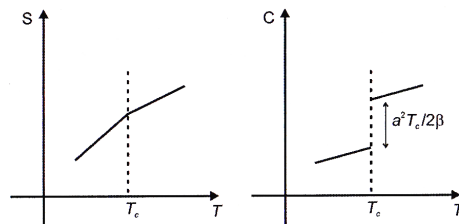
$$F(0, T) - \frac{a^2}{2\beta}(T - T_c)^2 + \frac{a^2}{4\beta}(T - T_c)^2 = F(0, T) - \frac{a^2}{4\beta}(T - T_c)^2$$

The entropy is

$$S = -\frac{\partial F(0, T)}{\partial T} + \frac{a^2}{2\beta}(T - T_c) = S_0 + \frac{a^2}{2\beta}(T - T_c)$$

The heat capacity is

$$C = T \frac{\partial S}{\partial T} + \frac{a^2}{2\beta}T = C_0 + \frac{a^2}{2\beta}T$$

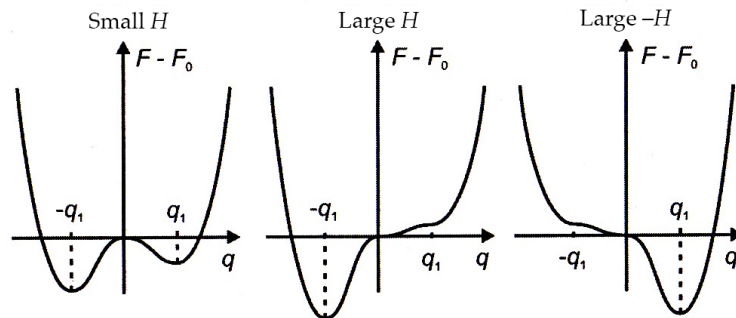


Thus we have a second order phase transition, there is a discontinuous derivative.

In presence of an external magnetic field H we have an extra term in the free energy that is γqH

$$F = F_0 + \gamma qH + \frac{aq^2}{2} + \frac{\beta q^4}{4} + \dots$$

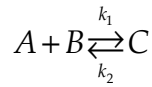
This extra term breaks the symmetry. We plot F for $T < T_c$



The system will exhibit hysteresis.

6.2 Chemical reactions

We study the chemical reaction



k_1 and k_2 determine the probabilities for the reaction. Let a , b , and c be the molar concentrations of A , B , and C respectively. We have then

$$\frac{dc}{dt} = k_1 ab - k_2 c$$

$$\frac{da}{dt} = -k_1 ab + k_2 c$$

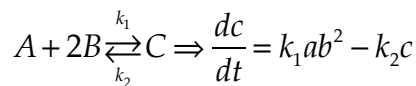
$$\frac{db}{dt} = -k_1 ab + k_2 c$$

The fix-point is given by $k_1 ab - k_2 c = 0$ or

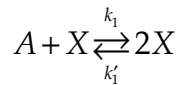
$$\frac{ab}{c} = \frac{k_2}{k_1} = \text{constant depending on temperature}$$

This is Gullberg-Waage's law or the law of mass action.

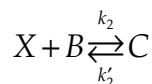
Example 6.1



Example 6.2 Autocatalytic reaction



Further assume that X is transformed into a new molecule C under influence of molecule B .



For the first reaction we have $\dot{x} = k_1 ax - k'_1 x^2$ and for the second one $\dot{x} = -k_2 bx + k'_2 c$.

Assume that A , B , and C are present in such large quantities that their concentrations can be taken as constant. After some rescaling (Exercise) we then have

$$\dot{x} = (1 - \beta)x - x^2 + \gamma = f(x)$$

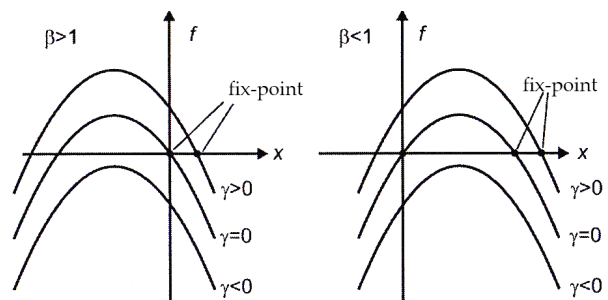
If $\gamma = 0$ we have

$$\dot{x} = x(1 - \beta - x)$$

Stable fix-points are given by

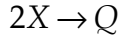
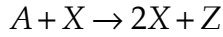
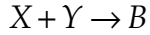
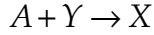
$$\begin{cases} x^* = 0 & \beta > 1 \\ x^* = 1 - \beta & \beta < 1 \end{cases}$$

We can plot $f(x)$ for $\gamma \neq 0$



Example 6.3 The Beluzov-Zhabotinski reaction, a chemical clock.

