

Solutions to exercises 1.

1. Calculate

$$S_a(x) = \sum_{i=1}^N L_i(x) \quad \text{and} \quad S_b(x) = \sum_{i=1}^N x_i L_i(x)$$

where

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^N \frac{x - x_j}{x_i - x_j} \quad (x_i \neq x_j \text{ if } i \neq j).$$

Lagrange's interpolation formula tells us that the equation

$$f(x) = \sum_{i=1}^N f(x_i) L_i(x)$$

holds for any choice of x_i as long as they are all different if $f(x)$ is a polynomial of order less than N . Hence, we're looking for a polynomial for which $f(x_i) = 1$ and $f(x_i) = x_i$ respectively for all x_i . So the answer is 1 and x respectively, where the latter only holds if $N > 1$.

2. Use the Euler-Maclaurin summation formula to show that

$$\sum_{m=1}^n m = \frac{1}{2}n(n+1) \quad \text{and} \quad \sum_{m=1}^n m^2 = \frac{1}{6}n(n+1)(2n+1)$$

Using $h = 1$ in the Euler-Maclaurin formula gives us

$$\frac{f(0)}{2} + \sum_{m=1}^{n-1} f(m) + \frac{f(n)}{2} = \int_0^n f(m) dm + \sum_{k=1}^{\infty} c_k$$

with

$$c_k = \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(0))$$

and where B_i are the Bernoulli numbers ($B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $\dots$). So in the first sum, we use $f(m) = m$. The integral gives $n^2/2$ and we need to add $f(n)/2 = n/2$. The first derivative of f is constant and all higher derivatives are zero so all c_k vanish and the result is $n(n+1)/2$. In the second sum we use $f(m) = m^2$. The integral becomes $n^3/3$ and we add $f(n)/2 = n^2/2$. The derivative of f is $2m$ and the $B_2 = 1/6$ so we need to add $c_1 = \frac{1/6}{2!}(2n - 0)$, all other c_k vanish. The result is $n(2n+1)(n+1)/6$.

3. Calculate

$$\int_3^4 \frac{dx}{x-2}$$

with the trapezoidal rule for the step sizes 0.5, 0.25 och 0.125. Improve the values by Richardson extrapolation.

The exact answer is, of course, $\log(2)$. The relative error for the trapezoidal rule is approximately 0.022, 0.0055 and 0.0014 respectively. Using step size 0.25 and one Richardson extrapolation gives the relative error 0.00015 and step size 0.125 with two Richardson extrapolation gives 0.000001.

4. Calculate the integral in exercise 3 by using two-point Gaussian integration, after a change of variable to $t = 2x - 7$. The integral becomes

$$\int_{-1}^1 \frac{dt}{t+3} \approx \frac{1}{-\sqrt{1/3}+3} + \frac{1}{\sqrt{1/3}+3}$$

giving a relative error of -0.0012 .

5. Determine an integration formula of the type

$$\int_{-1}^1 f(x)(1+x^2)dx \approx A_1 f(x_1) + A_2 f(x_2)$$

that is exact for all polynomials with degree less than 4.

We need the corresponding second order orthogonal polynomial, and we use the Gram-Schmidt method with $w(x) = 1 + x^2$. As always $\phi_0 = 1$.

$$\phi_1 = x - \frac{\langle \phi_0 | x \rangle}{\langle \phi_0 | \phi_0 \rangle} \phi_0 = x$$

$$\phi_2 = x^2 - \frac{\langle \phi_1 | x^2 \rangle}{\langle \phi_1 | \phi_1 \rangle} \phi_1 - \frac{\langle \phi_0 | x^2 \rangle}{\langle \phi_0 | \phi_0 \rangle} \phi_0 = x^2 - \frac{2}{5}$$

So the zeros are $\pm\sqrt{2/5}$. Inserting $f(x) = 1$ gives

$$\int_{-1}^1 (x^2 + 1)dx = \frac{8}{3} = A_1 + A_2$$

and $f(x) = x$ gives

$$\int_{-1}^1 x(x^2 + 1)dx = 0 = A_1(-\sqrt{2/5}) + A_2(\sqrt{2/5})$$

So $A_1 = A_2 = 4/3$ and

$$\int_{-1}^1 f(x)(1+x^2)dx \approx \frac{4}{3}f(-\sqrt{2/5}) + \frac{4}{3}f(\sqrt{2/5})$$

is exact for all polynomials with degree less than 4.