

Exercises 3

(To be completed before 08.09.24)

1. Importance sampling. One wants to make a Monte Carlo estimate of the integral

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

based on random numbers X with frequency function

$$p_a(x) = \begin{cases} (1-a)x^{-a} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

where $0 < a < 1$. Calculate the mean and variance of $Y = 1/[p_a(X)\sqrt{X}]$ as functions of a .

2. Consider a stationary two-state Markov chain $X_0, X_1, \dots$ ($X_n \in \{1, 2\}$) with transition probabilities $W_{ij} = P\{X_{n+1} = i | X_n = j\}$, $i, j \in \{1, 2\}$. Let $\mathbf{p}(n)$ be the probability distribution after step n , that is the vector with components $P\{X_n = 1\}$ and $P\{X_n = 2\}$.

- (a) Show that the transition matrix $\mathbf{W}$ can be written as

$$\mathbf{W} = \begin{pmatrix} \alpha & 1 - \beta \\ 1 - \alpha & \beta \end{pmatrix}$$

for some $0 \leq \alpha, \beta \leq 1$.

- (b) Solve the eigenvalue problem $\mathbf{W}\mathbf{v} = \lambda\mathbf{v}$.
- (c) Show that unless $\alpha = \beta = 0$ or $\alpha = \beta = 1$

$$\mathbf{p}(n) = \mathbf{W}^n \mathbf{p}(0) \rightarrow \frac{1}{2 - \alpha - \beta} \begin{pmatrix} 1 - \beta \\ 1 - \alpha \end{pmatrix} \quad n \rightarrow \infty$$

where the large-time limit is independent of the initial distribution $\mathbf{p}(0)$.

- (d) How does $\mathbf{p}(n)$ behave in the remaining cases $\alpha = \beta = 0$ and $\alpha = \beta = 1$?