We mix in a container $\text{Ce}(\text{SO}_4)_3$, KBrO_3 , $\text{CH}_2(\text{COOH})_2$, H_2SO_4 and some drops of ferrion (a redox indicator). After a short while the mixture displays periodic colour changes red (indicates surplus of Ce^{3+} ions) to blue (surplus of Ce^{4+} ions). You can also get spatial patterns like spirals with different colours. The reaction is often called the BZ reaction for short. A simplified model of the reaction is called the *oregonator*.



with $A = \text{BrO}_3^-$, $X = \text{HBrO}_2$, $Y = \text{Br}^-$, $Z = 2\text{Ce}^{4+}$, $B = \text{HOBr}$. Assuming that the concentrations of A and B are constant using suitable rescaling (Exercise) we can write the equations of the system

$$\dot{n}_1 = s(n_2 - n_2 n_1 + n_1 - q n_1^2)$$

$$\dot{n}_2 = t(-n_2 - n_2 n_1 + r n_3)$$

$$\dot{n}_3 = w(n_1 - n_3)$$

where n_1 , n_2 , and n_3 are proportional to the concentrations x , y , and z and s , t , q , r , and w are parameters. If you simulate this system on a computer you will see oscillations in the concentrations for suitable values of the parameters.

7 Systems with diffusion

Introducing diffusion in systems gives rise to many interesting phenomena. Here we have collected some examples.

7.1 Diffusion driven instability – the spots of the leopard

Models that describe patterns on the furs of animals are founded on that there is a morphogenetic *activator* that generates pigment changes and that spreads by diffusion and that there is an *inhibitor* that limits the pigment changes and is activated by the activator. We start with a rather easy example of this.

Example 7.1 Forest fires

Forest fires appear spontaneously and spread slowly – the diffusion constant D_A is small. The forest fires correspond to an activator. Firemen are aroused by the forest fires, attack it rapidly (D_B large) and pour water over the fire front, extinguishing the fire. The firemen function as an inhibitor. The result is a forest with burnt patches.

In 1952 Turing presented a model for diffusion driven pattern generation

$$\frac{\partial \mathbf{c}}{\partial t} = \mathbf{f}(\mathbf{c}) + D \nabla^2 \mathbf{c}$$

\mathbf{c} is a vector of morphogen concentrations. $\mathbf{f}(\mathbf{c})$ represents the different reactions that can take place (the reaction kinetics) and $D \nabla^2 \mathbf{c}$ represents the diffusion. D is a diagonal matrix for the different diffusion constants. A two-dimensional model (two morphogens) looks like this

$$\begin{aligned} \frac{\partial A}{\partial t} &= F(A, B) + D_A \nabla^2 A \\ \frac{\partial B}{\partial t} &= G(A, B) + D_B \nabla^2 B \end{aligned}$$

where F and G as a rule are non-linear functions. In the absence of diffusion $D_A = D_B = 0$, the system has stable fix-points. Then under certain conditions can spatial inhomogeneous patterns can develop by diffusion driven instability if $D_A \neq D_B$. The forest fire above is such an example.

We study a two-dimensional model in one *spatial* dimension. The model is given by Gierer-Meinhard in 1972.

$$F(A, B) = k_1 - k_2 A + k_3 A^2 / B$$

$$G(A, B) = k_4 A^2 - k_5 B$$

By suitable rescaling we can write

$$\partial_t u = a - bu + u^2 / v + \partial_x^2 u$$

$$\partial_t v = u^2 - v + d \partial_x^2 v$$

We first study the fix-points without diffusion

$$f(u, v) = a - bu + u^2 / v$$

$$g(u, v) = u^2 - v$$

Setting these functions to zero results in

$$\begin{cases} u^* = \frac{1+a}{b} \\ v^* = \left(\frac{1+a}{b}\right)^2 \end{cases}$$

We determine the derivatives in the fix-point

$$\begin{aligned} f_u &= -b + 2u/v & f_v &= -u^2/v^2 \\ g_u &= 2u & g_v &= -1 \end{aligned}$$

If we linearize around the fix-point we get the system matrix

$$\mathbf{A} = \begin{pmatrix} \frac{b(1-a)}{1+a} & -\frac{b^2}{(1+a)^2} \\ \frac{2(1+a)}{b} & -1 \end{pmatrix}$$

If we choose $a = b$ we have $\text{Tr } \mathbf{A} < 0$ and $\det \mathbf{A} > 0$, the fix-point is stable.

Let L be the linear size of the domain we are studying. Rescale $x \rightarrow xL$ such that distances are measures with L as the unit. Introduce $\gamma = L^2$. Then we have

$$\begin{aligned} \partial_t u &= \gamma(a - bu + u^2/v) + d \partial_x^2 u \\ \partial_t v &= \gamma(u^2 - v) + d \partial_x^2 v \end{aligned}$$

This does not change the sign of the trace or the determinant. Note that γ is directly related to the linear size of the system and represents the relative strength between the reaction kinetics and the diffusion.

We now study a general system of this type

$$\begin{aligned} \partial_t u &= \gamma f(u, v) + d \partial_x^2 u \\ \partial_t v &= \gamma g(u, v) + d \partial_x^2 v \end{aligned}$$

with a fix-point with $f_u > 0$, $\text{Tr } \mathbf{A} < 0$, and $\det \mathbf{A} \geq 0$ and $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

Further we require the boundary conditions $\nabla u = \nabla v = 0$ for $x = 0$ and $x = 1$.

We assume that $u(x, 0)$ and $v(x, 0)$ are given. Other possible conditions are for instance periodic conditions.

We linearize around fix-points without diffusion, $f(u^*, v^*) = g(u^*, v^*) = 0$.

$$u = u^* + w_1 \quad v = v^* + w_2 \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Still neglecting diffusion we have

$$\frac{\partial \mathbf{w}}{\partial t} = \gamma \mathbf{A} \mathbf{w} \quad \mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{u^*, v^*}$$

Try $\mathbf{w} \propto e^{\lambda t}$, this gives in the usual way stability if $\text{Tr } \mathbf{A} = f_u + g_v < 0$ and $\det \mathbf{A} = f_u g_v - f_v g_u > 0$. We assume from our earlier example that $f_u > 0$ and $g_v < 0$.

The full system is

$$\frac{\partial \mathbf{w}}{\partial t} = \gamma \mathbf{A} \mathbf{w} + D \nabla^2 \mathbf{w} = \gamma \mathbf{A} \mathbf{w} + D \frac{\partial^2 \mathbf{w}}{\partial x^2}$$

Assume solutions of type $\mathbf{w}(x, t) = \mathbf{W}(x)T(t)$. Separation of variables gives

$$\dot{T} = \lambda T$$

$$\lambda \mathbf{W} = \gamma \mathbf{A} \mathbf{W} + D \nabla^2 \mathbf{W}$$

The boundary condition says that $\frac{d\mathbf{W}}{dx} = 0$ at $x = 0$ and $x = 1$. Finite solutions are then of the type

$$\mathbf{W}_n \propto \cos n\pi x, \quad n \text{ integer}$$

We write $\mathbf{W}_n = \mathbf{c}_n \cos n\pi x$. Inserting gives

$$\lambda \mathbf{W}_n = \gamma \mathbf{A} \mathbf{W}_n + D \nabla^2 \mathbf{W}_n \Rightarrow \left(\gamma \mathbf{A} - D(n\pi)^2 - \lambda^{(n)} \cdot \mathbf{1} \right) \mathbf{W}_n = 0$$

that has a solution when and only when

$$\left| \gamma \mathbf{A} - D(n\pi)^2 - \lambda^{(n)} \cdot \mathbf{1} \right| = 0$$

This will give us two $\lambda^{(n)}$:s (\mathbf{A} is a two by two matrix) and the general solution is

$$\mathbf{w}(x, t) = \sum_{n,j=1,2} \mathbf{c}_n e^{\lambda_j^{(n)} t} \cos n\pi x$$

The secular equation is

$$\lambda^2 + \lambda \left[k^2(1+d) - \gamma(f_u + g_v) \right] + h(k^2) = 0$$

$$h(k^2) = dk^4 - \gamma(df_u + g_v)k^2 + \gamma^2|\mathbf{A}|, \quad k = n\pi$$

If $k^2 = 0$ we get back the system without diffusion with $\text{Re } \lambda < 0$ if the system is stable.

Introduce

$$\mathbf{A}' = \gamma \mathbf{A} - Dk^2 = \begin{pmatrix} \gamma f_u - k^2 & \lambda f_v \\ \gamma g_u & \gamma g_v - dk^2 \end{pmatrix}$$

The coefficient for the λ term in the secular equation is the negative of trace of the new diffusion-related matrix. As $\gamma(f_u + g_v) > 0$ and $k^2(1+d) > 0$, this trace is always negative. We want to have instable solutions, i.e. that small disturbances will pull the system away from the spatially homogeneous state. Thus we require that

$$|\mathbf{A}'| = h(k^2) < 0$$

for some value of k . A condition for this is that $\gamma(df_u + g_v) > 0$. As $\gamma(f_u + g_v) > 0$ and $f_u > 0$ ($g_v < 0$) this implies that $d > 1$ (at least) that is the diffusion constant of the inhibitors must be larger than that of the activator. The condition $\gamma(df_u + g_v) > 0$ is necessary but not enough. In order that $h(k^2) < 0$ the minimum h_{\min} must be negative.

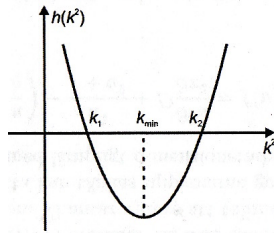
$$\begin{aligned}\frac{dh}{dk^2} &= 2dk^2 - \gamma(df_u + g_v) = 0 \Rightarrow \\ k_{\min}^2 &= \frac{\gamma(df_u + g_v)}{2d} \Rightarrow \\ h_{\min} &= \gamma^2 \left[|A| - \frac{(df_u + g_v)^2}{4d} \right] < 0 \Rightarrow \frac{(df_u + g_v)^2}{4d} > |A|\end{aligned}$$

At the bifurcation $h_{\max} = 0$ and we have a critical $d = d_c$ given by

$$d_c^2 f_u^2 + 2(2f_v g_u - f_u g_v) d_c + g_v^2 = 0$$

and

$$k_c^2 = \frac{\gamma(d_c f_u + g_v)}{2d_c} = \gamma \left[\frac{f_u g_v - f_v g_u}{d_c} \right]^{1/2}$$



For large times the solution will be dominated by the exponentially growing terms with allowed wave numbers $k \in (k_1, k_2)$. It is easy to modify our method for periodic boundary conditions.

The amplitudes of the exponentially growing regions will finally be limited by non-linear terms and we end up with periodic patterns with saturated activators and inhibitors respectively. For infinite domains all wave numbers $k \in (k_1, k_2)$ are allowed. Normally patterns with wave number k_c start to develop as soon as it is possible. That γ is small means that the scale in length is small.

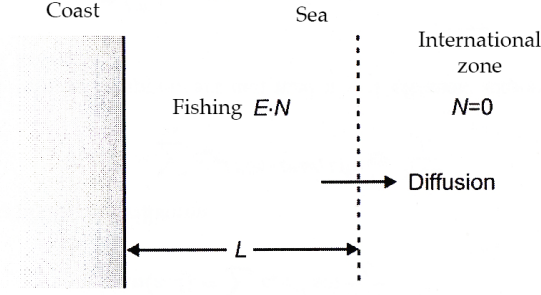
In two dimensions more patterns are possible: stripes, spots, etc. With these mechanisms we can explain the spots of the leopard and the stripes of the zebra. Also note that our model predicts that small animals would be non-patterned (no pattern will fit) like you have for mice. Also large animals would be without patterns (all wave numbers are allowed). The result is similar to what you get with standing waves in plates, something that has been tested practically with interesting results. A good reading is the article of James Murray

(www.resnet.wm.edu/~jxshix/math490/murray.doc)

7.2 Population model with diffusion

Example 7.2 Fishing zones

We consider the following system for fishing



Fishing is regulated in national waters but we assume that in international water the fishing is very intensive such that $N = 0$. The fish is spreading out into international waters by diffusion. We use the following equations:

$$\frac{\partial N}{\partial t} = rN(1 - N/K) - E \cdot N + D \frac{\partial^2 N}{\partial x^2}$$

with boundary values $N(x = L) = 0$ and $\frac{\partial N(x = 0)}{\partial x} = 0$.

If $D = 0$ we have a fix-point at $N = 0$. We linearize around this fix-point

$$\frac{\partial N}{\partial t} = rN - E \cdot N + D \frac{\partial^2 N}{\partial x^2}$$

Assume a solution $N(x, t) = T(t)W(x)$. This gives

$$\frac{dT}{dt} = \lambda T \Rightarrow T \propto e^{\lambda t} \quad \lambda W = (r - E) + D \frac{d^2 W}{dx^2}$$

The boundary conditions tell us that $W \propto \cos \frac{n\pi x}{2L}$ and we get

$$\lambda = (r - E) + D \frac{\pi^2}{4L^2}$$

This fix-point is stable ($\lambda < 0$), i.e. the population disappears if

$$(r - E) + D \frac{\pi^2}{4L^2} < 0 \quad \text{or} \quad L < \frac{\pi}{2} \sqrt{\frac{D}{r - E}}$$

7.3 Wave propagation in systems with diffusion

A system with diffusion is generally described by

$$\frac{\partial u}{\partial t} = f(u) + D \nabla^2 u$$

In one spatial dimension this is

$$\frac{\partial u}{\partial t} = f(u) + D \frac{\partial^2 u}{\partial x^2} \quad u = u(x, t)$$

In such system we can have waves with relatively high propagation speed. Below we show a specific example where we look at wave solutions, see also section 2 in the next chapter.

Example 7.3 Logistic growth with diffusion

Let $f(u) = ru(1 - u)$. Rescale the system $t \leftarrow rt$ $x \leftarrow \sqrt{\frac{r}{D}}x$ to get

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}$$

Without the diffusion we have fix-points in $u = 0$ (instable) and in $u = 1$ (stable). We search for solution in the form of waves that move without distortion (solitons) that have the form

$$u(x, t) = U(x - ct) \equiv U(z), \quad z = x - ct$$

Insertion in the equation gives

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} = -cU' \quad \frac{\partial^2 u}{\partial x^2} = U'' \quad \Rightarrow \quad -cU' = U(1-U) + U''$$

Introduce $V = U'$ such that we can write the system as two first order differential equations

$$\begin{aligned} \frac{dU}{dz} &= V & &= F(U, V) \\ \frac{dV}{dz} &= -cV - U(1-U) & &= G(U, V) \end{aligned}$$

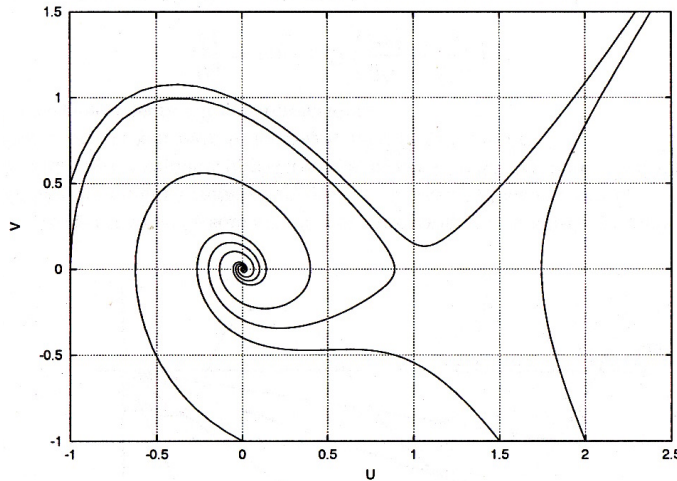
The fix-points are $(U, V) = (0, 0)$ and $(U, V) = (1, 0)$. We investigate the stability using the system matrix

$$\mathbf{A} = \begin{pmatrix} F_U & F_V \\ G_U & G_V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+2U & -c \end{pmatrix}$$

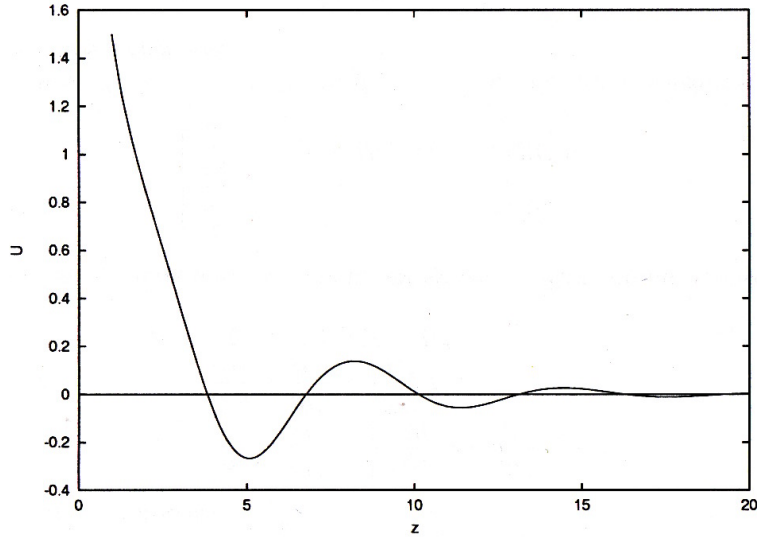
Fix-point $(0, 0)$: $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$, $\text{Tr } \mathbf{A} = -c$, $|\mathbf{A}| = 1$ (stable)

Fix-point $(1, 0)$: $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$, $\text{Tr } \mathbf{A} = -c$, $|\mathbf{A}| = -1$ (saddle point)

The trajectories are plotted below for $c = 0.5$ in order that the spirals are evident. It is also obvious that the second fix-point is a saddle point.



Spirals around $(0, 0)$ means problems as $U < 0$ at times, in many systems this is not allowed. This is also shown in the next graph of $U = U(z)$.



If we give up the requirement that the wave moves without distortion we have to solve the system numerically and we find that there are waves moving with exactly speed $c_{\min} = 2$. In the figure below we plot a simulation with the updating rule

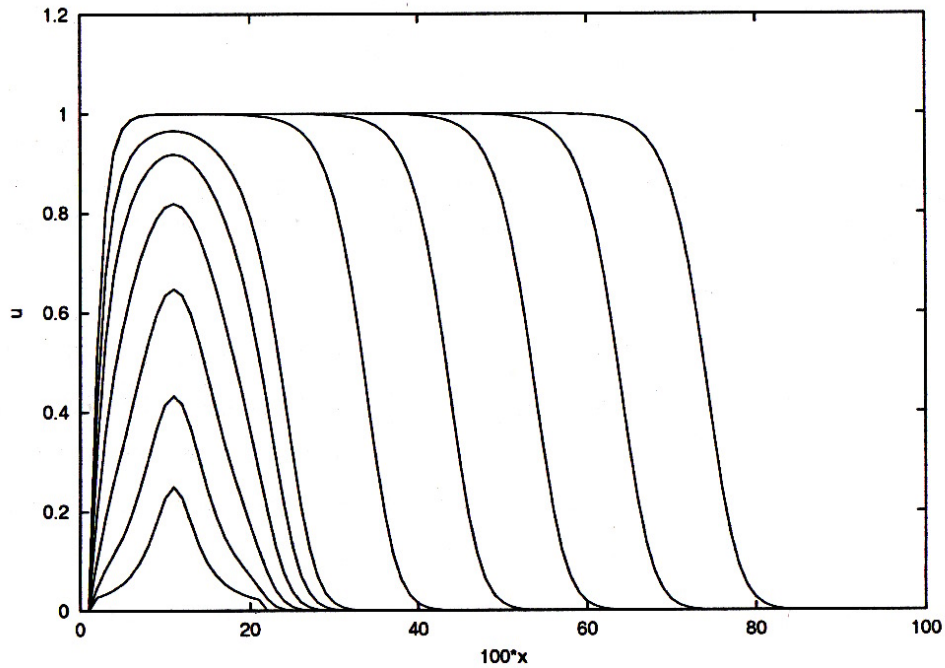
$$u(x, t) \leftarrow u(x, t) + \Delta t (u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)) / (\Delta x)^2 + u(x, t)(1 - u(x, t))$$

and $\Delta t = 0.1$ and $\Delta x = 1$. The boundary condition is

$$u(x, 0) = 1 / (1 + x^2) \quad 0 < x < 100$$

$$u(0, 0) = u(100, 0) = 0$$

In the beginning every 100 iteration is plotted in order to see the growth up to the fix-point. Then $\Delta t = 0.01$, and every 500 iteration is plotted. In the figure the horizontal distance between two fronts is 10 length units, the distance in time is 5 time units, i.e. the speed of the wave front is 2 as expected.



The original system was

$$\frac{\partial u}{\partial t} = ru(1-u) + D \frac{\partial^2 u}{\partial x^2}$$

with fix-points $u = 0$ and $u = u_1 > 0$. The wave speed in the system is given by $c = c_{\min} = 2\sqrt{rD}$. In many systems D is small, say $10^{-6} \text{ m}^2 \text{ s}^{-1}$ and $r \sim 1 \text{ s}^{-1}$. Then $c \approx 10^{-3} \text{ ms}^{-1}$ that is much larger than D , in this case by three orders of magnitude. Thus “information” spreads much faster in waves than by diffusion. This will be important in the description of epidemics in the next chapter.

The appearance of the waves in the graph above is typical of a *shock wave*.

8 Models for epidemics

Example 8.1 Measles

In 1921 Martini presented a model of immunizing diseases (*Berechnungen zur Epidemiologie der Malaria*) like measles, mumps, and rubella. The quantities he used were

I = fraction of the population infected and infective

S = fraction susceptible for infection

p = fraction infected/time

q = fraction cured/time

m = fraction losing their immunity/time

α = infectivity

where $p = \alpha / S$. The model is

$$\frac{dI}{dt} = \alpha IS - qI$$

$$\frac{dS}{dt} = -\alpha IS + m(1 - S)$$

The fix-points are given by

$$(I, S) = (0, 1) \text{ and } (I, S) = \left(\frac{m(\alpha - q)}{q\alpha}, \frac{q}{\alpha} \right), \quad \alpha > q$$

The system matrix is $\mathbf{A} = \begin{pmatrix} \alpha S - q & \alpha I \\ -\alpha S & -\alpha I - m \end{pmatrix}$

In the first fix-point we get

$$\mathbf{A} = \begin{pmatrix} \alpha - q & 0 \\ -\alpha & -m \end{pmatrix} \Rightarrow \begin{matrix} |\mathbf{A}| = -m(\alpha - q) \\ \text{Tr } \mathbf{A} = \alpha - q - m \end{matrix} \text{ stable if } \alpha < q$$

In the second fix-point

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{m(\alpha - q)}{q} \\ -q & \frac{-m\alpha}{q} \end{pmatrix} \Rightarrow \begin{matrix} |\mathbf{A}| = m(\alpha - q) \\ \text{Tr } \mathbf{A} = \frac{-m\alpha}{q} \end{matrix} \text{ stable if } \alpha > q$$

The behaviour of this point is determined by the function

$$\tilde{f} = \frac{\text{Tr}^2 \mathbf{A}}{2} - |\mathbf{A}| = \frac{m^2 \alpha^2}{4q^2} - m(\alpha - q) = \frac{m\alpha^2}{q} \left(\frac{m}{4q} - \frac{q}{\alpha} + \frac{q^2}{\alpha^2} \right)$$

Put $\frac{q}{\alpha} = \beta$ and study $f(\beta) = \beta^2 - \beta + \frac{m}{4q}$

We have $f(0) = \frac{m}{4q} > 0$ and find minima by $f'(\beta) = 2\beta - 1 = 0 \Leftrightarrow \beta = 1/2$. Thus

$f_{\min}(1/2) = \frac{1}{4}(m/q - 1)$. We will have oscillatory solutions when $m/q < 1$. This is quite common in reality.

8.1 SIR model for epidemics

SIR models for epidemics is founded on the following three quantities:

S = the number of susceptible

I = the number of infective and infected

R = the number not available for infection due to immunity or death

Schematically the models can be described by $S \rightarrow I \rightarrow R$

Assume that the different classes are homogeneously mixed with equal probability of being in contact with each other. A simple model by Kermack-McKendrick in 1927 is

$$\begin{aligned}\frac{dS}{dt} &= -rSI \\ \frac{dI}{dt} &= rSI - aI \\ \frac{dR}{dt} &= aI\end{aligned}$$

where $1/a > 0$ is the contagious time and $r > 0$ is the infectivity.

We have

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0 \Rightarrow S + I + R = N, \text{ constant}, \quad 0 < S, I, R < N$$

The starting conditions are given by $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = 0$.

and

$$\left. \frac{dI}{dt} \right|_{t=0} = I_0(rS_0 - a) \quad \begin{cases} > 0 \text{ if } S_0 > a/r \equiv \rho \\ < 0 \text{ if } S_0 < \rho \end{cases}$$

We also have

$$\left. \frac{dS}{dt} \right|_{t=0} < 0 \quad S(t + \varepsilon) < S_0$$

i.e. if $S_0 < \rho$ then $\frac{dI}{dt} = I(rS - a) \leq 0 \quad \forall t$

Thus the infection will die out if we start with too few susceptible, there is no

epidemic. If $S_0 > \rho$, to begin with $\frac{dI}{dt} > 0$ and we have an epidemic. We define an

epidemic as a process with $I(t) > I_0$ for some time $t > 0$. The critical parameter for an epidemic is ρ .

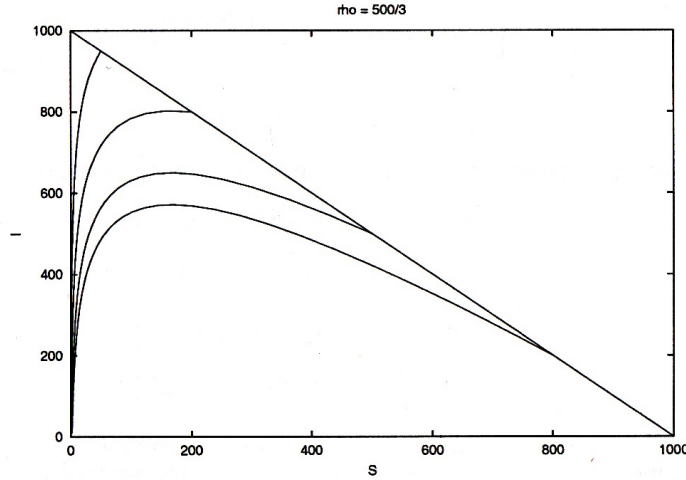
We study trajectories using

$$\frac{dI}{dS} = -\frac{(rS - a)I}{rSI} = -1 + \rho/S$$

This can be integrated

$$I = -S + \rho \ln S + \text{constant} = -S + \rho \ln S + S_0 - \rho \ln S_0 + I_0$$

As $R(0) = 0$, all trajectories start at $I_0 + S_0 = N$. The trajectories in the following graph are simulated using the system equations with $a = 500$ and $r = 3$ for different starting values.



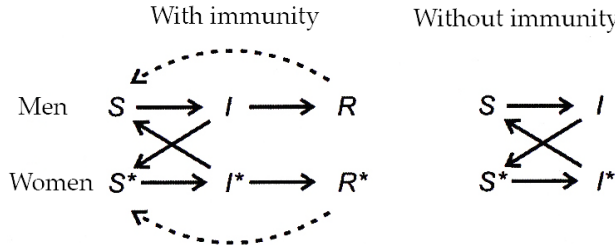
The intensity of the epidemic is determined by the maximum of the curve that occurs when $S = \rho$. We have

$$I_{\max} = N - \rho + \rho \ln$$

that is to be compared with I_0 .

Example 8.2 Model of gonorrhoea.

In models for venereal diseases we have to take into account that there are two different populations, women and men. We denote men by I, S and women by I^*, S^* . The relations between the different populations are described schematically below.



In this example we study gonorrhoea that does not give immunity and chose the simpler model to the right.

We also assume that $I + S = N$ and $I^* + S^* = N^*$ are constants. We use the following model

$$\begin{aligned} \frac{dS}{dt} &= -rSI^* & \frac{dS^*}{dt} &= -rS^*I + a^*I \\ \frac{dI}{dt} &= rSI^* - aI & \frac{dI^*}{dt} &= r^*S^*I - a^*I \end{aligned}$$

This can be simplified if we introduce N and N^* :

$$\frac{dI}{dt} = rI^*(N - I) - aI \quad \frac{dI^*}{dt} = r^*I(N^* - I^*) - a^*I$$

The fix-points are given by $I = I^* = 0$ and

$$I = \frac{NN^* - \rho\rho^*}{\rho + N^*}, \quad I^* = \frac{NN^* - \rho\rho^*}{\rho + N}, \quad \rho = a/r, \quad \rho^* = a^*/r^*$$

When $NN^* > \rho\rho^*$, the second fix-point becomes physical. A stability analysis will show that if $NN^* < \rho\rho^*$ the first fix-point is stable and if $NN^* > \rho\rho^*$, the second one is stable. Data from USA (1973) with $N \approx N^* \approx 20M$ gives an estimation of $N/\rho \approx 0.98$, $N^*/\rho^* \approx 1.15$ such that $NN^*/\rho\rho^* \approx 1.127$. This results in $I \approx 1.12M$, $I^* \approx 1.21M$ that fits well with the population data.

8.2 Spatial spread of epidemics by waves and diffusion

We will now give a model for spatial spreading of epidemics.

$$\begin{aligned} \frac{\partial S}{\partial t} &= -rIS + D\nabla^2 S \\ \frac{\partial I}{\partial t} &= rIS - aI + D\nabla^2 I \end{aligned}$$

Interpret aI as the mortality intensity i.e. the number of dead per time. Assume that at the start have a spatially homogeneous system $S = S_0$. We solve the equations in one dimension and rescale

$$I \leftarrow IS_0 \quad S \leftarrow SS_0 \quad x \leftarrow \sqrt{D/rS_0}x \quad t \leftarrow t/rS_0 \quad \lambda = a/rS_0$$

We then have

$$\begin{aligned} \frac{\partial S}{\partial t} &= -IS + \frac{\partial^2 S}{\partial x^2} \\ \frac{\partial I}{\partial t} &= IS - \lambda I + \frac{\partial^2 I}{\partial x^2} \end{aligned}$$

We study solutions that have the form of propagating waves $I(x, t) = I(z)$ and $S(x, t) = S(z)$ with $z = x - ct$, c is the wave speed. Insertion gives

$$\begin{aligned} I'' + cI' + I(S - \lambda) &= 0 \\ S'' + cS' - IS &= 0 \end{aligned}$$

We want solutions such that

$$I(-\infty) = I(\infty) = 0, \quad 0 \leq S(-\infty) < S(\infty) = 1$$

Study the leading part of the wave where $I \approx 0$ and $S \approx 1$. Linearization gives

$$I'' + cI' + I(1 - \lambda) = 0$$

with solution

$$I(z) \propto \exp\left(\frac{-c \pm \sqrt{c^2 - 4(1 - \lambda)}}{2} \cdot z\right)$$

As $I(z) > 0$ we cannot have oscillatory solutions, i.e. $c \geq 2\sqrt{1 - \lambda}$, $\lambda = a/rS_0 < 1$,

A more extensive (and very sophisticated) analysis shows that the wave under suitable conditions will propagate with the smaller speed. See the example in chapter 7. In dimensional units we have

$$c = 2\sqrt{rS_0 D(1 - a/rS_0)}$$

In order that there is a wave we require

- We must have more than a critical population density $S_0 = a / r$.
- Given S_0 there is a least critical infectivity $r_c = a / S_0$.
- Given r and S_0 there is a mortality threshold $a < a_c = rS_0$. A too high mortality or too rapid process will stop the epidemic.

To stop the epidemic we can

- Vaccinate $\Rightarrow S_0$ decreases
- Use quarantine $\Rightarrow r$ decreases

J. V Noble (*Geographic and temporal development of plagues*, Nature **250**, p. 726-729, 1974) studied the spread of the Black Death and used the parameters $D = 10^4$ miles²/year, $r = 0.4$ miles²/year, $S_0 = 50$ persons /miles², $a = 15$ /year (two weeks) that gave a wave speed of 400 miles/year that is rather reasonable and fit data quite well.

